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SYLLABUS

nit I : Series of positive terms - Convergence of infinite series, addition and noval of terms, n^{th} term test, Cauchy's criterion, series of positive terms, mparison theorem, Comparison test, Ratio and Root test, Comparison of io, Raabe's test, logarthmic test, Cauchy's Condensation test, De Morgan & rtrand's test, Higher logarthmic ratio test.

nit II: General Series - Convergence of arbitrary infinite series, Alternating ries, Absolute and conditional convergence, Abel's and Dirichlet's test, rearngement of series, Dirichlet's and Riemann's theorem on rearrangement of solutely and conditionally convergent series, convolution (Cauchy) prodt of series, Merten's theorem.

nit III: Riemann's theory of integraiton - Step functions and their integrals per and lower integrals of a bounded function, iintegrable functions, emann's condition of interabity, properties of integrals, Mean value theorem r interals, Differentiations of functions defines by integrals, Fundamental eorem of integral calculus, primitives, change of variables, second mean lue theorem, integral test.

nit IV: Convergence of Improper Integrals - Integrals over infinite intervals ith bounded integrands, convergence of such integrals, Necessary and sufcient conditions, Case of positive integrands, Comaprison test, μ-test, absote convergence, convergence of integrls of product of two functions, Abels's d Dirichlet's test, convergence of integrals with unbounded interands.

nit V: Metric Space - Definition and examples of Metric spaces, Open balls, terior, Boundary, Closure and accumulation points, Open and closed sets, terior and closure of sets, limits of sequences in metric spaces, Cauchy quence, completeness.

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Infinite Series: -

An expression of the form $a_1 + a_2 + \ldots + a_n + \ldots$ in which the succe sive terms are formed according to some definite law is called as an infi serie and it is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

n^{th} partial sum of a series: -

The sum of first n terms of an infinite series is called n^{th} partial sum the series and it is denotd by S_n.

i.e. $S_n = a_1 + a_2 + \ldots + a_n$.

Convergent infinite series: -

An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if sequence of it's n^{th} parti sum is convergent, i.e. limit of sequence n^{th} partial sum is finite and unique a that limit is called the sum of the series.

Note: -

If $\lim_{n\to\infty} S_n = S$, then sum of series is S.

Divergent Series: -

An infinite series $\sum_{n=1}^{\infty} a_n$ is divergent iff sequence of it's n^{th} parti sum is divergent, i.e. limit -

$$\lim_{n\to\infty} S_n = +\infty \text{ 'or' } -\infty$$

Oscillatory series of an infinite series: -

An infinite series $\sum_{n=1}^{\infty} a_n$ is oscillatory iff sequence of it's n^{th} parti sum is oscillatory.

e.g. $\sum_{n=1}^{\infty} (-1)^n$ is an oscillatory series.

Note: -

An infinite series of the positive term is either convergent or diver gent it can never oscillate.

Propertie of infinite series: -

(i) Addition or removal of finite number of terms in an infinite series does no change the nature of series.

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(ii) If series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is convergent and converges to S and S' respec tively, then $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$ is also convergent converges to $S \pm S'$.

(iii) If each term of a convrgent sequence $\sum_{n=1}^{\infty} a_n$ is multiplied by a real number k then sum of series is equal to k times the sum of series $\sum_{n=1}^{\infty} a_n$.

i.e. if $\sum_{n=1}^{\infty} a_n = S$

then $\sum_{n=1}^{\infty} k.a_n = k.S$

Cauchy's General principal of convergent of an infinite series: -

The series $\sum_{n=1}^{\infty} a_n$ is convergent iff for any $\varepsilon > 0$ there exists a positive integer N such that -

$|S_n - S_m| < \varepsilon$, for all $n > m > N$.

"or" $|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon$, for $n > m > N$.

Comparison Theorem: -

(i) If $\sum_{n=1}^{\infty} a_n$ is convergent series of positive terms and $\sum_{n=1}^{\infty} b_n$ be any series of positive terms such that $b_n \le k.a_n \ \forall n, k > 0$.

then series $\sum_{n=1}^{\infty} b_n$ is also convergent.

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent series and $\sum_{n=1}^{\infty} b_n$ be any series of positive terms such that

$$b_n \ge k.a_n, \ \forall n, k > 0$$

then $\sum_{n=1}^{\infty} b_n$ is also divergent.

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Comparision Test (Practical Comparision Theorem): -

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be the two series of positive terms such th

$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$, then

(i) If ℓ is finite and non zero then either noth series are convergent or diverge

(ii) If $\ell = 0$ then test fails.

Cauchy's n^{th} root test: -

If series $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that $\lim_{n\to\infty} (a_n)^{1/n} =$

then

(i) Series $\sum_{n=1}^{\infty} a_n$ is convergent if $\ell < 1$.

(ii) Series $\sum_{n=1}^{\infty} a_n$ is divergent if $\ell > 1$.

(iii) Test fails if $\ell = 1$.

D'Alembert ratio test: -

If series $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$

then series $\sum_{n=1}^{\infty} a_n$ is -

(i) Convergent if $\ell < 1$.
(ii) Divergent if $\ell > 1$.
(iii) Test fails if $\ell = 1$.

Note: -

D'Alembert ratio test can also be stated as - If series $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that -

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \ell$, then series $\sum_{n=1}^{\infty} a_n$ is -

(i) Convergent if $\ell > 1$.
(ii) Divergent if $\ell < 1$.
(iii) Test fails if $\ell = 1$.

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Comparision of ratio: -

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be the two series of positive terms such that

$\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}}$, $\forall n > m$ (m is fixed positive integer) then

(i) Series $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} b_n$ is convrgent.

(ii) Series $\sum_{n=1}^{\infty} b_n$ is divergent if $\sum_{n=1}^{\infty} a_n$ is divergent.

Raabe's Test: - If series $\sum_{n=1}^{\infty} a_n$ be series of positive terms such that

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \ell$, then $\sum_{n=1}^{\infty} a_n$ is -

(i) Convergent if $\ell > 1$.
(ii) Divergent if $\ell < 1$.
(iii) Test fails if $\ell = 1$.

Logarthmic test: -

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that $\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = \ell$,

then -

(i) Series $\sum_{n=1}^{\infty} a_n$ is convergent if $\ell > 1$.

(ii) Series $\sum_{n=1}^{\infty} a_n$ is divergent if $\ell < 1$.

(iii) Test fails if $\ell = 1$.

De-Morgan's and Bertrand's test: -

If $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that

$\lim_{n\to\infty} \left\{ n\left(\frac{a_n}{a_{n-1}} - 1\right) - 1 \right\} \log_e n = \ell$ then series $\sum_{n=1}^{\infty} a_n$ is -

(i) Convergent if $\ell > 1$.

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(ii) Divergent if $\ell < 1$.
(iii) Test fails if $\ell = 1$

Higher logarithmic test: -

If $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that $\lim_{n\to\infty}\left(n\log_e \frac{a_n}{a_{n+1}} - 1\right)\log_e n = \ell$, then series $\sum_{n=1}^{\infty} a_n$ is

(i) Convergent if $\ell > 1$.
(ii) Divergent if $\ell < 1$.
(iii) Test fails if $\ell = 1$.

Cauchy's condensation test: -

If ϕ be a function such that -

(i) $\phi(n)$ is positive for all positive integers n.

(ii) The sequence $\{\phi(n)\}$ is non increasing then the two series $\sum_{n=1}^{\infty}\phi(n)$ and $\sum_{n=1}^{\infty} a^n\phi(a^n)$ are both divergent or convergent, where 'a' is a positive integer greater than 1.

Q.1. When do we say that an infinite series $\sum_{n=1}^{\infty} a_n$ converges and has the sum S? (2011)

Ans. An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if sequence of its n^{th} partial sum is convergent i.e. limit of sequence of the n^{th} partial sum is finite and unique and that limits is called the sum of series.

Q.2. Test the convergence of the series

(i) $\sum_{n=1}^{\infty}\left(n^{1/n}-1\right)^n$ **(ii)** $1+x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\ldots\ (x>0)$ (2011)

Ans.(i) $\sum_{n=1}^{\infty}\left(n^{1/n}-1\right)^n$ .....(i)

Let $a_n = (n^{1/n} - 1)^n$

Hence $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(n^{1/n}-1\right) = \lim_{n\to\infty}\left(n^{1/n}\right) - \lim_{n\to\infty} 1$

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$= 1 - 1 = 0 < 1$

Hence By Cauchy's n^{th} root test series is convergent.

(ii) $1+x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\ldots\ (x>0)$ .....(i)

Omitting first term we get -

$$a_n = x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\ldots+\frac{x^n}{n} \quad \text{.....(ii)}$$

and $$a_{n+1} = x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots+\frac{x^n}{n}+\frac{x^{n+1}}{(n+1)} \quad \text{.....(iii)}$$

Now $$\frac{a_n}{a_{n+1}} = \frac{x^n}{n}\times\frac{(n+1)}{x^{n+1}} = \frac{n+1}{nx} = \frac{n\left(1+\frac{1}{n}\right)}{nx} = \frac{\left(1+\frac{1}{n}\right)}{x}$$

$\therefore$ $\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \frac{1}{x}$

Hence By D'Alembert Ratio test series (i) is convergent if $\frac{1}{x} > 1$ or $x < 1$ and divergent if $\frac{1}{x} < 1$ or $x > 1$ and test fails if $\frac{1}{x} = 1$ i.e. $x = 1$.

Now put $x = 1$ in eq.(ii) we get

$$a_n = \frac{1}{n}$$

$\therefore$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$, Hence convergent.

Hence given series (i) is convergent if $x \le 1$ and divergent if $x > 1$.

Q.3. If an infinite series of positive terms $\sum u_n$ is a convergent series; then prove that $\lim_{n\to\infty} u_n = 0$.

Deduce that the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{1/n}$ diverges. (2012)

Ans. Let series $\sum u_n$ is convergent.

Therefore sequence of its n^{th} partial sum i.e. S_n is also convergent.

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$$\therefore \quad \lim_{n\to\infty} S_n = S(\text{finite}) \qquad \ldots\ldots(1)$$

Since $S_n = u_1 + u_2 + \ldots\ldots + u_n$,

therefore $S_{n-1} = u_1 + u_2 + \ldots\ldots + u_{n-1}$

$$\therefore \quad u_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

Hence $\lim_{n\to\infty} u_n = 0$

Let $$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{1/n} \qquad \ldots\ldots(2)$$

Let $$u_n = \left(\frac{1}{n}\right)^{1/n} = \frac{1}{(n)^{1/n}}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(n)^{1/n}} = 1 \neq 0$$

Hence given series (1) is divergent.

Q.4. If $\alpha > 0, \beta > 0$, then test the convregence of the series:

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots\ldots \quad \textbf{(2012)}$$

Ans. Here we have -

$$u_n = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots\{(n-1)\alpha+1\}}{(\beta+1)(2\beta+1)\ldots\ldots\{(n-1)\beta+1\}} \qquad \ldots\ldots(i)$$

and $$u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots\{(n-1)\alpha+1\}(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots\{(n-1)\beta+1\}(n\beta+1)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n\beta+1}{n\alpha+1} = \lim_{n\to\infty} \frac{\beta + \frac{1}{n}}{\alpha + \frac{1}{n}} = \frac{\beta}{\alpha}$$

Hence By ratio test, the given series converges if $\frac{\beta}{\alpha} > 1$ i.e. $\beta > \alpha$ and diverges if $\frac{\beta}{\alpha} < 1$ i.e. $\beta < \alpha$ and test fails if $\beta/\alpha = 1$ i.e. if $\beta = \alpha$.

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When $\beta = \alpha$, then from eq.(1)

$$u_n = 1$$

So $\lim_{n\to\infty} u_n = 1 \neq 0$, showing that series is diverges when $\beta = \alpha$.

Thus the given series is converges if $\beta > \alpha$ and diverges if $\beta \leq \alpha$.

Q.5. Use Cauchy's condensation test, to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$ **converges for $p > 1$ and diverges if $p \leq 1$.** **(2012)**

Sol. Case (I) - Let $\phi(n) = \frac{1}{n(\log n)^p}$, then

(i) $\phi(n) > 0$

(ii) $\{\phi(n)\}$ is monotonically decreasing.

Let $a > 0$ be an integer.

Therefore by Cauchy's condensation test series $\sum\phi(n)$ and $\sum a^n\phi(a^n)$ is either both convergent or divergent.

Since $$\sum a^n\phi(a^n) = \sum a^n \cdot \frac{1}{a^n(\log a^n)^p} = \sum \frac{1}{n^p(\log a)^p}$$

$$\sum a^n\phi(a^n) = \frac{1}{(\log a)^p}\sum \frac{1}{n^p}$$

$\Rightarrow$ $\sum a^n\phi(a^n)$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Therefore By Cauchy condensation test $\sum\phi(n)$ is convergent if $p > 1$ and divergent if $0 \leq p < 1$.

Case (ii) - If $p \leq 0$

$$\phi(n) = \frac{1}{n(\log n)^p} \geq \frac{1}{n}$$

Therefore $\sum\phi(n)$ is divergent, since $\sum\frac{1}{n}$ is divergent.

Q.6. Show that the series

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$$1^p + \left[\frac{1}{2}\right]^p + \left[\frac{1.3}{2.4}\right]^p + \left[\frac{1.3.5}{2.4.6}\right]^p + \ldots\ldots$$

converges if $p > 2$ and diverges if $p \le 2$. (2012, 19)

Sol. Omitting the first term

$$a_n = \left(\frac{1.3.5.\ldots\ldots(2n-1)}{2.4.6.\ldots\ldots(2n)}\right)^p$$

$$a_{n+1} = \left(\frac{1.3.5.\ldots\ldots(2n-1)(2n+1)}{2.4.6.\ldots\ldots(2n)(2n+2)}\right)^p$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\left(\frac{2n+2}{2n+1}\right)^p = \lim_{n\to\infty}\left(\frac{2+\frac{2}{n}}{2+\frac{1}{n}}\right)^p = 1$$

Therefore D'Alembert ratio test fails.
By Logarthmic ratio test.

$$\lim_{n\to\infty} n\log\frac{a_n}{a_{n+1}} = \lim_{n\to\infty} n\log\left(\frac{2n+2}{2n+1}\right)^p = \lim_{n\to\infty} n.p\log\left(\frac{2n+2}{2n+1}\right)$$

$$= \lim_{n\to\infty} pn\left[\log\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right)\right] = \lim_{n\to\infty} pn\left[\log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{1}{2n}\right)\right]$$

$$= \lim_{n\to\infty} pn\left[\frac{1}{n} + \frac{1}{2n^2} + \ldots\ldots - \frac{1}{2n} + \frac{1}{2.4n^2} + \ldots\right] = \frac{p}{2}$$

Therefore by logarthmic test series is convergent if $\frac{p}{2} > 1$ or $p > 2$ and diver-

gent if $\frac{p}{2} < 1$ or $p < 2$.

And test fails if $p = 2$
Therefore By higher logarthmic test

$$\lim_{n\to\infty}\left[n\log\frac{a_n}{a_{n+1}} - 1\right]\log_e n = \lim_{n\to\infty} 2\left[n\log\left(\frac{2n+2}{2n+1}\right) - 1\right]\log_e n$$

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$$= \lim_{n\to\infty} 2\left[n\log\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right) - 1\right]\log_e n$$

$$= \lim_{n\to\infty} 2\left[n\log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{1}{2n}\right) - 1\right]\log_e n$$

$$= \lim_{n\to\infty} 2n\left[1 - \frac{1}{2n} + \frac{1}{3n^2} + \ldots\ldots + \frac{1}{2} + \frac{1}{8n} + \ldots\ldots 1\right]\log_e n$$

$$= \lim_{n\to\infty} 2\left[-\frac{1}{2} - \frac{3}{8n} + \ldots\right]\log_e n = -\infty < 1$$

Hence by Higher logarthmic test series is divergent. Hence series is converse if $p > 2$ and diverges if $p \le 2$.

Q.7. Prove that the infinite series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$ is convergent if $p > 1$, and divergent if $p \le 1$. (2013)

Other Related Questions -

Q. Prove that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is diverges if $-\infty < p \le 1$. (2011)

Q. Prove that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

is convergent for $p > 1$ and divergent for $p \le 1$. (2018)

Ans. $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots = \sum\frac{1}{n^p}$

Case (I) - If $p > 1$ -

then $\sum\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} + \ldots\ldots$

Now grouping the terms of series -

$$\sum\frac{1}{n^p} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$+ \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots \text{stay } 8^{th} \text{ term}\right) + \ldots\ldots$$

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$$< \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right)$$

$$+ \left(\frac{1}{8^p} + \frac{1}{8^p} + \text{.....stay } 8^{th} \text{ term}\right) + \text{.....}$$

$$\sum \frac{1}{n^p} < \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \text{.....}$$

$$\sum \frac{1}{n^p} < \frac{1}{1^p} + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \text{.....} \quad \text{.....(1)}$$

The series on R.H.S. of eq.(i) is a conversent geometric series because common ratio is $\frac{1}{2^{p-1}} < 1$ therefore by comparision test series $\sum \frac{1}{n^p}$ convergent.

Case (II) If p = 1

then series $\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{....}$

Grouping these terms -

$$\therefore \sum \frac{1}{n^p} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{.....}\right)$$

$$\sum \frac{1}{n^p} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \text{.....}\right) + \text{....}$$

$$\sum \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \text{.....} \quad \text{.....(2)}$$

Let a_n be the n^{th} term of series on R.H.S. of eq.(2) and $a_n = 1/2$.

$$\therefore \quad \lim a_n = \frac{1}{2} \neq 0$$

Hence series on R.H.S. of eq.(2) is divergent.

$\therefore$ By comparison theorem series $\sum \frac{1}{n}$ is also divergent.

Case (iii) if p < 1 -

$$\frac{1}{n^p} > \frac{1}{n}$$

$$\Rightarrow \quad \sum \frac{1}{n^p} > \sum \frac{1}{n} \quad \text{.....(3)}$$

Series on R.H.S. of eq. (3) is divergent therefore by comparision theorem series $\sum \frac{1}{n^p}$ is also divergent.

Q.8. Test the convergence of the following series:

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \text{.....}, \text{ for } x \geq 0. \qquad \textbf{(2011, 17)}$$

Other Related Questions -

Q. Test the convergence of the infinite series

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \text{.....} \qquad \textbf{(2013)}$$

Q. State Raabe's test for the convergence of a series of positive real numbers. Test the convergence of $\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$ for $x > 0$. **(2015)**

Ans. Raabe's Test: -

If series $\sum_{n=1}^{\infty} a_n$ be a series of positive terms, then $\lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \ell$, then series $\sum_{i=1}^{\infty} a_n$

(i) convergent if $\ell > 1$
(ii) divergent if $\ell < 1$
(iii) and test fails if $\ell = 1$

The given series is

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \text{.....} \quad \text{for } x > 0$$

Here $a_n = \frac{n^n x^n}{n!}$ and $a_{n+1} = \frac{(n+1)^{n-1} x^{n+1}}{(n+1)!}$

Hence $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} . x^{n+1}}{(n+1)!} \times \frac{n!}{n^n . x^n}$

$$= \frac{(n+1)^{n+1}}{(n+1) n^n} . x = \frac{(n+1)^n}{n^n} . x = \left(1 + \frac{1}{n}\right)^n x$$

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So $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n x=e.x \qquad \left[\text{since } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\right]$

Hence by first ratio test we have $\sum_{n=1}^{\infty}a_n$ is convergent if $e.x<1$ or $x<\frac{1}{e}$ and series $\sum_{n=1}^{\infty}a_n$ is divergent if $e.x>1$ or $x>\frac{1}{e}$. Also test is fails if $e.x=1$ such that $x=\frac{1}{e}$.

Since $\frac{a_n}{a_{n+1}}=\frac{n^n}{(n+1)^n}.\frac{1}{x}$ .....(ii)

Put $x=1/e$ in eq.(ii) we get

$$\frac{a_n}{a_{n+1}}=\frac{n^n}{(n+1)^n}.e=\left(\frac{n}{n+1}\right)^n.e$$

Now apply logarthmic test for $x=1/e$ we have

$$\lim_{n\to\infty} n\log\frac{a_n}{a_{n+1}}=\lim_{n\to\infty} n\log\left[\left(\frac{n}{n+1}\right)^n.e\right] \qquad \left[\text{Since putting the value of } \frac{a_n}{a_{n+1}}\right]$$

$$=\lim_{n\to\infty} n\log\left[\frac{e}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$=\lim_{n\to\infty} n\left[\log e-n\log\left(1+\frac{1}{n}\right)\right] \qquad \left[\begin{array}{l}\text{Since applying}\\ \log\frac{m}{n}=\log m-\log n\\ \text{and } \log m^n=n\log m\end{array}\right]$$

$$=\lim_{n\to\infty} n\left[\log e-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}.....\right)\right] \qquad \left[\text{Since using expansion of } \log\left(1+\frac{1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\left[1-1+\frac{1}{2n}-\frac{1}{3n^2}+....\right] \qquad [\text{Since } \log_e e=1]$$

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$$=\lim_{n\to\infty} n\left[\frac{1}{2n}-\frac{1}{3n^2}+.....\right]$$

$$=\frac{1}{2}<1$$

Hence by logarthmic test series $\sum_{n=1}^{\infty}a_n$ is divergent if $x=\frac{1}{e}$.

Hence series (I) is convergent if $x<1$ and divergent if $x\geq 1$.

Q.9 Test the convergence of the following series:

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+..... \quad x>0 \qquad (2017)$$

Ans. The series is given as -

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+..... \quad x>0 \qquad ....(I)$$

Omitting the firt term, we get the n^{th} term

$$a_n=\frac{3.6.9.12.....3n}{7.10.13.16.....(3n+4)}x^n$$

$$a_{n+1}=\frac{3.6.9.12.....3n(3n+3)}{7.10.13.16.....(3n+4)(3n+7)}x^{n+1}$$

So $\frac{a_n}{a_{n+1}}=\left(\frac{3n+7}{3n+3}\right)\frac{1}{x}$

Hence $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\left[\left(\frac{3n+7}{3n+3}\right)\frac{1}{x}\right]=1.\frac{1}{x}$

Therefore by first ratio test, we get $\sum_{n=1}^{\infty}a_n$ is convergent if $1/x>1$ or $x<1$ and $\sum_{n=1}^{\infty}a_n$ is divergent if $1/x<1$ or $x>1$.

The test fails when $x=1$.

Now we apply Raabe's test when $x=1$.

When $x=1$, then $\frac{a_n}{a_{n+1}}=\frac{3n+7}{3n+3}$

So $\frac{a_n}{a_{n+1}}-1=\frac{3n+7}{3n+3}-1=\frac{4}{3n+3}$

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So $n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{4n}{3n+3}$.

So $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{4}{3}>1$

Hence By Raabe's test $\sum a_n$ is convergent if $x=1$. Hence series (1) is convergent if $x \le 1$ and divergent if $x>1$.

Q.10. Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series of positive terms such that $\lim_{n\to\infty}\frac{U_n}{V_n}=\ell$. If ℓ is finite and non-zero then prove that the two series are either both convergent or both divergent. Test convergence of series $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{4^n+1}}$. (2014)

Ans. Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series of positive term such that

$\lim_{n\to\infty}\frac{u_n}{v_n}=\ell$

Case (i) If $\ell \neq 0$:-

Choose $\varepsilon>0$ as $0<\varepsilon<\ell$, then for above $\varepsilon>0\ \exists$ a positive integer such that

$$\left|\frac{u_n}{v_n}-\ell\right|<\varepsilon, \quad \forall n>N$$

$$\Rightarrow \quad \ell-\varepsilon<\frac{u_n}{v_n}<\ell+\varepsilon, \quad \forall n>N$$

(a) If $\ell-\varepsilon<\frac{u_n}{v_n}$, $\forall n\in N$:-

$$\Rightarrow \quad (\ell-\varepsilon)v_n<u_n, \quad \forall n>N$$

$$\Rightarrow \quad u_n>kv_n \qquad \forall n>N$$

$$k=\ell-\varepsilon>0$$

By comparision theorem if series $\sum_{n=1}^{\infty} v_n$ is divergent then $\sum_{n=1}^{\infty} u_n$ is also divergent (Neglecting I^{st} N terms.)

(b) If $\frac{u_n}{v_n}<\ell+\varepsilon$, $\forall n\in N$:-

$$\Rightarrow \quad u_n<(\ell+\varepsilon)v_n$$

$$\Rightarrow \quad u_n<k'v_n \qquad \forall n>N, k=\ell+\varepsilon>0$$

Neglecting I^{st} N-term we get

$$u_n<k'v_n \qquad \forall n$$

By comparison theorem series $\sum_{n=1}^{\infty} u_n$ is convergent if $\sum_{n=1}^{\infty} v_n$ is convergent.

Let $a_n=\frac{2^n}{\sqrt{4^n+1}}$

Then $a_n=\frac{2^n}{2^n\sqrt{\left(1+\frac{1}{4^n}\right)}}=\frac{1}{\sqrt{1+\frac{1}{4^n}}}$

Take $b_n=\frac{1}{n^0}$

$\sum b_n=\sum\frac{1}{n^0}$ is divergent.

And $\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1}{\sqrt{1+\frac{1}{4^n}}}=\frac{1}{1}=1$

Which is finite and non zero.

By comparision test $\sum_{n=1}^{\infty} a_n$ is divergent.

Q.11. Test convergence of series.

$$x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots\ldots(x>0)$$ (2014)

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Ans. $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{2.4.5.6.7.8}x^8+\ldots\ldots$ $(x>0)$

We can write n^{th} term of given series as

$$a_n=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2}{3.4.5.6.7.8\ldots\ldots(2n-1)(2n+2)}x^{2n+2}$$

(omitting first term)

$$a_{n+1}=\frac{2^2.4^2.6^2\ldots\ldots(2n+2)^2}{3.4.5.6.7.8\ldots\ldots(2n+3)(2n+4)}x^{2n+4}$$

$$\Rightarrow \quad \frac{a_n}{a_{n+1}}=\frac{(2n+3)(2n+4)}{(2n+2)^2x^2} \qquad \ldots\ldots(i)$$

$$\Rightarrow \quad \frac{a_n}{a_{n+1}}=\frac{\left(2+\frac{3}{n}\right)\left(2+\frac{4}{n}\right)}{\left(2+\frac{2}{n}\right)^2}.\frac{1}{x^2} \qquad \ldots\ldots(ii)$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\right)=\frac{1}{x^2}$$

Hence given series is convergent if $\frac{1}{x^2}>1$ or $x^2<1$ and diverge

$\frac{1}{x^2}<1$ or $x^2>1$ and test fails if $\frac{1}{x^2}=1$ or $x^2=1$.

Put $x^2=1$ in eq^n(1) we get

$$\frac{a_n}{a_{n+1}}=\frac{(2n+3)(2n+4)}{(2n+2)^2}$$

And $$\frac{a_n}{a_{n+1}}-1=\frac{6n+8}{(2n+2)^2}$$

And $$n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{n(6n+8)}{(2n+2)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty}\frac{6+8/n}{(2+2/n)^2}=\frac{6}{4}>1$$

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Hence given series is convergent when $x^2=1$.

Hence series is convergent if $x^2\le 1$ and divergent if $x^2>1$.

Q.12. Test convergence of series. (2014)

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\ldots$$

Ans. Now we have to check the convergence of series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots.$$

Here $$a_n=\left\{\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right\}^{-n}$$

Therefore

$$(a_n)^{1/n}=\left\{\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right\}^{\frac{-n}{n}}=\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right\}^{-1}$$

So $$\lim_{n\to\infty}(a_n)^{1/n}=(e-1)^{-1}<1$$

Therefore by Cauchy root test, series is convergent

Q.13. Test convergence of series (2014)

$$x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+\ldots\ldots\ldots(x>0)$$

Other Related Questions -

Q. Test the convergence of the series $\sum_{n=1}^{\infty}x^{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}}$ $(x>0)$ (2015)

Ans. n^{th} term of the series

$$x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}+\ldots\ldots\ldots\quad x>0 \qquad \ldots\ldots(i)$$

is given by

$$a_n=x^{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}$$

$$a_{n+1}=x^{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}}$$

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Now $\frac{a_n}{a_{n+1}} = \frac{1}{x^{\frac{1}{n+1}}}$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

So ratio test fails.

Again $\log\left(\frac{a_n}{a_{n+1}}\right) = \frac{1}{n+1}\log\frac{1}{x}$

$$n\left[\log\left(\frac{a_n}{a_{n+1}}\right)\right] = \frac{n}{n+1}\log\frac{1}{x}$$

$$\lim_{n\to\infty} n\left[\log\left(\frac{a_n}{a_{n+1}}\right)\right] = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)}\log\frac{1}{x} = \log\frac{1}{x}$$

So by logarthmic test the series is convegent if

$$\log\frac{1}{x} > 1 \text{ or } \frac{1}{x} > e \text{ or } x < \frac{1}{e}$$

and divergent if $\log\frac{1}{x} < 1$ or $\frac{1}{x} < e$ or $x > \frac{1}{e}$

Test fails if $x = \frac{1}{e}$, i.e. $\log\frac{1}{x} = 1$

Then take $x = \frac{1}{e}$

$$n\log\frac{a_n}{a_{n+1}} = \frac{n}{n+1}\log_e e = \frac{n}{n+1}$$

$$n\log\frac{a_n}{a_{n+1}} - 1 = \frac{n}{n+1} - 1 = -\frac{1}{n+1}$$

$$\therefore \quad \log n\left\{n\log\frac{a_n}{a_{n+1}} - 1\right\} = -\frac{\log n}{n}\left\{-\frac{1}{1+\frac{1}{n}}\right\}$$

$$\therefore \quad \lim_{n\to\infty} \log n\left\{n\log\frac{a_n}{a_{n+1}} - 1\right\} = 0 < 1$$

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So by Higher logarthmic test seris is divergent if $x = \frac{1}{e}$.

Hence given series (1) is convergent if $x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

Q.14. Let $\sum_{n=1}^{\infty} a_n$ be convergent series of positive real numbers. Test the convergence of $\sum_{n=1}^{\infty} a_n^2$, $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$. (2015)

Ans. Suppose $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers.

$$\Rightarrow \quad \lim_{n\to\infty} a_n = 0$$

Hence for any $\varepsilon > 0$, $\exists$ a positive integer N such that

$$a - \varepsilon < a_n < a + \varepsilon \qquad \forall n \geq N$$

$$\Rightarrow \quad -\varepsilon < a_n < \varepsilon \qquad \forall n \geq N$$

Using this inequality for $n = N+1, N+2, \ldots\ldots\infty$ and adding all we get

$$\sum_{n=N+1}^{\infty} a_n^2 < \sum_{n=N+1}^{\infty} a_n$$

But we are given that $\sum_{n=1}^{\infty} a_n$ is convergent, therefore by comparision test the infinite series $\sum_{n=1}^{\infty} a_n^2$ is convergent.

(ii) $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$

Since given $\sum_{n=1}^{\infty} a_n$ is convergent.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test)

$$\Rightarrow \quad a_n = \frac{1}{n^2}$$

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Consider $\frac{1}{1+a_n} = \frac{1}{1+\frac{1}{n^2}}$

$$\sum_{n=1}^{\infty}\frac{1}{1+a_n} = \sum_{n=1}^{\infty}\frac{1}{1+\frac{1}{n^2}}$$

Let $U_n = \frac{1}{1+\frac{1}{n^2}}$

$\lim_{n\to\infty} U_n = 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty}\frac{1}{1+a_n}$ is divergent.

(iii) $\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$

Since $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow$ $\lim_{n\to\infty} = 0$

Consider $\lim_{n\to\infty}\frac{\left(\frac{u_n}{1+u_n}\right)}{u_n} = \lim_{n\to\infty}\frac{1}{1+u_n} = 1 \neq 0$ (finite)

So $\sum_{n=1}^{\infty}\frac{u_n}{1+u_n}$ and $\sum_{n=1}^{\infty} u_n$ have same nature by comparision test.

Since $\sum_{n=1}^{\infty} u_n$ is convergent.

So $\sum_{n=1}^{\infty}\frac{u_n}{1+u_n}$ is also convergent.

Q.15. State and prove a necessary and sufficient condition for a series positive real numbers to be convergent. (2015

Ans. Statement: -

A necessary and sufficient condition for a series $\sum u_n$ converges is that to each $\varepsilon > 0$, there exist a positive integer m such that

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$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| < \varepsilon$

Proof:-

Let $\langle S_n \rangle$ be sequence of partial sums of the series

have $\sum u_n$ converges.

$\Leftrightarrow$ $\langle S_n \rangle$ converges.

$\Leftrightarrow$ to each $\varepsilon > 0$ $\exists$ a positive integer m such that

$|S_n - S_m| < \varepsilon$ $\forall n \geq m$, by Cauchy's principle of convergence for sequences.

$\Leftrightarrow$ $|(u_1 + u_2 + \ldots\ldots + u_m + u_{m+1} + \ldots\ldots u_n) - (u_1 + u_2 + \ldots\ldots + u_m)| < \varepsilon$

$\Leftrightarrow$ $|(u_{m+1} + u_{m+2} + \ldots\ldots + u_n)| < \varepsilon$, $\forall n \geq m$

Hence the result is proved.

Q.16. State Cauchy's integral test for the convergence of a series of positive real numbers. Test the convergence of $\sum_{n=2}^{\infty}\frac{1}{n\log(\log n)}$. (2015)

Ans. Cauchy Integral Test: -

If u(x) is a non-negative decreasing integrable function such that $u(n) = u_n$ for all positive integral values of n, then $\sum_{n=1}^{\infty} u_n$ is convergent iff the infinite integral $\int_1^{\infty} u(x)dx$ convergent.

$\because$ $\sum_{n=1}^{\infty}\frac{1}{n\log(\log n)}$ .....(i)

Let $a_n = \frac{1}{n\log(\log n)}$ where $n = 2, 3, \ldots\ldots$

$\Rightarrow$ $a_n = \frac{1}{[\log(\log n)]^n}$ $\left(\because \log m^n = n\log m\right)$

Now $(a_n)^{1/n} = \frac{1}{[\log(\log n)]}$ where $n = 2, 3, \ldots\ldots$

$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{1}{[\log(\log n)]}$

$= 0 < 1$

Hence by nth root test series (i) is convergent.

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Q.17. Prove that an infinite series of positive real numbers is convergent and only if the sequence of its partial sums is bounded above.

Other Related Questions -

Q. Prove that a series positive terms converges if and only if the sequence of its partial sums is bounded above.

Sol. Let $\sum_{n=1}^{\infty} a_n$, be a series of non negative term and $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ be the n^{th} partical sum.

Necessary Part: -

Let sreies $\sum a_n$ is convergent.

$\therefore$ $\langle S_n \rangle$ is convergent.

$\Rightarrow$ $\langle S_n \rangle$ is bounded ($\because$ Every convergent sequence is bounded).

Also hence "S_n" is bounded above.

Sufficient Part: -

Let sequence S_n is bounded.

$\Rightarrow$ $\langle S_n \rangle$ is bounded above

$S_{n+1} \geq S_n, \forall n$

$\Rightarrow$ $\langle S_n \rangle$ is monotonically increasing.

Since every sequence which is monotonically increasing and bounded above is always convergent.

$\therefore \langle a_n \rangle$ is convergent. Hence series $\sum a_n$ is also convergent.

Q.18. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of positive terms and $\{S_n\}_{n=1}^{\infty}$ note the sequence of its partial sums. Test the convergence of series $\sum_{n=1}^{\infty}$ (2

Other Related Questions -

Q. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real number. Sh that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}$ is convergent if $p > \frac{1}{2}$. Give a counter example for $p = \frac{1}{2}$. (20

Sol. $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n$ which is divergent.

and $S_n = a_1 + a_2 + a_3 + \ldots + a_n$.

$S_n = 1 + 2 + 3 + \ldots + n$.

$$S_n = \frac{n}{2}(n+1)$$

Consider

$$\frac{a_n}{S_n} = \frac{n}{\frac{n}{2}(n+1)} = \frac{2}{(n+1)}$$

$$\lim_{n\to\infty}\left(\frac{a_n}{S_n}\right) = \lim_{n\to\infty}\frac{2}{n+1} = 2 \neq 0$$

$\because$ We known that

If $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$.

So $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ is divergent.

Q.19. State and prove Cauchy condensation test for the convergence of an infinite series of psitive terms. (2016)

Other Related Questions - **(2014)**

Q. State Cauchy's condensation test.

Ans. Cauchy's Condensation Test: -

If ϕ be a function such that -

(i) $\phi(n)$ is positive for all positive integers "n".

(ii) The sequence $\{\phi(n)\}$ is non increasing then the two series $\sum \phi(n)$ and $\sum a^n \phi(a^n)$ are both divergent or convergent, where "a" is a positive ineger greater than 1.

Proof -

Case (I) - $\sum \phi(n) = \phi(1) + \phi(2) + \ldots + \phi(a)$

$+ \phi(a+1) + \phi(a+2) + \ldots + \phi(a^2)$

$+ \phi(a^2+1) + \phi(a^2+2) + \ldots + \phi(a^3)$

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$$+\phi\left(a^{k}+1\right)+\ldots\ldots\ldots\ldots+\phi\left(a^{k+1}\right)$$

First row in eq.(1) $> a\phi(a)$
Second row in eq.(1) $> (a^2 - a)\phi(a^2)$ $\{\because \{\phi(n)\}$ is monotonically decre
Third row in eq.(1) $> (a^3 - a^2)\phi(a^3)$

adding above equation we get

$$\sum\phi(n) > a\phi(a) + (a^2 - a)\phi(a^2) + (a^3 - a^2)\phi(a^3) + \ldots\ldots$$

$$\sum\phi(n) > \phi(a) + (a-1)\phi(a) + a(a-1)\phi(a^2) + a^2(a-1)\phi(a^3) + \ldots\ldots$$

$$\sum\phi(n) > \phi(a) + (a-1)[\phi(a) + a\phi(a^2) + \ldots\ldots + a^2\phi(a^3) + \ldots\ldots]$$

$$\sum\phi(n) > \phi(a) + \frac{(a-1)}{a}\sum_{n=1}^{\infty} a^n\phi\left(a^n\right).$$

$\Rightarrow \sum\phi(n)$ is divergent if $\sum a^n\phi\left(a^n\right)$ is divergent (By comparision t
rem).

Case (II) - Also first row of eq.(1) $< a\phi(1)$
Second row of eq.(1) $< (a^2 - a)\phi(a)$
Third row of eq.(1) $< (a^3 - a^2)\phi(a^2)$

adding above equation we get

$$\sum\phi(n) < a\phi(1) + (a^2 - a)\phi(a) + (a^3 - a^2)\phi(a^2) + \ldots\ldots$$

$$\sum\phi(n) < a\phi(1) + (a-1)[a\phi(a) + a^2\phi(a^2) + \ldots\ldots]$$

$$\sum\phi(n) < a\phi(1) + (a-1)\sum_{n=1}^{\infty} a^n\phi\left(a^n\right)$$

$\Rightarrow \sum\phi(n)$ is convergent if $\sum_{n=1}^{\infty} a^n\phi\left(a^n\right)$ is convergent. (By comparisio
theorem).

Q.20. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_n + 1}{a_n} \le \frac{b_n + 1}{b_n}$, for all $n \in N$. If $\sum_{n=1}^{\infty} b_n$ is convergent, then prove that $\sum_{n=1}^{\infty} a_n$ is also convergent. Test the convergence of $\sum$

Ans. We can write $S_n = a_1 + \ldots\ldots$
and suppose that $n \ge m$
We also write
$a_1 + \ldots\ldots + a_m = a$ and $b_1 + \ldots\ldots$
we have $n \ge m$

$$S_n = a_1 + \ldots\ldots + a_m + a_{m+1}\left[1 + \frac{a_{m+2}}{a_{m+1}}\right.$$

Now - $\frac{a_{m+2}}{a_{m+1}} \le \frac{b_{m+2}}{b_{m+1}}$

$$\frac{a_{m+3}}{a_{m+1}} = \frac{a_{m+3}}{a_{m+2}}\frac{a_{m+2}}{a_{m+1}} \le \frac{b_{m+3}}{b_{m+2}}\frac{b_{m+2}}{b_{m+1}} = \frac{b_{m+3}}{b_{m+1}}$$

Proceeding thus, we may see that

$$\frac{a_n}{a_{m+1}} \le \frac{b_n}{b_{m+1}}$$

Thus, we have

$$S_n \le a + a_{m+1}\left[1 + \frac{b_{m+2}}{b_{m+1}} + \ldots\ldots + \frac{b_n}{b_{m+1}}\right]$$

$$= a + \frac{a_{m+1}}{b_{m+1}}(T_n - b)$$

$$\Rightarrow \quad S_n \le a - b\frac{a_{m+1}}{b_{m+1}} + T_n\frac{a_{m+1}}{b_{m+1}} \quad \forall n \ge m$$

Since $\sum b_n$ is convergent, the sequence $\langle T_n \rangle$ is bounded. Thus sequence
$\langle S_n \rangle$ is also bounded and as such the series $\sum a_n$ is convergent.

Let $u_n = \frac{\sqrt{n}+1}{n^5 + n^2} = \frac{\sqrt{n}\left(1 + \frac{1}{\sqrt{n}}\right)}{n^5\left(1 + \frac{1}{n^3}\right)}$

Let $v_n = \frac{\sqrt{n}}{n^5} = \frac{n^{1/2}}{n^5} = \frac{1}{n^5 n^{-1/2}} = \frac{1}{n^{9/2}}$

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By p-test $\sum_{n=1}^{\infty} u_n$ is convergent.

Q.21. State and prove D Alembert's Ratio Test for the convergence of series $\sum_{n=1}^{\infty} u_n$ of positive terms.

Ans. Statement: -

If series $\sum_{n=1}^{\infty} u_n$ be series of positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$, then $\sum u_n$ is

(i) convergent if $\ell < 1$.
(ii) divergent if $\ell > 1$.
(iii) Test fails if $\ell = 1$.

Proof - Case I: If $\ell < 1$: -

We can choose $\varepsilon > 0$, such that $r = \ell + \varepsilon < 1$.

Now since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$, so for the above choosen $\varepsilon > 0$ there exist a positive integer N depending on ε such that -

$$\frac{u_{n+1}}{u_n} < \ell + \varepsilon (= r), \qquad \forall n > N$$

$$\frac{u_{n+1}}{u_n} < r, \qquad \forall n > N \qquad \text{.....(i)}$$

Hence putting $n = N+1, N+2, N+3, \ldots\ldots$ in (i) we have

$u_{N+2} < r\, u_{N+1}$
$u_{N+3} < r\, u_{N+2} < r^2\, u_{N+1}$
$u_{N+4} < r\, u_{N+3} < r^3\, u_{N+1}$
.....
.....

Now

$$u_{N+2} + u_{N+3} + u_{N+4} + \ldots\ldots < r u_{N+1}(1 + r + r^2 + \ldots\ldots)$$

$$\sum_{p=2}^{\infty} u_{N+p} < k \sum_{p=0}^{\infty} r^p, \qquad \text{Where } k = r u_{N+1} > 0$$

Since series $\sum_{p=0}^{\infty} r^p$ is a geometric series with common ratio $r < 1$.

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nce by comparision theorem $\sum_{p=2}^{\infty} u_{N+p}$ is also convergent. Hence $\sum_{p=1}^{\infty} u_n$ lso convergent.

(ase (II) - If $\ell > 1$ -

We can choose $\varepsilon > 0$ such that $R = \ell - \varepsilon > 1$.

w for the above choosen $\varepsilon > 0$, there exist a positive integer N depending ε such that

$$\ell - \varepsilon (= R) < \frac{u_{n+1}}{u_n}, \qquad \forall n > N \qquad \left[\text{Since } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell\right]$$

$$R < \frac{u_{n+1}}{u_n}, \qquad \forall n > N$$

$$u_{n+1} > R u_n, \qquad \forall n > N \qquad \text{.....(ii)}$$

w putting $n = N+1, N+2, \ldots\ldots$ etc. in (ii) we have

$u_{N+2} > R\, u_{N+1}$
$u_{N+3} > R\, u_{N+2} > R^2\, u_{N+1}$
$u_{N+4} > R\, u_{N+3} > R^3\, u_{N+1}$
.....
.....

ow $\sum_{p=2}^{\infty} u_{N+p} > R.u_{N+1} \sum_{p=0}^{\infty} R^p$

$$\sum_{p=2}^{\infty} u_{N+p} > K' \sum_{p=0}^{\infty} R^p, \qquad \text{Where } K' = RU_{N+1}.$$

nce $\sum_{p=0}^{\infty} R^p$ is a geometric series with common ratio $R > 1$. Therefore $\sum_{p=0}^{\infty} R^p$ divergent. Hence by comparison theorem $\sum_{p=2}^{\infty} u_{N+p}$ is also divergent. Hence $\sum_{n=1}^{\infty} U_n$ is also divergent.

ase (III) if $\ell = 1$ -

Now if $\ell = 1$ then we can not choose $\varepsilon > 0$, such that $\ell + \varepsilon < 1$ or $\ell - \varepsilon$ 1. Hence D'Alembert's ratio test fails when $\ell = 1$.

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Q.22 Prove that the series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent or divergent according as $\lim_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n > 1$ or < 1. If for this series, $\lim_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n = 1$ then what is the nature of $\sum_{n=1}^{\infty} u_n$ (20)

justify the answer.

Ans. Let $\sum_{n=1}^{\infty} U_n$ be a series of positive terms and also let

$$\operatorname{Lim}_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n = \ell$$

Then we have to prove that series $\sum_{n=1}^{\infty} U_n$ is

(i) convergent if $\ell > 1$. (ii) Divergent if $\ell < 1$.

Case (i) If $\ell > 1$ -

We may choose p such that $\ell > p > 1$

Take $b_n = \frac{1}{n(\log n)^p}$,

Now the $\sum b_n$ is convergent if $p > 1$ and divergent if $p \le 1$.

Clearly series $\sum b_n$ is convergent, then by comparision of ratio's serie

$\sum U_n$ is convergent if

$$\frac{U_n}{U_{n+1}} > \frac{b_n}{b_{n+1}}$$

or if $$\frac{U_n}{U_{n+1}} > \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}$$

or if $$\frac{U_n}{U_{n+1}} > \left(1+\frac{1}{n}\right)\left\{\frac{\log(n+1)}{\log(n)}\right\}^p$$

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or if $$\frac{U_n}{U_{n+1}} > \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

or if $$\frac{U_n}{U_{n+1}} > \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\cdots\right)\right\}^p$$

or if $$\frac{U_n}{U_{n+1}} > \left(1+\frac{1}{n}\right)\left\{1+\frac{p}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)+\cdots\right\}$$

or if $$\frac{U_n}{U_{n+1}} > 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}-\cdots\cdots$$

or if $$\frac{U_n}{U_{n+1}}-1 > \frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}-\cdots\cdots$$

or if $$n\left(\frac{U_n}{U_{n+1}}-1\right) > 1+\frac{p}{\log n}+\frac{p}{n\log n}+\cdots\cdots$$

or if $$\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\} > \frac{p}{\log n}+\frac{p}{n\log n}+\cdots\cdots$$

or if $$\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n > p+\frac{p}{n}+\cdots\cdots$$

or if $$\operatorname{Lim}_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n > p$$

or if $\ell > p > 1$

or if $\ell > 1$ (Since $p > 1$)

Hence series $\sum_{n=1}^{\infty} U_n$ is convrgent if $\ell > 1$, or the series $\sum_{n=1}^{\infty} U_n$ is convergent

for $\operatorname{Lim}_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n > 1$

Case (II) If $\ell < 1$ -

We may choose $p < 1$ such that $\ell < p < 1$.

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Take $b_n = \frac{1}{n(\log n)^p}$

Now the series $\sum b_n$ is divergent if $p < 1$ then by comparision of ra

$\sum U_n$ is divergent if $\frac{U_n}{U_{n+1}} < \frac{b_n}{b_{n+1}}$

or if $$\frac{U_n}{U_{n+1}} < \frac{(n+1)\{\log(1+n)\}^p}{n(\log n)^p}$$

or if $$\frac{U_n}{U_{n+1}} < \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

or if $$\frac{U_n}{U_{n+1}} < \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{3n^3}+.....\right)\right\}^p$$

or if $$\frac{U_n}{U_{n+1}} < \left(1+\frac{1}{n}\right)\left\{1+\frac{p}{\log n}\left(1+\frac{1}{2n^2}-\frac{1}{3n^3}+.....\right)+...\right\}$$

or if $$\frac{U_n}{U_{n+1}} < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}-.....$$

or if $$\frac{U_n}{U_{n+1}}-1 < \frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}-.....$$

or if $$n\left(\frac{U_n}{U_{n+1}}-1\right) < 1+\frac{p}{\log n}+\frac{p}{n\log n}-.....$$

or if $$\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\} < \frac{p}{\log n}+\frac{p}{n\log n}-.....$$

or if $$\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n < p+\frac{p}{n}-.....$$

or if $$\lim_{n\to\infty}\left[\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n\right] < p$$

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if $\ell < p < 1$

if $\ell < 1$

ence series $\sum_{n=1}^{\infty} U_n$ is divergent if $\ell < 1$.

lso if $\lim_{n\to\infty}\left\{n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right\}\log n = 1$, then the series $\sum_{n=1}^{\infty} U_n$ is neither con-

rgent nor divergent, because we can not choose p such that $\ell > p > 1$ or $\ell <$ < 1.

.23. Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms and $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \ell$.

rove that the series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent accordingly as $\ell > 1$ $\ell < 1$ respectively. (2018)

ns. **Case (i) if $\ell > 1$: -**

Choose a 'p' such that $\ell > p > 1$

ke $\sum b_n = \sum \frac{1}{n^p}$

$\sum b_n$ is convergent.

by comparision of ratio's series $\sum u_n$ is convergent.

$$\frac{u_n}{u_{n+1}} > \frac{b_n}{b_{n+1}}$$

if $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$$

if $$\log\frac{u_n}{u_{n+1}} > p\log\left(\frac{n+1}{n}\right)$$

if $$\log\frac{u_n}{u_{n+1}} > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-.....\right)$$

if $$n\log\frac{u_n}{u_{n+1}} > p\left(1-\frac{1}{2n}+\frac{1}{3n^2}-.....\right)$$

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or if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} > p$ or if $\ell > p$

or if $\ell > p > 1$

or if $\ell > 1$

Case (ii) If $\ell < 1$: -

Choose a 'p' such that $\ell < p < 1$.

$\therefore \quad \sum b_n = \sum \frac{1}{n^p}$ is divergent.

$\therefore$ By comparision of ratio's $\sum u_n$ is divergent.

If $\frac{u_n}{u_{n+1}} < \frac{b_n}{b_{n+1}}$

or if $\frac{u_n}{u_{n+1}} < \frac{(n+1)^p}{n^p}$

or if $\log\frac{u_n}{u_{n+1}} < p\log\left(1+\frac{1}{n}\right)$

or if $\log\frac{u_n}{u_{n+1}} < p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)$

or if $n\log\frac{u_n}{u_{n+1}} < p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right)$

or if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} < p$

or if $\ell < p$ or if $\ell < p < 1$

or if $\ell < 1$

Q.24. Test the convergence of the following series:

$$1 + \frac{2^2}{3^2} + \frac{2^2}{3^2}\cdot\frac{4^2}{5^2} + \frac{2^2}{3^2}\cdot\frac{4^2}{5^2}\cdot\frac{6^2}{7^2} + \ldots\ldots\ldots$$

Sol. Omitting the first term, we get

$$u_n = \frac{2^2.4^2.6^2\ldots\ldots(2n)^2}{3^2.5^2.7^2\ldots\ldots(2n+1)^2}$$

$$u_{n-1} \quad \frac{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}{3^2.5^2.7^2\ldots\ldots(2n+1)^2(2n+3)^2}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2} \qquad \ldots\ldots\ldots(i)$$

Hence $\lim_{n\to\infty}\frac{u_n}{u_{n-1}} = 1$ and so ratio test fails.

From eq.(i)

$$\frac{u_n}{u_{n-1}} = \left(1+\frac{3}{2n}\right)^2\left(1+\frac{1}{n}\right)^{-2}$$

$$= \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left(1-\frac{2}{n}+\frac{3}{n^2}+\ldots\right) \qquad \left(\begin{matrix}\text{On expanding by the}\\ \text{Binomial Theorem}\end{matrix}\right)$$

$$= 1+\frac{1}{n}+\frac{1}{n^2}\left(-\frac{15}{4}+\ldots\right) = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right),$$

Where $O(1/n^2)$ stands for terms of order two or more in $1/n$.

Here ρ = coeff. of $1/n = 1$. So by the Gauss's test, the given series diverges.

Q.25. Test the convergence of the following series:

$$\sum_{n=1}^{\infty}\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) \qquad \textbf{(2018)}$$

Sol. Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$, Rewritting it, we have

$$u_n = \frac{\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\sqrt{n^4+1} + \sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

(2 nd take

$$= \frac{2}{\sqrt{n^4} + \sqrt{n^4}} = \frac{1}{n^2}$$

ow. $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2n^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$

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$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+\left(\frac{1}{n}\right)^4}+\sqrt{1-\left(\frac{1}{n}\right)^4}}$$

Thus, $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=1$, which is finite and non-zero.

The series $\sum v_n$ being convergent, we see that the series $\sum u_n$ is convergent.

Q.26. State and prove that root test for convergence of series of posi... (20) real numbers.

Ans. Cauchy n^{th} root test (Root test): -

If series $\sum a_n$ be a series of postive terms such that $\lim_{n\to\infty}(a_n)^{1/n}=\ell$, then

(I) Series $\sum a_n$ is convergent if $\ell<1$

(II) Series $\sum a_n$ is divergent if $\ell>1$

(III) Test fails if $\ell=1$

Proof :- case (I) - Let $\ell<1$

Choose $\epsilon>0$ such that $\ell+\epsilon<1$.

Also $\lim_{n\to\infty}(a_n)^{1/n}=\ell$ .....(i)

$\therefore$ for above $\epsilon>0$, $\exists$ a positive integer N such that $(a_n)^{1/n}<\ell+\epsilon$, $\forall n>N$

$\Rightarrow$ $a_n^{1/n}<R$, Where $R=\ell+\epsilon<1$, $\forall n>N$

$\Rightarrow$ $a_n<R^n$, $\forall n>N$

$\Rightarrow$ $a_{N+1}<R^{N+1}$

$a_{N+2}<R^{N+2}$

$a_{N+3}<R^{N+3}$

.

.

$a_{N+1}+a_{N+2}+a_{N+3}+.....<R^{N+1}(1+R+R^2+....)$

$$\sum_{n=N+1}^{\infty} a_n < k\sum_{n=0}^{\infty} R^n \qquad .....(ii)$$

The series on R.H.S of eq. (ii) is a geometric series with common rati...

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...1, therefore it is convergent. Hence by comparison theorem series on L.H.S ...f eq. (ii) such that $\sum_{n=N+1}^{\infty} a_n$ is also convergent.

Therefore $\sum_{n=1}^{\infty} a_n$ is also convergent, because addition of finite num-...er of terms in an infinite series does not change the nature of the series.

...ase (II) - If $\ell>1$

Choose $\epsilon>0$, such that $\ell-\epsilon>1$.

$\lim_{n\to\infty}(a_n)^{1/n}=\ell$

for above $\epsilon>0$, $\exists$ a posstive integer N such that

$\ell-\epsilon<a_n^{1/n}$, $\forall n>N$ (by definition of sequence)

$r<a_n^{1/n}$, $\forall n>N$ (take $r=\ell-\epsilon$)

$r^n<a_n$, $\forall n>N$

$a_{N+1}>r^{N+1}$

$a_{N+2}>r^{N+2}$

$a_{N+3}>r^{N+3}$

.

.

.

...dding we get

$$\sum_{n=N+1}^{\infty} a_n > r^{N+1}(1+r+r^2+.....)$$

$$\sum_{n=N+1}^{\infty} a_n > k'\sum_{n=0}^{\infty} r^n \qquad .....(iii)$$

Since series on R.H.S of eq.(iii) i.e. $\sum_{n=0}^{\infty} r^n$ is geometric series with ...ommon ratio $r>1$ therefore it is divergent, therefore by comparison therom ...eries on L.H.S of eq.(iii) i.e. $\sum_{n=N+1}^{\infty} a_n$ is also divergent.

$\therefore$ $\sum_{n=1}^{\infty} a_n$ is also divergent because addition of finite number of terms in ...n infinite series does not change the nature of the series.

...ase (III) - If $\ell=0$

Then we can not choose $\epsilon>0$ such that

$\ell+\epsilon<1$ and $\ell-\epsilon>1$

...ence test fails.

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Unit II

Alternating Series: -

A series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots (u_n > 0)$ $\forall$ n is called alternating se

Leibnitz's Test: -

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots (u_n > 0, \forall n)$ is conv

if -

(i) $u_{n+1} \le u_n$, $\forall n$, i.e. sequence $\langle u_n \rangle$ is monotonically decreasing.

(ii) $\lim_{n\to\infty} u_n = 0$

Absolute convergence of a series: -

A series $\sum_{n=1}^{\infty} u_n$ is called absolutely convergent if series $\sum_{n=1}^{\infty} |u_n|$ convergent.

Conditionally convergent series: -

A series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent series if se

$\sum_{n=1}^{\infty} u_n$ is convergent but $\sum_{n=1}^{\infty} |u_n|$ is divergent.

Rearrangment of a series: -

A rearrangment of an infinite series if $\sum_{n=1}^{\infty} b_n$ whose terms are sam

those of series $\sum_{n=1}^{\infty} a_n$ but in different order.

Abel's test: -

If $\langle b_n \rangle$ is positive and monotonically decreasing and series $\sum_{n=1}^{\infty} u_n$

convergent series then series $\sum_{n=1}^{\infty} u_n . b_n$ is also convergent.

Dirichlet's Test: -

If $\langle b_n \rangle$ is monotonically decreasing and conversing to zero for se

$\sum_{n=1}^{\infty} u_n$, its nth partial sum S_m is bounded then series $\sum_{n=1}^{\infty} u_n b_n$ is converge

Cauchy product of two series: -

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series, then their Cauchy prod

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is defined as -

$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_n b_0$

Merten's theorem: -

Let $\sum_{n=0}^{\infty} a_n = \ell$ and $\sum_{n=1}^{\infty} b_n = m$. If one of these is absolutely conver-

gent then series of their Cauchy product $\sum_{n=0}^{\infty} c_n$ is also convergent and con-

verges to ℓ.m.

Integral test (Cauchy integral test): -

Let f(x) be continuous, positive and non increasing for $x \ge 1$ then the

series $\sum_{1}^{\infty} f(k)$ converges or diverges according as sequence $\{I_n\}$ where

$I_n = \int_1^n F(t)\,dt$, is convergent to a finite limit as $n \to \infty$ or divergent.

Riemann's rearrangment theorem: -

If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series then we have get a

rearrangment $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} b_n = x$ (i.e. $\sum_{n=1}^{\infty} b_n$ converges to

x). Also we can get a rearrangment of $\sum_{n=1}^{\infty} a_n$ which is divergent $(-\infty \text{ or } +\infty)$.

Q.1. **Test the convergence of the series** $\sum \frac{(-1)^{n+1}}{n}\left(1+\frac{1}{n}\right)^{-n}$. **(2011)**

Ans. Let $a_n = \frac{1}{n}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{n}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}$, $a_{n+1} = \frac{1}{n+1}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$

$$a_n - a_{n+1} = \frac{1}{n}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} - \frac{1}{n+1}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} \quad \ldots\ldots\ldots(1)$$

$\frac{1}{n} > \frac{1}{n+1}$ and $\left(\frac{1}{n} \text{ and } \frac{1}{n+1} \text{ are positive}\right)$

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And $\left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1}$

$$\Rightarrow \frac{1}{n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n}} > \frac{1}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$$

$$\Rightarrow \frac{1}{n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n}} - \frac{1}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} > 0$$

Hence eq.(1) becomes $\Rightarrow \quad a_n > a_{n+1}$

$a_n - a_{n+1} > 0$

thus a_n is monotonically decreasing.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = 0.\frac{1}{e} = 0$$

Thus by Leibnitz test series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\left(1+\frac{1}{n}\right)^{n}$ convergent.

Q.2. Prove that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$ is conditionally convergent if $0 < p \le 1$. (20

Sol. Let the given series be bounded by $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$

Since $p > 0$ and $(n+1)^p > n^p$, therefore

$$\frac{1}{(n+1)^p} < \frac{1}{n^p} \quad \Rightarrow \quad u_{n+1} < u_n \quad \forall n \in N$$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^p} = 0$, Since $p > 0$

$\therefore$ By Leibnitz test, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$ is convergent.

Now $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent if

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$p \le 1$.

Hence given series is conditionally convergent if

$0 < p \le 1$.

Q.3. If $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ are absolutely convergent and their sums are A and B respectively, then prove that the Cauchy's product series $\sum_{n=0}^{\infty}C_n$, where $C_n = \sum_{k=0}^{n}a_k b_{n-k}\ \forall n \ge 0$, is absolutely convergent and its sum is AB. (2017)

Other Related Question -

Q. If the two series $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ converge to the sums A and B respectively and also the series $\sum_{n=1}^{\infty}C_n$ converges to C where $C_n = \sum_{k=0}^{n}a_k b_{n-k}$ then prove that C = AB. (2011)

Q. If one of two series $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ is absolutely convergent while the other is convergent and their sums are A and B respectively then show that Cauchy product series $\sum_{n=1}^{\infty}a_n$, where $C_n = a_0b_n + a_1b_{n-1} + \ldots\ldots + a_nb_0 = \sum_{k=0}^{n}a_n a_k b_{n-k}\ \forall n \ge 0$ is convergent and its sum is AB. (2013)

Q. Let $\sum_{n=1}^{\infty}u_n$ be an absolutely convergent series and $\sum_{n=1}^{\infty}v_n$ be a convergent series converging to zero. Then prove that Cauchy product series will converge to zero. (2014)

Q. Let $\sum_{n=1}^{\infty}a_n$ be an absolutely convergent series and $\sum_{n=1}^{\infty}b_n$ be a convergent sereis converging to zero prove that their Cauchy product series

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will converge to zero.

Ans. Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are sequence of partial sum of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ respectively.

then $A_n = a_0 + a_1 + a_2 + \ldots\ldots$
$B_n = b_0 + b_1 + b_2 + \ldots\ldots$ .....(i)

Let $\beta_n = B_n - B$

Then $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$ and $\lim_{n\to\infty} \beta_n = 0$

Now Cauchy product of above series is $\sum_{n=0}^{\infty} C_n$ where

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \forall n \geq 0$$

Let $\{C_n\}_{n\in\mathbb{N}} \cup \{0\}$ be sequence of partial sum of sequence $\sum_{n=0}^{\infty} C_n$.

Then we have to show that $C_n \to AB$ as $n \to \infty$, so that Cauchy product is convergent.

$$\begin{aligned}
C_n &= c_0 + c_1 + \ldots\ldots + c_n \\
&= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + b_0a_2) + (a_0b_3 + a_1b_2 + a_2b_1 + a_3 \\
&\quad + \ldots\ldots + (a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots\ldots + a_nb_0) \\
&= a_0(b_0 + b_1 + b_2 + \ldots\ldots + b_n) + a_1(b_0 + b_1 + b_2 + \ldots\ldots + b_{n-1}) + a_2(b_0 + \\
&\quad + b_2 + \ldots\ldots + b_{n-2}) + a_3(b_0 + b_1 + b_2 + \ldots\ldots + b_{n-3}) + \ldots\ldots + a_nb_0 \\
&= a_0B_n + a_1B_{n-1} + a_2B_{n-2} + \ldots\ldots\ldots + a_nB_0 \\
&= a_0B_n + a_1B_{n-1} + a_2B_{n-2} + \ldots\ldots\ldots + a_n(\beta_0 + B) \\
&= a_0(\beta_n + B) + a_1(\beta_{n-1} + B) + a_2(\beta_{n-2} + B) + \ldots\ldots\ldots + a_n(\beta_0 + B) \\
&= a_0(\beta_n + B) + a_1(\beta_{n-1} + B) + a_2(\beta_{n-2} + B) + \ldots\ldots\ldots + a_n\beta_0 \\
&= B(a_0 + a_1 + \ldots\ldots + a_n) + a_0\beta_n + a_1\beta_{n-1} + a_2\beta_{n-2} + \ldots\ldots\ldots + a_n\beta_0 \\
&= BA_n + \nu_n, \quad \text{where } \nu_n = \beta_0a_n + \beta_1a_{n-1} + \ldots\ldots\ldots + \beta_na_0
\end{aligned}$$

To prove theorem it is sufficient to show that $\nu_n \to 0$ as $n \to \infty$ then $C_n \to AB$ as $n \to \infty$.

Since $\sum a_n$ converges absolutely, then we may take

$$\alpha = \sum_{n=0}^{\infty} |a_n| \text{ and } \beta_n \to 0 \text{ as } n \to \infty$$

Then for a given $\varepsilon > 0$ $\exists$ $m \in \mathbb{N}$ such that

$$|\beta_n| < \frac{\varepsilon}{\alpha}, \quad \forall n > m$$

$$\begin{aligned}
\text{Now } |\nu_n| &= |\beta_0a_n + \beta_1a_{n-1} + \beta_2a_{n-2} + \ldots\ldots + \beta_na_0| \\
&= |\beta_0a_n + \beta_1a_{n-1} + \ldots\ldots + \beta_ma_{n-m} + \beta_{m+1}a_{n-m-1} + \ldots\ldots + \beta_na_0|
\end{aligned}$$

$$\begin{aligned}
&\leq |\beta_0a_n + \beta_1a_{n-1} + \ldots\ldots + \beta_{n-m}a_m| + |\beta_{n-m+1}a_{m-1}| + |\beta_{n-m+2}a_{m-2}| \\
&\quad + |\beta_{n-1}a_1| + |\beta_na_0| \\
&< |\beta_0a_n + \beta_1a_{n-1} + \ldots\ldots + \beta_{n-m}a_m| + \frac{\varepsilon}{\alpha}(|a_{m-1}| + |a_{m-2}| + \ldots\ldots + |a_0|)
\end{aligned}$$

$$|\nu_n| < |\beta_0a_n + \beta_1a_{n-1} + \ldots\ldots + \beta_{n-m}a_m| + \frac{\varepsilon}{\alpha}\cdot\alpha$$

Keeping m fixed and taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} |\nu| < \frac{\varepsilon}{\alpha}\alpha \qquad \text{as } a_k \to 0, k \to \infty.$$

Since $\varepsilon > 0$ is arbitrary so $\nu_n \to 0$ as $n \to \infty$.
Therefore $C_n \to AB$ as $n \to \infty$.
Thus Cauchy product of above series is convergent and has sum AB.

Note: -

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutly convergent therefore $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{\infty} |b_n|$ are convergent. Therefore by Merten's theorem we have -

$$\sum (|a_0||b_n| + |a_1||b_{n-1}| + \ldots\ldots + |a_n||b_0|) \text{ is convergent}$$

$$|C_n| = |a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots\ldots + a_nb_0|$$

$$|C_n| \leq |a_0||b_n| + |a_1||b_{n-1}| + \ldots\ldots + |a_n||b_0|$$

Therefore comparison theorem $\sum_{n=0}^{\infty} |C_n|$ is convergent.

$\Rightarrow \sum C_n$ is absolutely convergent.

Hence $\sum C_n$ is convergent.

Q.4. If $\sum_{n=1}^{\infty} u_n$ is an absolutely Convergent series, then prove that the series of positive terms and the series of negative terms of $\sum u_n$ are both convergent. Also prove that if $\sum_{n=1}^{\infty} u_n$ is conditionally convergent then the

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series of its positive terms and the series of its negative terms are
divergent.

Ans. **(I) Part -**

Since series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

i.e. $\sum_{n=1}^{\infty} |u_n|$ is convergent and $\sum_{n=1}^{\infty} u_n$ is also convregent.

Let $\sum p_n = \frac{\sum u_n + \sum |u_n|}{2}$ and $\sum q_n = \frac{\sum u_n - \sum |u_n|}{2}$

where $\sum p_n$ and $\sum q_n$ is series of positive term and negative term re
tively.

$\Rightarrow$ $\sum p_n$ and $\sum q_n$ both are convergent because sum and differen
two convregence series is also convergence.

(II) Part: -

Let series $\sum u_n$ is conditionally convergent i.e. $\sum u_n$ is con
gent and $\sum |u_n|$ is divergent. If possible let $\sum p_n$ is series of positive te
convergent therefore

$\sum |u_n| = 2\sum p_n - \sum u_n$ = convergent – convergent = convergen
(Because difference and sum of two convergent series is also converge
Which is a contraduction because $\sum |u_n|$ is divergent therefore our assump
that series $\sum p_n$ is convergent is wrong. Hence $\sum p_n$ is divergent. Simil
we can prove that $\sum q_n$ is divergent.

Q.5. Prove that the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$

Converges if and only if $x \in (-1,1]$. (20

Ans. Here $|u_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$ and $|u_{n+1}| = \left|\frac{x^{2n+1}}{2n+1}\right|$

$\Rightarrow$ $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{1}{x^2} \frac{2n+1}{2n-1} = \frac{1}{x^2}$

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By ratio test. $\sum u_n$ is convergent if

$\frac{1}{x^2} > 1 \Rightarrow x^2 < 1 \Rightarrow |x| < 1$

and $\sum |u_n|$ is divergent if $|x| > 1$.

Thus, $\sum u_n$ is absolutely convergent if $|x| < 1$.

$\Rightarrow$ $\sum |u_n|$ is convergent if $|x| < 1$.

If $|x| = 1$, the ratio test fails.

Now $|x| = 1 \Rightarrow x = 1$ or $x = -1$

Putting $x = 1$, the given series becomes

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

which converges by Leibnitz test

Putting $x = -1$, the given series becomes

$-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \ldots\ldots$

Which is again convergent by Leibnitz's test. Hence the given series is con-
vergent if $|x| \le 1$ or $-1 \le x \le 1$.

If $x > 1$, then u_n does not tend to zero as $n \to \infty$, and so $\sum u_n$ does not
converge when $x > 1$.

Since $\sum |u_n|$ diverges when $x < -1$ (i.e. $|x| > 1$), so $\sum u_n$ also diverges when
$x < -1$.

Hence $\sum u_n$ converges iff $-1 \le x \le 1$.

Q.6. Test the convergence of the series for $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ for positive values of x. (2012)

Ans. we take

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin nx$ and $(b_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

Let $(S_n)_{n\in\mathbb{N}}$ be sequence of partial sum of the series $\sum_{n=1}^{\infty} a_n$

$S_n = \sin x + \sin 2x + \ldots\ldots + \sin nx$

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$$= \frac{\sin\frac{(n-1)x}{2} - \sin\frac{nx}{2}}{\sin\frac{x}{2}} \quad \text{i.e.} \quad x \neq 0, 2k\pi$$

$$\Rightarrow \quad |S_n| = \frac{\left|\sin\frac{(n-1)x}{2}\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|}$$

$$|S_n| \leq 1 \quad \text{if } x \neq 0, 2k\pi$$

and $(b_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is monotonic decreasing sequence and $\lim_{n\to\infty}\frac{1}{n} = 0$ $\Rightarrow$

Thus $(S_n)_{n\in\mathbb{N}}$ sequence of partial sum of the series $\sum_{n=1}^{\infty} a_n$ is bounded and

$(b_n)_{n\in\mathbb{N}}$ is monotonically decreasing sequence of positive trem and tend to zero.

Then by Dirichelt test -

$$\sum_{n=1}^{\infty} a_n.b_n = \sum_{n=1}^{\infty}\frac{1}{n}\sin x \text{ is convergent if } x \neq 0, 2k\pi$$

Now, if $x = 0, 2k\pi$
then $\sin nx = 0$

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin x = 0$$

So, convergent.

Thus $\sum_{n=1}^{\infty}\frac{1}{n}\sin x$ is convergent for all x:

Q.7. Define alternating series and give an example. Prove that the ser

$\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ **is convergent** $(u_n \geq 0\ \forall n)$ **if**

(i) $(u_n \geq u_{n+1}\ \forall n \geq 1)$ **and** **(ii)** $\lim_{n\to\infty} u_n = 0$ **(20**

Other Related Questions -

Q. State and prove Leibnitz's test for convergence of an alternat series. (20

Ans. We will prove that $\langle S_n\rangle$ i.e. n^{th} partial sum of an alternating series

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$u_2 + u_3 - u_4 + \ldots\ldots$ is convergent.

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + u_{2n-1} - u_{2n}$$
$$S_{2n+1} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + u_{2n-1} - u_{2n} + u_{2n+1}$$
$$S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$$
$$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$$

since $\langle u_n\rangle$ is monotonically decreasing)

Consider

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + u_{2n-1} - u_{2n}$$
$$S_{2n} = u_1 - \{(u_2 - u_3) + (u_4 - u_5) + \ldots\ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}\}$$
$$S_{2n} < u_1\ \forall n$$

$\langle S_{2n}\rangle$ is bounded above therefore $\langle S_{2n}\rangle$ is convergent.

$\lim_{n\to\infty} S_{2n} = \beta$ (finite and unique)

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1} = \beta + 0 = \beta$$

$$\lim_{n\to\infty} S_{2n+1} = \beta$$

$\therefore$ Series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

Q.8. State and prove Riemann's rearrangment Theorem for a conditinally convergent series. (2013)

Other Related Questions -

Q. Prove that by a suitable reaaragement of the terms of a conditionally convergent series, it can be made to converge to any given number α. (2017)

Ans. Riemann Rearrangment Theorem: -

Let $\sum a_n$ be a conditionally convergent series and α be any number, then there is a rearrangement $\sum b_n$ of $\sum a_n$ which converges to sum α.

Proof - Let us define

$$P_n = \frac{|a_n| + a_n}{2} = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases}$$

$$q_n = \frac{|a_n| - a_n}{2} = \begin{cases} 0, & a_n \geq 0 \\ -a_n, & a_n < 0 \end{cases}$$

Then both series P_n and q_n are divergent because $\sum a_n$ is conditionally convergent.

We re-arrange a_n as follows -

Take in order just enough positive term of P_n so that their sum exceeds α.

For it p_1 positive term are required, we have

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$\sum_{n=1}^{p_1} P_n > \alpha$ but $\sum_{n=1}^{q} P_n \leq \alpha$. $q < p_1$

This is always possible since the partial sum of the $\sum P_n$ tends to $+\infty$.

To this sum, we add just enough -ve form of $\sum q_n$ (say n_1) so that re sum is less than α.

This is possible because the partial sum of $\sum q_n$ tends to $-\infty$.

Thus we have

$$\sum_{n=1}^{p_1} P_n + \sum_{n=1}^{n} P_q < \alpha \text{ but } \sum_{n=1}^{p_1} P_n + \sum_{n=1}^{m} q_n \geq \alpha$$

Now, we repeat the same process adding just enough new term to m sum less than α.

Proceeding in this way we obtained a rearrangment $\sum b_n$ of

Each partial sum of $\sum b_n$ differ from α by at most one term of $\sum P_n$ or

But $a_n \to 0$ as $n \to \infty$. Since $\sum a_n$ is convergent, so the partial sum of tends to α. Therefore $\sum b_n$ which is a rearrangement series of $\sum a_n$ convergent and converges to sum α.

Q.9. Show that the series $\sum_{n=2}^{\infty}\left\{\frac{\left(n^4+1\right)^{1/4}-n}{\log n}\right\}$ is convergent.

Ans. $\sum_{n=2}^{\infty}\left\{\frac{\left(n^4+1\right)^{1/4}-n}{\log n}\right\}$

Let $u_n = \left(n^4+1\right)^{1/4} - n$, $b_n = \frac{1}{\log n}$

$$u_n = \left(n^4+1\right)^{1/4} - n = n\left\{\left(1+\frac{1}{n^4}\right)^{1/4} - 1\right\}$$

$$= n\left\{1 + \frac{1}{n^4}\cdot\frac{1}{4} + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!\left(n^4\right)^2} + \ldots\ldots - 1\right\} = \frac{1}{n^3}\left\{\frac{1}{4} - \frac{3}{32}\frac{1}{n^4} + \ldots\ldots\right.$$

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ake $\sum v_n = \sum \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{4}$ (finite)

hen by comparision test both series $\sum u_n$ and $\sum v_n$ convergent or diver- ent together.

hen $\sum v_n$ is convergent by p -test (p>1).

hus $\sum u_n = \sum\left(n^4+1\right)^{1/4} - 1$ is also convergent.

ince $(b_n)_n$ is monotonically decreasing and positive real valued terms. hen by abel's test the series

$$\sum u_n b_n = \sum \frac{\left(n^4+1\right)^{1/4} - n}{\log n} \text{ is convergent.}$$

Q.10. Prove that if a series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent and series $\sum_{n=1}^{\infty} v_n$ is rearrangement of $\sum_{n=1}^{\infty} u_n$ then $\sum_{n=1}^{\infty} v_n$ also absolutely convergent and has the same sum as $\sum_{n=1}^{\infty} u_n$. **(2014)**

Other Related Questions -

Q. Prove that if the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered. (2018)

Ans. Let series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent and $\sum_{n=1}^{\infty} v_n$ be its rearrangment.

$$p_n = \begin{cases} u_n & \text{if } u_n > 0 \\ 0 & \text{if } u_n \leq 0 \end{cases}$$

$$q_n = \begin{cases} 0 & \text{if } u_n > 0 \\ u_n & \text{if } u_n \leq 0 \end{cases}$$

$\Rightarrow \sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both are convergent. $\left(\because \sum_{n=1}^{\infty} u_n \text{ is absolutly convergent}\right)$

$\Rightarrow \sum_{n=1}^{\infty} p_n = P$

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And $\sum_{n=1}^{\infty} q_n = Q$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} p_n + \sum_{n=1}^{\infty} q_n = P + Q \qquad \text{.....(i)}$$

Since $\sum_{n=1}^{\infty} v_n$ is arrangment of $\sum_{n=1}^{\infty} u_n$

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} p_{n_1} + \sum_{n=1}^{\infty} q_{n_1}$

Where series $\sum p_{n_1}$ and $\sum q_{n_1}$ are rearrangment of series $\sum p_n$ and $\sum q_n$ respectively.

$$\sum_{n=1}^{\infty} v_n = P + Q \qquad \text{.....(ii)}$$

$\Rightarrow$ Series $\sum_{n=1}^{\infty} v_n$ is convergent.

Now we will show that $\sum_{n=1}^{\infty} v_n$ is absolutely convergent.

For this we will show that $\sum_{n=1}^{\infty} |v_n|$ is convergent.

$$\sum_{n=1}^{\infty} |v_n| = \sum_{n=1}^{\infty} p_{n_1} - \sum_{n=1}^{\infty} q_{n_1} = P - Q$$

Therefore $\sum_{n=1}^{\infty} |v_n|$ is also convergent.

Hence series $\sum_{n=1}^{\infty} v_n$ is absolutely convergent, and has same sum as $\sum$

Q.11. Define a conditionally convergent series giving an example.

Ans. Conditionally convergent series: -

A series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent series if $\sum$

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convergent but $\sum |u_n|$ is divergent.

Example - $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ is conditionally convergent series because by Leibnitz's test series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent. But $\sum u_n$ is divergent.

Q.12. Prove that if $\{b_n\}$ is positive monotonic decreasing sequence and series $\sum_{n=1}^{\infty} u_n$ is convergent then series $\sum_{n=1}^{\infty} u_n b_n$ is also convergent. Test convergence of series : $\frac{1}{4^0.1} - \frac{1}{4.3} + \frac{1}{4^2.5} - \frac{1}{4^3.7} + \ldots\ldots$ **(2014)**

Ans. Let $S_n = u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n$ be the nth partial sum of series $\sum_{n=1}^{\infty} u_n$.

$S_1 = u_1$, $S_2 = u_1 + u_2$.

$u_2 = S_1 - S_2$, $u_3 = S_3 - S_2$

.....

Let $S'_n = u_1 b_1 + u_2 b_2 + \ldots\ldots\ldots + u_n b_n$ be the nth partial sum of series $\sum_{n=1}^{\infty} u_n b_n$.

$$S'_n = S_1 b_1 + (S_2 - S_1) b_2 + (S_3 - S_2) b_3 + \ldots\ldots\ldots + (S_n - S_{n-1}) b_n.$$

$$= S_1 (b_1 - b_2) + S_2 (b_2 - b_3) + S_{n-1} (b_{n-1} - b_n) + S_n b_n.$$

$$S'_n = \sum_{r=1}^{n-1} S_r (b_r - b_{r+1}) + S_n b_n$$

Since $\langle b_n \rangle$ is monotonacally decreasing and (+) ve and also series $\sum u_n$ is convergent.

$\Rightarrow$ $\langle S_n \rangle$ is convergent.

$\Rightarrow$ $\langle S_n \rangle$ is bounded.

$\therefore$ By above lemma series $\sum_{i=1}^{\infty} S_r (b_r - b_{r+1})$ is convergent.

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$\lim_{n\to\infty}\sum_{r=1}^{n-1} S_r(b_r - b_{r+1})$ = finite

Also $\langle S_n \rangle$ is convergent.

$\lim_{n\to\infty} b_n$ = finite

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty}\sum_{r=1}^{n-1} S_r(b_r - b_{r+1}) + \lim_{n\to\infty} S_n b_n$$

= finite + finite = finite

$\therefore \langle S'_n \rangle$ is convergent.

Hence $\sum_{n=1}^{\infty} u_n b_n$ is convergent.

$$\frac{1}{4^0.1} - \frac{1}{4.3} + \frac{1}{4^2.5} - \frac{1}{4^3.7} + \ldots\ldots$$

The given series is -

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^{n-1}} \cdot \frac{1}{2n-1}$$

Let $u_n = \frac{(-1)^{n-1}}{4^{n-1}}$ and $a_n = \frac{1}{2n-1}$

Then, the given series can be rewritten as $\sum a_n u_n$.

Now -

$$\sum |u_n| = \sum \left|\frac{(-1)^{n-1}}{4^{n-1}}\right| = \sum \left(\frac{1}{4}\right)^{n-1}$$

which is a geometric series with common ratio $\frac{1}{4}(<1)$ and so $\sum$ convergent. Hence $\sum u_n$ is an absolutely convergent series.

Since every absolutely convergent series is convergent, it f that $\sum u_n$ is convergent.

Again $\langle a_n \rangle$ i.e. $\left\langle \frac{1}{(2n-1)} \right\rangle$ is a positive monotonically dec sequence. Hence, by Abel's test, the given series is convergent.

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Q.13. State Leibnitz's test for convergence of an alternating series. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is a conditionally convergent series. (2015)

Ans. Leibniz's Test: -

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ $(u_n > 0, \forall n)$ is convergent if

(i) $u_{n+1} \le u_n$. $\forall n$ i.e. sequence $\langle u_n \rangle$ is monotonically decreasing.

(ii) $\lim_{n\to\infty} u_n = 0$

Now we show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \ldots\ldots(i)$$

is conditionally convegent series.

Here let $u_n = \frac{1}{\sqrt{n}}$

$\Rightarrow$ $(u_n)_{n\in N}$ is monotonically decreasing sequence and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So by Leibnitz test given series (i) is convergent. As we see it that series (i) is not absolutly convergent because $\sum \left|(-1)^{n-1} \cdot \frac{1}{\sqrt{n}}\right| = \sum \frac{1}{\sqrt{n}}$ is divergent.

So eq.(i) is conditionally convergent series.

Q.14. Define absolutely convergent series. if $\{a_n\}$ is a sequence of real numbers such that $\{n^2 a_n\}$ is convergent, prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent. (2015)

Ans. Absolutely Convergent Series: -

A Series $\sum_{n=1}^{\infty} u_n$ is called absolutely convergent if series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

Let $S_n = n^2 a_n$

$\because$ $\langle S_n \rangle = \langle n^2 a_n \rangle$ is convergent.

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Let a sequence $\{S_n\}$ converges to the limit ℓ. Let $\varepsilon > 0$ be given number, a positive integer m such that

$$|S_n - \ell| < \varepsilon, \quad \forall n \geq m$$

$$\Leftrightarrow \quad \ell - \varepsilon < S_n < \ell + \varepsilon, \quad \forall n \geq m$$

Let $g = \min\{\ell - \varepsilon, S_1, S_2, \ldots\ldots\ldots\ldots S_{m-1}\}$

And $v = \max\{\ell + \varepsilon, S_1, S_2, \ldots\ldots\ldots\ldots S_{m-1}\}$

Thus we have $g \leq S_n \leq v, \quad \forall n$

$$\Rightarrow \quad |S_n| \leq v, \quad \forall n$$

$$\Rightarrow \quad |n^2 a_n| \leq v, \quad \forall n$$

$$\Rightarrow \quad |a_1| \leq v, \qquad n = 1$$

$$\Rightarrow \quad |4a_2| \leq v, \qquad n = 2$$

$$\vdots$$

$$|(n-1)^2 a_{n-1}| \leq v,$$

So $\exists\ k > 0$ such that

$$|a_1| + |4a_2| + \ldots\ldots\ldots + |n^2 a_n| \leq k$$

$$\Rightarrow \quad |a_1| + 4|a_2| + \ldots\ldots\ldots + n^2|a_n| \leq k \qquad \ldots\ldots\ldots(i)$$

Consider

$$|a_1 + a_2 + \ldots\ldots + a_n| \leq |a_1| + |a_2| + \ldots\ldots\ldots + |a_n|$$

$$\leq |a_1| + 4|a_2| + \ldots\ldots\ldots + n^2|a_n|$$

$$\leq k \qquad \text{using eq.(i)}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

Q.15. Prove that $\sum_{k=1}^{\infty} \sin kx$ has bounded partial sums, for each real number x. Determine all real number x for which

$$\sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}\right)\frac{\sin nx}{n}, \text{ convergent.}$$

(2

Othre Related Questions -

Q. Prove that

$\sum_{k=1}^{\infty} \sin kx$

has bounded partial sum, for each $x \in \mathbb{R}$. Determine all real numbers x for which

$$\sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}\right)\frac{\sin nx}{n}$$

is convergent.

Sol. Let S_n be the n^{th} partial sum of the series $\sum_{k=1}^{\infty} \sin kx$ (2018)

So $S_n = \sum_{k=1}^{\infty} \sin kx = \sin x + \sin 2x + \ldots\ldots + \sin nx$

$$\Rightarrow \quad S_n \leq |\sin x| + |\sin 2x| + \ldots\ldots + |\sin nx| \qquad (-1 \leq \sin x \leq 1)$$

$$S_n \leq 1 + 1 + \ldots\ldots + 1$$

$\Rightarrow$ S_n is bounded.

So $\sum_{k=1}^{\infty} \sin kx$ has bounded partial sums.

Consider series - $\sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}\right)\frac{\sin nx}{n}$

Let $a_n = \frac{\sin nx}{x} \quad \Rightarrow \quad \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$

Let S_n be the nth partial sum of the term -

$$S_n = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$$

$$\therefore \quad S_n = \sin x + \frac{\sin 2x}{2} + \ldots\ldots\ldots + \frac{\sin nx}{n}$$

$$\Rightarrow \quad |S_n| \leq |\sin x| + \left|\frac{\sin 2x}{2}\right| + \ldots\ldots + \left|\frac{\sin nx}{n}\right|$$

$$\leq 1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}$$

Since $1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}$ are H.P. so -

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$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots+\frac{1}{n}=\frac{2}{n(n+1)}<2$$

$\Rightarrow \quad |S_n|<2 \quad \Rightarrow \quad S_n$ is bounded.

Let $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right) \quad \Rightarrow \quad b_n=1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}$

Since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots+\frac{1}{n}=\frac{2}{n(n+1)}$

$$\Rightarrow \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}=\frac{2}{(n+1)(n+2)}$$

$$\because \quad \frac{1}{(n-1)(n+2)}<\frac{1}{n(n+1)}$$

$$\Rightarrow \quad \frac{2}{(n+1)(n+2)}<\frac{2}{n(n+1)}$$

$\Rightarrow \quad b_{n+1}<b_n$

Thus $\{b_n\}$ is monotonically decreasing sequence

and $\quad \lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{2}{n(n+1)}=0$

$$\Rightarrow \quad \lim_{n\to\infty} b_n=0$$

Hence By Dirichlet test $\sum a_n b_n$ is convergent.

i.e. $\quad \sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)\frac{\sin nx}{n}$ is convergent for all x.

Q.16. Define Cauchy product of two series of real numbers. Show example that Cauchy product of two convergent series need be conve

Ans. Cauchy Product: -

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series then the series

where $c_n=\sum_{n=0}^{n} a_k b_{n-k}$, $n\geq 0$ is called Cauchy product of $\sum_{n=0}^{\infty} a_n$ and

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Now we show by an example that cauchy product of two convergent series need not be convergent.

Let us take $\sum_{n=0}^{\infty} a_n=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$, which is convergent by Leibnitz test and

$$a_n=b_n=\frac{(-1)^n}{\sqrt{n+1}}$$

Then both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, Now Cauchy product of above series is

$$\sum_{n=0}^{\infty} c_n=a_0b_0+(a_0b_1+a_1b_0)+(a_0b_2+a_1b_1+a_2b_0)+\ldots\ldots$$

$$=1+\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}\sqrt{2}}+\frac{1}{\sqrt{3}}\right)+\ldots\ldots$$

Then $\sum_{n=0}^{\infty} c_n$, where

$$c_n=\sum_{k=0}^{n} a_k b_{n-k}=\sum_{k=0}^{n}\frac{(-1)^k}{\sqrt{k+1}}\cdot\frac{(-1)^{n-k}}{\sqrt{n-k+1}}$$

$$c_n=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}\sqrt{n-k+1}}$$

$$(n-k+1)(k+1)=\left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-k\right)^2 \quad \left\{\because (a^2-b^2)=(a+b)(a-b)\right\}$$

$$(n-k+1)(k+1)\leq\left(\frac{n}{2}+1\right)^2$$

$$\Rightarrow \quad \sqrt{(n-k+1)(k+1)}\leq\left(\frac{n}{2}+1\right)$$

$$\Rightarrow \quad \frac{1}{\sqrt{(n-k+1)(k+1)}}\geq\frac{1}{\frac{n}{2}+1}$$

$$\Rightarrow \quad \sum_{k=0}^{n}\frac{1}{\sqrt{(n-k+1)(k+1)}}\geq\sum_{k=0}^{n}\frac{1}{\frac{n}{2}+1}$$

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$$\Rightarrow \quad |c_n| \geq \sum_{r=0}^{n} \frac{2}{(n+2)}$$

$$\Rightarrow \quad |c_n| \geq \frac{2(n+1)}{(n+2)}$$

$\sum c_n$ is divergent because $c_n \not\to 0$ or c_n does not converges to zero, as $n \to \infty$ thus Cauchy product of two convergent series is non necessary conve

Q.17. Define absolutely convergent series and conditionally conve series. Prove that every absolutely convergent series is convergent. S

that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2}$ is conditionally convergent. (2

Other Related Questions -

Q. Define absolute and conditional convergence of infinite series. P that every absolute convergent series is convergent. (201

Ans. Absolutely convergent series: -

A series $\sum u_n$ is called absolutely convergent if $\sum u_n$ is convergent but $\sum |u_n|$ is convergent.

Conditionaly Convergent Series: -

A series $\sum u_n$ is called conditionally convergent series. If $\sum u_n$ is convergent but $\sum |u_n|$ is divergent.

Proof - Let series $\sum u_n$ be absolutely convergent

$$\Rightarrow \quad \sum |u_n| \text{ is convergent.}$$

$\therefore$ By Cauchy's general principle of convergence of infinite series $\forall \varepsilon >$ positive integer N such that.

$$\left| |u_{n+1}| + |u_{n+2}| + \ldots\ldots\ldots |u_{n+p}| \right| < \varepsilon, \quad \forall n > N$$

$$\Rightarrow \quad |u_{n+1}| + |u_{n+2}| + \ldots\ldots\ldots |u_{n+p}| < \varepsilon, \quad \forall n > N \qquad \ldots\ldots(1)$$

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Now we will show that $\sum u_n$ is also convergent.

For this -

$$|u_{n+1} + u_{n+2} + \ldots\ldots + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + \ldots\ldots + |u_{n+p}| < \varepsilon, \ \forall n > N$$

(using eq.(1))

$\therefore$ series $\sum u_n$ is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2}$$

Consider $\sum_{n=1}^{\infty} \frac{1}{3n-2}$

$\because$ $$\sum_{n=1}^{\infty} \frac{1}{3n-2} \geq \sum_{n=1}^{\infty} \frac{1}{3n}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{3n-2} \geq \sum_{n=1}^{\infty} \frac{1}{3n} \qquad \ldots\ldots(1)$$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-test.

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{3n} \text{ is also divergent (because 1/3 does not depend on "n").}$$

from eq.(1)

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{3n-2} \text{ is also divergent by comperision test}$$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2}$ is not absolutely convergent.

Test for conditionally convergent -

Let $a_n = \frac{1}{3n-2}$

$$a_{n+1} = \frac{1}{3(n+1)-2} = \frac{1}{3n+1}$$

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$$a_{n+1} - a_n = \frac{1}{3n+1} - \frac{1}{3n-2} = \frac{3n-2-3n-1}{(3n+1)(3n-2)} = \frac{-3}{(3n+1)(3n-2)}$$

$\Rightarrow a_{n+1} - a_n < 0 \quad \Rightarrow a_{n+1} < a_n$

$\Rightarrow$ monotonically decreasing (From M - by theorem)

$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{3n-2} = 0$

So by Leibneitz test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} \text{ is convergent}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} \text{ is conditionally convergent}$$

Q.18. Let $\sum_{n=1}^{\infty} a_n$ be conditioanlly convergent series of real numbers, $\ell \in \mathbb{R}$. Prove that there is a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} b_n = \ell$. (20

Other Related Questions -

Q. prove that if a series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent then it be rearranged to converge to a real number α. (20

Ans. Let $\sum u_n$ be given conditionally convergent series and let "ℓ" given number.

We suppose that "ℓ" is positive, we associate two positive term series $\sum$ and $\sum b_n$ to the series $\sum u_n$. Thus -

$$a_n = \begin{cases} u_n \text{ if } u_n \geq 0 \\ 0 \text{ if } u_n < 0 \end{cases} \text{ and } b_n = \begin{cases} -u_n \text{ if } u_n \leq 0 \\ 0 \text{ if } u_n > 0 \end{cases}$$

We have

$|u_n| = a_n + b_n$ and $u_n = a_n - b_n$ so that

$$a_n = \frac{\{|u_n| + u_n\}}{2} \text{ and } b_n = \frac{\{|u_n| - u_n\}}{2}$$

The series $\sum u_n$ being conditionally convergent. The sum of $\sum u_n$ is finite and that of $\sum |u_n|$ is infinity. It follows that each of the series $\sum a_n$ and $\sum b_n$ diverges to infinity.

We now give the mode of construction of the required rearrangement of $\sum u_n$. We add positive term of the series $\sum u_n$: i.e. the sum of the series $\sum a_n$ so that the sum of the finite number of term thus obtained is just greater than ℓ.

This process is possible because of the series $\sum a_n$ being divergent to infinity.

We now add the negative term of the series $\sum u_n$ i.e. the term of the series $\sum(-b_n)$ so that the total sum of the positive and negative term used is just less than "ℓ". This process is possible because of the series $\sum b_n$ diversing to infinity.

Again we consider the positive term and continue with the term already obtained so that the sum of all the term so far obtained is just greater than "ℓ".

We again consider the negative term of the series and continue with the process.

Surely this new series is a re-arrangement of the given series which will be shown to be convergent with sum "ℓ".

Now the series $\sum u_n$ being convergent.

$\text{Lim } u_n = 0$

so that if "ε" is a given positive number, there exist $m \in N$ such that

$|u_n| < \varepsilon \quad \forall\ n \geq m$

By the method of formation of the series, we see that in due course, we would have included in the rearrangement all those term of the given series for which $n \geq m$.

Thus, $\forall n \geq m$, the sum of the re-arrangement differes from ℓ by $< \varepsilon$ so that the rearrangement is convergent with sum "ℓ".

Q.19. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers whose partial sums form

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a bounded sequence. Let $\{b_n\}_{n=1}^{\infty}$ be a monotonic decreasing sequence

nonnegative real numbers which converges to 0. Prove that the series $\sum_{n=1}^{\infty} a_n$

(2016)

is convergent.

Other Related Questions -

Q. Prove that if the sequence $\{a_1 + \ldots\ldots + a_n\}$ is bounded and the sequence $\{b_n\}$ be monotonically decreasing and converging to zero then the series

(20

$\sum_{n=1}^{\infty} a_n b_n$ converges.

Q. If the series $\sum_{n=0}^{\infty} a_n$ is convergent and $\{v_n\}_{n=0}^{\infty}$ is monotonic

quence that converges to zero then show that the series $\sum_{n=0}^{\infty} a_n v_n$ is conv

(20

gent.

Q. If the series $\sum_{n=0}^{\infty} a_n$ is convergent and $\{v_n\}_{n=0}^{\infty}$ is a monotonic sequ

that converges to zero then show that the series $\sum_{n=0}^{\infty} a_n v_n$ is convergent. (20

Sol. Let $S_n = a_1 + a_2 + \ldots\ldots + a_n$ be the n^{th} partial sum of the series $\sum a$

$S_1 = a_1, S_2 = a_1 + a_2, a_2 = S_2 - S_1, a_3 = S_3 - S_2$

Let $S'_n = a_1b_1 + a_2b_2 + \ldots\ldots + a_nb_n$ be the n^{th} partical sum of series $\sum a_nb$

$S'_n = S_1b_1 + (S_2 - S_1)b_2 + (S_3 - S_2)b_3 + \ldots\ldots + (S_n - S_{n-1})b_n$

$S'_n = S_1(b_1 - b_2) + S_2(b_2 - b_3) + \ldots\ldots + S_{n-1}(b_{n-1} - b_n) + S_nb_n$

$$S'_n = \sum_{r=1}^{n-1} Sr(b_r - b_{r+1}) + S_nb_n \quad \ldots\ldots(i)$$

$\because$ $<S_n>$ bounded and $<b_n>$ is decreases therefore series $\sum_{1}^{\infty} S_r(b_r - b_{r+1})$

convergent.

$$\therefore \quad \lim_{n\to\infty} \sum_{r=1}^{n-1} S_r(b_r - b_{r+1}) = \text{finite}$$

$\because$ $< S_n >$ is bounded

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$\therefore$ $\exists\ k > 0$ such that

$|S_n| \le k \quad \forall n$

$<b_n>$ is convergent to zero. .....(ii)

$\therefore$ $\forall \varepsilon > 0$, $\exists$ a positive integer N such that

$$|b_n - 0| < \frac{\varepsilon}{k}, \ \forall\ n > N$$

.....(iii)

Consider $|S_n b_n - 0| = |S_n||b_n| < k.\frac{\varepsilon}{k}, \ \forall n > N$ (Using eq.(ii) and (iii)).

$\Rightarrow$ $<S_nb_n>$ is convergent and converges to zero.

$$\lim_{n\to\infty} S_n b_n = 0$$

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty} \sum_{r=1}^{r} S_r(b_r - b_{r+1}) + \lim_{n\to\infty} S_n b_n$$

$\because$ $<S_n>$ is convergent.

$\therefore$ Series $\sum a_nb_n$ is convergent.

Q.20. State Merten's theorem. Find Cauchy product $\sum_{n=0}^{\infty} c_n$ of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, where $a_n = b_n = r^n$, for all n and $|r| < 1$. Show that $\sum_{n=0}^{\infty} c_n$ is absolutely convergent and find its sum. (2016)

Sol. Let $\sum_{n=0}^{\infty} a_n = \ell$ and $\sum_{n=0}^{\infty} b_n = m$. If one of there is absolutely convergent then series of their Cauchy product $\sum_{n=0}^{\infty} c_n$ is also convergent and converges to lower limit.

$\because$ Series $\sum a_n$ and $\sum b_n$ are absolutely convergent therefore $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{\infty} |b_n|$ are convergent, therefore by Merten's theorem

$$\sum_{n=0}^{r} (|a_0||b_n| + |a_1||b_{n-1}| + \ldots\ldots + |a_n||b_n|) \text{ is convergent}$$

$$|c_n| = |a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots\ldots + a_nb_0|$$

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$$|c_n| \le |a_0||b_n| + |a_1||b_{n-1}| + \ldots\ldots\ldots + |a_n||b_0|$$

$\therefore$ By comparision theorem $\sum|C_n|$ is convergent.

$\Rightarrow \sum C_n$ is absolutely convrgent.

Hence $\sum C_n$ is convergent.

Q.21. Assuming the expansion

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 < x \le 1$$

Prove that $\left(\tan^{-1}x\right)^2 = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots+\frac{1}{2n+1}\right)$. (20

Ans. Given that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 < x < 1$$

The series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ is absolutely convergent for $-1 < x < 1$. Hen

Cauchy product of given series by itsefl will converges absolutely to t square of its sum.

Thus -

$$\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)^2 = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

$$= x^2 - \left(\frac{1}{3}+\frac{1}{3}\right)x^4 + \left(\frac{1}{1\times5}+\frac{1}{3\times3}+\frac{1}{5\times1}\right)x^6 + \ldots\ldots$$

$$+(-1)^n\left[\frac{1}{1\times(2n+1)}+\frac{1}{3\times(2n-1)}+\frac{1}{(2n+1)\times1}+\ldots+\frac{1}{3\times(2n-1)}+\frac{1}{(2n+1)\times1}\right]x^{2n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\left\{\frac{1}{1\times(2n+1)}+\frac{1}{3\times(2n-1)}+\ldots+\frac{1}{3\times(2n-1)}+\frac{1}{(2n+1)\times1}\right\}x^{2n}$$

$$= \sum_{n=0}^{\infty}(-1)^n C_n x^{2n+2}, \text{ say} \qquad \ldots\ldots\ldots(i)$$

Then

$$C_n = \frac{1}{1\times(2n+1)} + \frac{1}{3\times(2n-1)} + \ldots\ldots + \frac{1}{3\times(2n-1)} + \frac{1}{1\times(2n+1)}$$

multiply and divide by $(2n+2)$ we get

$$C_n = \frac{1}{2n+2}\left\{\frac{2n+2}{1\times(2n+1)} + \frac{2n+2}{3\times(2n-1)} + \ldots\ldots + \frac{2n+2}{3\times(2n-1)} + \frac{2n+2}{1\times(2n+1)}\right\}$$

$$= \frac{1}{2n+2}\left\{\left(1+\frac{1}{2n+1}\right)+\left(\frac{1}{3}+\frac{1}{2n-1}\right)+\ldots+\left(\frac{1}{3}+\frac{1}{2n-1}\right)+\left(\frac{1}{2n+1}+1\right)\right\}$$

$$= \frac{2}{2n+2}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1}\right)$$

$$C_n = \frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1}\right)$$

Thus using eq.(ii) eq.(i) reduces to $\qquad \ldots\ldots(ii)$

$$\left(\tan^{-1}x\right)^2 = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1}\right)$$

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Riemann Integration: -

Integration of discontinuous functions bounded by Riemann

Riemann integration.

Partition of closed interval [a, b]: -

Let [a, b] be a closed interval. Consider 'n-1' points $x_1, x_2, \ldots$

that $a_0 = x_0 < x_1 < x_2 < \ldots\ldots < x_{n-1} < x_n = b$ then

$P = \{x_0, x_1, x_2, \ldots\ldots, x_{k-1}, x_k, \ldots\ldots, x_n\}$ is called partition of [a, b]

$, x_k]$ $(k = 1, 2, \ldots\ldots n)$ is called subinterval of closed interval [a, b].

For example: -

Partition of [-2, 3] is

$P_1 = \{-2, -1, 0, 1, 2, 3\}$

$P_2 = \{-2, 0, 3\}$

$P_3 = \{-2, -1.5, -1, 0, 0.5, 1.5, 2.5, 3\}$

i.e. a closed interval [a, b] have infinite number of partion.

Step Function: -

Let $f:[a, b] \to E^1$ ($\mathbb{R}$) defined as $f(x) = C_K$, $\forall x \in (x_{k-1}, x_k)$ $k =$

n. where $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{k-1}, x_k, \ldots\ldots, x_n\}$ be the partition of closed

[a, b] then f(x) is called step function.

Example: -

Let $f: [-2, 2] \to E^1$ be defined as

$f(x) = [x]$, $\forall x \in [-2, 2]$.

where [ ] denotes greatest integer function.

Let $P = \{-2, -1, 0, 1, 2\}$ be the partition of [-2, 2]. Then

$f(x) = -2$, $\forall x \in (-2, -1)$

$f(x) = -1$, $\forall x \in (-1, 0)$

$f(x) = 0$, $\forall x \in (0, 1)$

$f(x) = 1$, $\forall x \in (1, 2)$

Therefore f(x) is a step function.

Integration of step function: -

Let $f:[a, b] \to E^1$ be a step function defined as -

$f(x) = C_k$, $\forall x \in (x_{k-1}, x_k)$

where $P = \{x_0, x_1, x_2, \ldots\ldots x_{k-1}, x_k, \ldots\ldots, x_n\}$ be the partition of [a, b] then in

of f(x) is defined as -

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$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} C_k (x_k - x_{k-1})$$

$$= C_1(x_1 - x_0) + C_2(x_2 - x_1) + \ldots\ldots + C_n(x_n - x_{n-1})$$

ote: 1. If $f:[a, b] \to E^1$ and $g:[a, b] \to E^1$ are step functions then $f \pm g$ is also step function and

$$\int_a^b (f \pm g)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Product of two step function $f:[a, b] \to E^1$ and $g:[a, b] \to E^1$ is also a step nction but

$$\int_a^b (f.g)(x)dx \neq \int_a^b f(x)dx . \int_a^b g(x)dx$$

tegration of bounded function: -

ower integration of a function -

Let $f:[a, b] \to E^1$ be a bounded function such that

$m \le f(x) \le M$, $\forall x \in [a, b]$ where m and M are real number. Defined set.

$S_f = \{$step function $s:[a, b] \to E^1$, such that $s(x) \le f(x)$, $\forall x \in [a, b]\}$

so $f(x) \le M, \forall x \in [a, b]$

$s(x) \le f(x)$

$s(x) \le M$, $\forall x \in [a, b]$

$$\int_a^b s(x)dx \le M(b - a) \qquad \ldots\ldots\ldots(i)$$

efine a set

$$S_1 = \left\{\int_a^b s(x)d : s(x) \in S_f\right\} \neq \phi$$

$\neq \phi$ and bounded above by M(b - a), therefore axiom of completeness of al number, least upper bound of set S_1 exist and this least upper bound primum) is defined as lower integration of function f(x) from limit a to b and is denoted by

$$\underline{\int_a^b} f(x)dx = \text{least upper bound } \left\{\int_a^b s(x)dx : s(x) \in S_f\right\}$$

per integration of a function: -

Define a set

$$T_f = \left\{\text{Step function } t:[a, b] \to E^1 : t(x) \ge f(x), \forall x \in [a, b]\right\}$$

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Also $m \le f(x), \quad \forall x \in [a,b]$

$f(x) \le t(x), \quad \forall x \in [a,b]$

$m \le t(x), \quad \forall x \in [a,b]$

$m(a-b) \le \int_a^b t(x)dx$ .....(2)

Define $S_2 = \left\{\int_a^b t(x)dx : t(x) \in T_f\right\}$

$\neq \phi$

Define a set

$T_f = \{\text{Step function } t : [a,b] \to E^1,\ t(x) \ge f(x),\ \forall x \in [a,b]\}$

$\because S_2 \neq \phi$ and bounded below my m(b - a) therefore by axiom of comple of real number greatest lower bound or infimum of set S_2 exist and this g lower bound is called as upper integral of function f(x) from limit a to b, a denoted by -

$$\int_a^{\bar{b}} f(x)dx = \text{greatest lower bound}\left\{\int_a^b t(x)dx : t(x) \in T_f\right\}$$

Riemann integration: -

A bounded function f:[a, b] → E^1 is called Riemann integrable tion if lower integral is equal to upper integral and it's common value is de by

$$\int_{\underline{a}}^b f(x)dx = \int_a^{\bar{b}} f(x)dx$$

First mean value theorem: -

If f:[a, b] → E^1 be a continuous function then ∃ r belonging to such that -

$$\int_a^b f(x)dx = (b-a)f(r)$$

Indefinite integral: -

Let f : [a, b] → E^1 be integrable then f is also integrable in any subinterval [a, x], $\forall x \in [a,b]$

Define a function G on [a, b] as

$$G(x) = \int_a^x f(t)dt, \quad \forall x \in [a,b]$$

then G is called indefinite integral.

Fundamental theorem of integral calculus: -

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If f : [a, b] → E^1 be continuous function and G : [a, b] → E^1 be a function such that

$$G(x) = \int_a^x f(t)dt$$

then $\frac{d}{dt}[G(x)] = f(x)$

Primitive function: -

If f:[a, b] → E^1 is a continuous function and F:[a, b] → E^1 is another function such that

$$\frac{d}{dt}[F(x)] = f(x)$$

then F(x) is called primitive of f(x).

For example -

$$\frac{d}{dx}[\sin x] = \cos x$$

or $\int \cos x dx = \sin x$

then sinx is primitive function of cosx.

Lagrange's Mean Value theorem(LMTV): -

If f:[a, b] → E^1 is continuous funtion and F:[a, b] → E^1 and G : [a, b] → E^1 be two function such that F(x) and G(x) are primitive of f(x) then F(x) - G(x) = constant.

Bonnet's form of 2nd mean value theorem: -

If f:[a, b] → E^1 and g:[a, b] → E^1 are integrable function and g is monotonically decreasing and non-negative in [a, b] then exist $\xi \in [a,b]$ such that

$$\int_a^b (f.g)(x)dx = g(a)\int_a^{\xi} f(x)dx$$

Second mean value theorem: -

If f:[a, b] → E^1 and g:[a, b] → E^1 are integrable function and g is monotonic in [a, b] then $\exists \xi \in [a,b]$ such that

$$\int_a^b (g.f)(x)dx = g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^b f(x)dx$$

Substitution Method: -

If g : [a, b] → $\mathbb{R}$ and f(g[a, b]) → $\mathbb{R}$ be two functions and 'f' be continuous and g'(x) exist and continuous for all x ∈ [a, b] then

$$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(x)).g'(x)dx$$

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Uniform Continuity :-

A function f is said to be uniform continuous on a set $A \subseteq \mathbb{R}$ if for

$\varepsilon > 0\ \exists\ \delta > 0$ such that

for any $x, y \in A$

$|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ (δ depends only on ε)

Note: - (1) Every continuous function defined on a closed and bounded subset A of $\mathbb{R}$ is uniformly continuous.

(2) Let A be a closed and bounded subset of $\mathbb{R}$ and f is continuous then f is bounded and $\exists\ u, v \in A$ such that -

$$f(u) \le f(x) \le f(v), \quad \forall x \in A$$

Q.1. If a function f is bounded on [a, b], prove that

$$\int_a^b f(x)dx \le \int_a^{\bar{b}} f(x)dx$$

, where the quantities above are lower and upper integrals of f over [a, b] respectively. (2011)

Ans. If $f : [a, b] \to E^1$ be bounded function then for a given $\varepsilon > 0\ \exists$ step function $s_0(x)$ and $t_0(x)$ such that

$$\int_{\underline{a}}^{b} f(x)dx - \int_a^b s_0(x)dx < \frac{\varepsilon}{2} \quad \text{.....(i)}$$

$$\int_a^t t_0(x)dx - \int_a^{\bar{b}} f(x)dx < \frac{\varepsilon}{2} \quad \text{.....(ii)}$$

Adding eq.(i) and (ii) we get

$$\left(\int_{\underline{a}}^{b} f(x)dx - \int_a^b s_0(x)dx\right) + \left(\int_a^t t_0(x)dx - \int_a^{\bar{b}} f(x)dx\right) < \varepsilon \quad \text{.....(iii)}$$

Since $s_0 \in s_f$

$$\Rightarrow \quad s_0(x) \le f(x), \quad \forall x \in [a,b] \text{ and } t_0(x) \in T_f$$

$$\Rightarrow \quad f(x) \le t_0(x), \quad \forall x \in [a,b]$$

$$s_0(x) \le t_0(x), \quad \forall x \in [a,b]$$

$$\Rightarrow \quad \int_a^b s_0(x)dx \le \int_a^b t_0(x)dx$$

$$\Rightarrow \quad \int_a^b t_0(x)dx - \int_a^b s_0(x)dx \ge 0 \quad \text{.....(iv)}$$

From eq.(iii) and (iv) we get

$$\int_{\underline{a}}^{b} f(x)dx - \int_a^{\bar{b}} f(x)dx < \varepsilon$$

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$$\Rightarrow \quad \int_a^b f(x)dx - \int_a^{\bar{b}} f(x)dx \le 0$$

$$\Rightarrow \quad \int_a^b f(x)dx \le \int_a^{\bar{b}} f(x)dx$$

Q.2. If f and g are bounded and integrable over [a, b]. Prove that fg given by (fg)(x) = f(x) g(x) is also integrable over [a, b] and if further, f is differentiable over [a, b] prove that

$$\int_a^b f(x)g(x)dx = \left[f(b)g_1(b) - f(a)g_1(a)\right] - \int_a^b f'(x)g_1(x)dx$$

where $g_1'(x) = g(x)$ for every $x \in [a, b]$.

Ans. Since f and g are bounded, there exist k, such that $\forall x \in [a,b]$ (2011)

$$|f(x)| \le k, \ |g(x)| \le k$$

$$\Rightarrow \quad |f(x)g(x)| \le k^2, \quad \forall x \in [a,b]$$

$$\Rightarrow$$ f.g is bounded.

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, x_{r+1}, \ldots\ldots x_n = b\}$ be any partition of [a, b]. Let $M_r', m_r'; M_r'', m_r''; M_r, m_r$ be the bounds of f, g and fg respectively in $I_r = [x_{r-1}, x_r]$. We have $\forall \alpha_1, \alpha_2 \in I_r$

$$f(\alpha_2)g(\alpha_2) - f(\alpha_1)g(\alpha_1) = g(\alpha_2)[f(\alpha_2) - f(\alpha_1)] + f(\alpha_1)[g(\alpha_2) - g(\alpha_1)]$$

$$\Rightarrow \quad |f(\alpha_2)g(\alpha_2) - f(\alpha_1)g(\alpha_1)| \le |g(\alpha_2)||f(\alpha_2) - f(\alpha_1)| + |f(\alpha_1)||g(\alpha_2) - g(\alpha_1)|$$

$$\le k(M_r' - m_r') + k(M_r'' - m_r'')$$

$$\Rightarrow \quad (M_r - m_r) \le k(M_r' - m_r') + k(M_r'' - m_r'')$$

Now let ε be nay positive number. .....(i)

Since f and g are integrable, there exist $d > 0$ such that for every partition of norm $\le \delta$, the corresponding oscillatory sums for f and g are both $< \varepsilon/2k$, Now for every partition P of norm $\le \delta$, we have

$$\sum(M_r - m_r) < k\sum(M_r' - m_r') + k\sum(M_r'' - m_r'')\delta r$$

$$< k(\varepsilon/2k) + k(\varepsilon/2k) < \varepsilon$$

implying that fg is integrable in [a, b].

Since we know that if f and g are integrable on [a, b] and

$$F(x) = A + \int_a^x f(x)dx, \ G(x) = B + \int_a^x g(x)dx$$

where A and B are constants, then

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$$\int_a^b f(x)g(x)dx = \left[F(x)G(x)\right]_a^b - \int_a^b G(x)f(x)dx$$

where $\left[F(x)G(x)\right]_a^b$ denotes the difference F(b)G(b) - F(a)G(a)

Now if function g is bounded and integrable on [a, b] and function f is differentiable and its derivative f' is bounded and integrable on [a, b], then

$$\int_a^b f(x)g(x)dx = \left[f(x)\int_a^x g(x)dx\right]_a^b - \int_a^b f'(x)\int_a^x g(x)dx$$

$$\Rightarrow \quad \int_a^b f(x)g(x)dx = \left[f(x)g_1(x)\right]_a^b - \int_a^b f'(x)g_1(x)dx$$

$$= \left[f(b)g_1(b) - f(a)g_1(a)\right] - \int_a^b f'(x)g_1(x)dx$$

where $g_1'(x) = g(x)$ for every $x \in [a,b]$.

Q.3. If f is integrable over [a, b] then show that for each $\varepsilon > 0$, there partition p_ε of [a, b], for which each partition p of [a, b] which is a refinem of p_ε,

$$\left|\sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) - \int_a^b f(x)dx\right| < \varepsilon$$

for every choice of $\xi_j \in |x_{j-1}, x_j|$ where p is the partition

P : $a = x_0 < x_1 < \ldots\ldots < x_n = b$.

Ans. Since f is bounded and integrable, therefore

$$\int_a^{\bar{b}} f(x)dx = \int_{\underline{a}}^b f(x)dx = \int_a^b f(x)dx$$

Let ε be any positive number. By Darboux's theorem, there exist $\delta > 0$ such for every partition P whose norm is $\leq \delta$

$$U(P_\varepsilon, f) < \int_a^{\bar{b}} f(x)dx + \varepsilon = \int_a^b f(x)dx + \varepsilon \qquad \ldots\ldots\ldots(i)$$

and $$L(P_\varepsilon, f) > \int_{\underline{a}}^b f(x)dx - \varepsilon = \int_a^b f(x)dx - \varepsilon \qquad \ldots\ldots\ldots(ii)$$

Where $P_\varepsilon = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n = b\}$, is the partition of b].

If ξ_j be any point of the interval I_j of P, we have

$$L(P_\varepsilon, f) \leq \sum_{j=1}^{j=n} f(\xi_j)\delta_j \leq U(P_\varepsilon, f) \qquad \ldots\ldots\ldots(iii)$$

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From eq.(i), (ii) and (iii) we deduce that for every partition P whose norm is $\leq \delta$.

$$\int_a^b f(x)dx - \varepsilon < \sum_{j=1}^{n} f(\xi_j)\delta_j < \int_a^b f(x)dx + \varepsilon$$

$$\Rightarrow \quad \left|\sum_{j=1}^{n} f(\xi_j)\delta_j - \int_a^b (x)dx\right| < \varepsilon$$

where $\delta_j = (x_j - x_{j-1})$

Q.4. Apply change of variable theorem of integration to integrate $\int_0^{\pi/2}\left(x^2 \sin x \cos x + x \sin^2 x\right)dx$.

Ans. $\int_0^{\pi/2}\left(x^2 \sin x \cos x + x \sin^2 x\right)dx = \int_0^{\pi/2} x \sin x\left(x \cos x + \sin x\right)dx$ **(2012)**

We take $g(x) = x\sin x$ $\Rightarrow$ $g'(x) = x\cos x + \sin x$

We define $f(u) = u$

then $f(g(x)) = x\sin x$

thus $\int_a^b f(gx)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

$$\int_0^{\pi/2} x \sin x\left(x \cos x + \sin x\right)dx = \int_0^{\pi/2} u.du$$

$$\Rightarrow \quad \left[\frac{u^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}$$

Q.5. If f is positive and monotinically decreasing in $[1,\infty]$. Show that the sequence $\{A_n\}$ where $A_n = \left\{f(1) + f(2) + \ldots.f(n) - \int_1^n f(n)dx\right\}$ is convergent. Deduce the convergence of

$$\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n\right\}$$ **(2012)**

Ans. We have

$$A_n = \left[f(1) - \int_1^2 f(x)dx\right] + \left[f(2) - \int_2^3 f(x)dx\right] + \ldots.$$

$$+ \left[f(n-1) - \int_{n-1}^n f(x)dx\right] + f(n)$$

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Now because of the monotonic character of the function f, each of the expressions within brackets is positive. Also f(n) is positive. Thus A_n is positive $\forall$ n. Again $\forall$ n.

$$A_{n+1} - A_n = f(n+1) - \int_1^{n+1} f(x)dx + \int_1^n f(x)dx$$

$$= f(n-1) - \int_n^{n+1} f(x)dx < 0$$

$$\Rightarrow \quad A_{n+1} < A_n \quad \forall n$$

Thus $\{A_n\}$ is monotonically decreasing. Also being positive, A_n is bounded below for each $n \in \mathbb{N}$. Hence $\{A_n\}$ is convergent

Taking $f(x) = 1/x$, we can now deduce that

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}-\log n\right) \text{ exist} \qquad \ldots\ldots\ldots(i)$$

We note that eq.(i) is known as Euler's constant whose numerical value is 0.577215.

Q.6. If $\int_a^b f(x)dx$ and $\int_a^b \phi(x)dx$ exist and $\phi(x)$ keeps the same sign, positive or negative throughout the interval of integration, then show that there exists a number μ lying between the infimum and supermum of f over [a, b] such that $\int_a^b f(x)\phi(x)dx = \mu\int_a^b \phi(x)dx$. (2012)

<u>Other Related Questions -</u>

Q. If $\int_a^b f(x)dx$ and $\int_a^b \phi(x)dx$ both exist and $\phi(x)$ keeps the same sign throughout the interval of integration, prove that there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)\phi(x)dx = \mu\int_a^b \phi(x)dx \qquad (2018)$$

Ans. First suppose that $\phi(x)$ is positive $\forall x \in [a,b]$ If M and m be two bounds of f, we have $m \le f(x) \le M$

$$\Rightarrow \quad m\phi(x) \le f(x)\phi(x) \le M\phi(x) \qquad \text{for } \phi(x) \ge 0, \ \forall x \in [a,b]$$

$$\Rightarrow \quad m\int_a^b \phi(x)dx \le \int_a^b f(x)\phi(x)dx \le M\int_a^b \phi(x)dx \quad \text{if } b \ge a$$

and $\quad m\int_a^b \phi(x)dx \ge \int_a^b f(x)\phi(x) \ge M\int_a^b \phi(x)dx \quad$ if $b \le a$

. Then there exist a number μ, lying between M and m, such that

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$\Rightarrow$ the function $|f|$ is bounded.

Let $\varepsilon > 0$ be any positive number.

Now f is integrable implies that there exist a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$, such that the corresponding oscillatory sum for f is $< \varepsilon$.

Let M_r' and m_r', M_r, m_r be the bounds of f and $|f|$ in $I_r = [x_{r-1}, x_r]$

Now $\forall \alpha_1, \alpha_2 \in I_r$ we have

$$\big||f(\alpha_2)| - |f(\alpha_1)|\big| \le |f(\alpha_2) - f(\alpha_1)| \le M_r' - m_r'$$

$$\Rightarrow \quad M_r - m_r \le M_r' - m_r'$$

$$\Rightarrow \quad \sum (M_r - m_r)\delta_r \le (M_r' - m_r')\delta_r < \varepsilon$$

Hence $|f|$ is integrable in $[a, b]$.

Since $|f| = \max\{f, -f\}$

$$\therefore \quad f(x) \le |f(x)| = |f|(x) \text{ and } -f(x) \le |f(x)| = |f|(x), \quad \forall x \in [a,b]$$

$$\Rightarrow \quad \int_a^b f(x)dx \le \int_a^b |f|(x)dx \quad \text{and} \quad \int_a^b \{-f(x)\}dx \le \int_a^b |f|(x)dx$$

$$\Rightarrow \quad \int_a^b f(x)dx \le \int_a^b |f|(x)dx \quad \text{and} \quad -\int_a^b f(x)dx \le \int_a^b |f|(x)dx$$

$$\Rightarrow \quad \left|\int_a^b f(x)dx\right| \le \int_a^b |f|(x)dx$$

$$\Rightarrow \quad \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

Hence $|f|$ is Riemann integrable.

Remark - The converse of above theorem is not always true, that is if $|f|$ is integrable, then f may not the integrable. To this end, consider the following function.

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be any partition of $[a, b]$.

Let m_r and M_r be the infimum and superimum of f.

$I_r = [x_{r-1}, x_r]$, then

$m_r = -1$ and $M_r = 1, r = 1, 2, 3 \ldots\ldots r.$

$$\therefore \quad L(P,f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n}(-1)\delta_r = -(b-a) = a-b$$

and $$U(P,f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n}(1)\delta_r = b-a$$

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Now, $\int_a^b f(x)dx = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$

$= \sup\{(a-b)\} = (a-b)$

and $\int_a^{\bar{b}} f(x)dx = \inf\{U(P,f) : P \text{ is a partion of } [a,b]\}$

$= \inf\{(b-a)\} = b-a$

Since $\int_a^b f(x)dx \ne \int_a^{\bar{b}} f(x)dx$, R is not Riemann integrable.

But $|f|(x) = |f(x)| = 1, \ \forall x \in [a,b]$. Since $|f|$ is a constant function, $|f|$ is integrable.

Q.9. If a function f is bounded in [a,b] and the set of its points of discontinuity in [a,b] has only a finite number of limit points then prove that f is Riemann integrable in [a,b]. Hence or otherwise, show that the function f defined by

$$f(x) = \frac{1}{2^n}, \text{when} \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, n = 0,1,2,3....$$

is Riemann integrable in [0,1].

Other Related Questions -

Q. Prove that a bounded function f: [a,b] → ℝ which has only a finite number of points of discontinuity in [a,b] is Riemann integrable in [a,b]. (2013)

Ans. Let $\{a_1, a_2, a_3, \ldots\ldots a_p\}$ be the finite set of limit points of the set of the points of discontinuity of f in [a, b]. We inclose them in p, non-overlapping intervals. (2018)

$[a'_1, a''_1], [a'_2, a''_2], \ldots\ldots\ldots [a'_p, a''_p]$

such that the sum of their length is $< 2/2(M - m)$; M, m being the bounds of f in [a, b]. The oscillation of f each of these intervals being $\ge (M - m)$, their total contribution to the oscillatory sum of these intervals is

$$< \left\{\frac{\varepsilon}{2(M-m)}\right\} \times (M-m) = \frac{\varepsilon}{2}.$$

Only a finite number of points of discontinuity of f can lie in each of the $(p+1)$ intervals.

$[a, a'_1], [a''_1, a'_2], \ldots\ldots\ldots [a''_p, b]$

So that, as each of them can be so sub-divided that the contribution of the oscillatory sum of the sub-intervals of each of these $(p+1)$ intervals is separately $< \varepsilon / 2(p+1)$.

Thus there exist a partition of [a, b] such that corresponding oscillatory sum is

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$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(p+1)}(p+1) = \varepsilon$$

hence f is integrable in [a, b].

$f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0, 1, 2, 3, \ldots$

For $n = 1, 2, 3, \ldots$ we have

$$f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}} \text{ and } f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}$$

Showing that f is discontinuous at $x = \frac{1}{2^n}$, $n = 1, 2, 3, \ldots$. Again $f(1-0) = 1$

$f(1) \Rightarrow f(x)$ is continuous at $x = 1$. Clearly, $f(x)$ is not continuous at $x = 0$.

Thus, we notice that f is boundeed and continuous in [0, 1] except the set of points

$$0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots, \frac{1}{2^n}, \ldots\ldots$$

which has only one limit point, namely 0, and hence f is integrable.

Q.10. If $f: [a, b] \to \mathbb{R}$ is bounded function and r be a real number such that $a < r < b$, if $\int_a^r f(x)dx$ and $\int_r^b f(x)dx$ exist, then prove that $\int_a^b f(x)dx$ exists and $\int_a^b f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$. (2014)

Ans. Since $\int_a^r f(x)dx$ is exist.

$$\int_{\underline{a}}^{r} f(x)dx = \int_a^{\bar{r}} f(x)dx = \int_a^r f(x)dx \qquad \forall x \in [a, b] \quad \ldots\ldots(i)$$

Since $f: [a, b] \to \mathbb{R}$ be bounded therefore it is bounded in [a, r], therefore

$\forall \varepsilon > 0, \exists$ step function, $s_1(x) \in S_f$ and $t_1(x) \in T_f$ such that.

$$\int_{\underline{a}}^{r} f(x)dx - \int_a^r s_1(x)dx < \frac{\varepsilon}{2}$$

And $$\int_a^r t_1(x)dx - \int_a^{\bar{r}} f(x)dx < \frac{\varepsilon}{2}$$

$$\Rightarrow \int_a^r f(x)dx - \int_a^r s_1(x)dx < \frac{\varepsilon}{2} \quad \ldots\ldots(ii) \quad (\because \text{using eq.(i)})$$

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And $$\int_a^r t_1(x)dx - \int_a^r f(x)dx < \frac{\varepsilon}{2} \quad \ldots\ldots(iii)$$

Since $\int_r^b f(x)dx$ exist.

Therefore $\int_r^b f(x)dx = \int_{\underline{r}}^{b} f(x)dx = \int_r^{\bar{b}} f(x)dx$; $\forall x \in [a, b]$ ……(iv)

Again $\forall \varepsilon > 0$ $\exists$ step functions $s_2(x) \in S_f$ and $t_2(x) \in T_f$ such that.

$$\int_r^b f(x)dx - \int_r^b s_2(x)dx < \frac{\varepsilon}{2}, \quad \ldots\ldots(v) \quad (\because \text{using (iv)})$$

$$\int_r^b t_2(x)dx - \int_r^b f(x)dx < \frac{\varepsilon}{2}, \quad \ldots\ldots(vi)$$

Define step function, $s_0: [a, b] \to E^1$ as

$$s_0(x) = \begin{cases} s_1(x), & \forall x \in [a, r] \\ s_2(x), & \forall x \in [r, b] \end{cases}$$

$$s_0(x) \le f(x) \qquad \forall x \in [a, b]$$

$$\Rightarrow \quad s_0(x) \in S_f$$

$$\Rightarrow$$

Define step function $t_0: [a, b] \to \mathbb{R}$ as $t_0(x) = \begin{cases} t_1(x), & \forall x \in [a, r] \\ t_2(x), & \forall x \in [r, b] \end{cases}$

$$t_0(x) \ge f(x), \qquad \forall x \in [a, b]$$

$$\therefore \quad t_0(x) \in T_f$$

$$\Rightarrow$$

Consider $\int_a^b t_0(x)dx - \int_a^b s_0(x)dx$

$$= \left[\int_a^r t_1(x)dx + \int_r^b t_2(x)dx\right] - \left[\int_a^r s_1(x)dx + \int_r^b s_2(x)dx\right]$$

$$= \left[\int_a^r t_1(x)dx - \int_a^r s_1(x)dx\right] + \left[\int_r^b t_2(x)dx - \int_r^b s_2(x)dx\right]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad \forall \varepsilon > 0$$

$$< \varepsilon, \qquad \forall \varepsilon > 0$$

$\Rightarrow$ $\int_a^b f(x)dx$ is Riemann integrable. (By Riemann condition of integra-

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bility).

Part II - Adding eq.(ii) and (v).

$$\int_a^r f(x)dx + \int_r^b f(x)dx - \int_a^b s_0(x)dx < \varepsilon$$

$$\Rightarrow \int_a^r f(x)dx + \int_r^b f(x)dx - \int_a^b f(x)dx < \varepsilon \quad \left(\because \int_a^b s_0(x)dx \le \int_a^b f(x)\right.$$

$$\Rightarrow \int_a^r f(x)dx + \int_r^b f(x)dx - \int_a^b f(x)dx \le 0 \quad (\because A \le \varepsilon, \forall \varepsilon > 0 \Rightarrow A$$

$$\Rightarrow \int_a^r f(x)dx + \int_r^b f(x)dx \le \int_a^b f(x)dx \quad \text{.....(vii)} \left(\because \int_a^b f(x)dx = \int_a^b f(x)\right.$$

Similarly adding equation (iii) and (vi), we get

$$\int_a^b f(x)dx \le \int_a^r f(x)dx + \int_a^b f(x)dx \qquad \text{.....(viii)}$$

From eq.(vii) and (viii) we get

$$\Rightarrow \int_a^b f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$$

Q.11. Define a step function giving an example.

Evaluate $\int_{-2}^{2}\left[x^2+1\right]dx$ **where [y] denotes largers integer less than or equal to y.** (2)

Ans. Step Function: -

Let $f:[a,b] \to E^1(\mathbb{R})$ is defined as

$$f\ [\ x_k\ ] \qquad \forall x \in (x_{k-1}, x_k) \qquad k = 1, 2, \ldots\ldots, n$$

where $P = \{a = x_0, x_1, x_2, \ldots\ldots x_{k-1}, x_k, \ldots\ldots, x_n\}$ be the partition of cl

interval [a, b] then f is called step function.

Example - Let $f:[-2,2] \to E^1$ defined as

$$f(x) = [x], \qquad \forall x \in [-2,2]$$

Let P = {-2, -1, 0, 1, 2} be an partion of [-2, 2]

then $f(x) = -2, \quad \forall x \in (-2,-1)$

$f(x) = -1, \quad \forall x \in (-1,0)$

$f(x) = 0, \quad \forall x \in (0,1)$

$f(x) = 1, \quad \forall x \in (1,2)$

$\therefore$ f(x) is a step function.

$$\int_{-2}^{2}\left[x^2+1\right]dx$$

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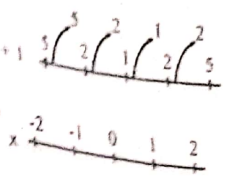
Let P = {-2, -1, 0, 1, 2} be partition of [-2, 2]. Then

$f(x) = 5, \quad \forall x \in (-2,-1)$

$f(x) = 2, \quad \forall x \in (-1,0]$

$f(x) = 1 \quad \forall x \in (0,1)$

$f(x) = 2 \quad \forall x \in (1,2)$



Therefore

$$\int_{-2}^{2}\left[x^2+1\right]dx = \int_{-2}^{-1}5dx + \int_{-1}^{0}2dx + \int_{0}^{1}1dx + \int_{1}^{2}2dx$$

$$= 5[x]_{-2}^{-1} + 2[x]_{-1}^{0} + [x]_{0}^{1} + 2[x]_{1}^{2}$$

$$= 5(1) + 2(1) + 1 + 2(1) = 10$$

Hence $\int_{-2}^{2}\left[x^2+1\right]dx = 10$

Q.12. Define primitive of a function and state and prove theorem on change of variable of integration. (2014)

Other Related Questions -

Q. State and prove the change of variable theorem of integration. (2012)

Ans. Primitive Of Function: -

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a given function then

$f:[a,b] \longrightarrow \mathbb{R}$ is called primitive of f if

$$\frac{d}{dx}F(x) = f(x)$$

Theorem: -

Change of variable of integration: -

Let A be a subset of $\mathbb{R}$. If $g:[a,b] \longrightarrow A$ be a function such that g(x) exists and is continuous $\forall x \in [a,b]$ and $f: A \longrightarrow \mathbb{R}$ be continuous on A, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$$

Proof - The composition map $fog:[a,b] \longrightarrow \mathbb{R}$ is well defined and continuous because f and g are continuous and composition of two continuous

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$\Rightarrow$ $\int_a^b f(g(x))g'(x)dx$ exists.

Let $\psi(x)=\int_a^x f(g(t))g'(t)dt$ where $x\in[a,b]$

$\Rightarrow$ $\frac{d\psi(x)}{dx}=f(g(x).g'(x))$ .....(i)

Since f is integrable on A, so f is also integrable on [g(a), g(b)] being a interval.

Hence $\int_{g(a)}^{g(b)} f(x)dx$ exists.

Let $\phi(u)=\int_{g(a)}^{u} f(t)dt$ .....(ii)

$\Rightarrow$ $\frac{d\phi(u)}{du}=F(u)$ .....(iii)

Put u = g(x) is equation (ii) we get

$$\phi(g(x))=\int_{g(a)}^{g(x)} f(t)dt$$

Now $\frac{d}{dx}\phi(g(x))=\frac{d}{du}\phi(u).\frac{du}{dx}=f(u).g'(x)$

$=f(g(x)).g'(x)$ .....(iv)

By eq.(i) and (iv), $\psi(x)$ and $\phi(g(x))$ are primitives of same functi

$f(g(x)).g'(x)$. Therefore

$\phi(g(x))-\psi(x)=K$ (constant) .....(v)

Put x = a, in eq.(v)

$\phi(g(a))-\psi(a)=K$

$\Rightarrow$ $k=0$

$\phi(g(a))=\psi(a)$

$\Rightarrow$ $\phi(g(x))=\psi(x)$

Put x = b in eq.(v) we get

$\phi(g(b))-\psi(b)=K$

$\Rightarrow$ $k=0$

$\therefore$ $\phi(g(b))=\psi(b)$

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$\int_{g(a)}^{g(b)} f(u)du=\int_a^b f(g(t)).g'(t)dt$

$\Rightarrow$

$\int_a^b f(g(x)).g'(x)dx=\int_{g(a)}^{g(b)} f(x)dx$

$\Rightarrow$

Q.13. What do we say that a bounded function f:[a b] $\to$ $\mathbb{R}$ is Riemann integrable? Prove that the function f:[0, 2] $\to$ $\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } 0\le x\le 1,\\ 2 & \text{if } 1<x\le 2,\end{cases}$$

is Riemann integrable and find its integral.

Ans. A bounded function f:[a, b] $\to$ E′ is called Riemann integrable if lower integrable if lower integral = upper integral and its common value is denoted by (2015)

$$\int_{\underline{a}}^{b} f(x)dx=\int_a^{\overline{b}} f(x)dx$$

Define a function f:[0, 2] $\to$ $\mathbb{R}$, defined by

$$f(x)=\begin{cases}1 & \text{if } 0\le x\le 1\\ 2 & \text{if } 1<x\le 2\end{cases}$$

which is a step function for the partition {0, 1, 2} and hence f is Riemann integrable on [0, 2].

Now $\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$

$=1[1-0]+2[2-1]=1+2=3$

Q.14. Define a step function and its integral. If f and g are Riemann integrable functions on [a, b] such that f(x) $\le$ g(x) for all x in [a, b] prove that

$$\int_a^b f(x)dx\le\int_a^b g(x)dx$$ (2015)

Ans. **Step Function: -**

Let f:[a, b] $\to$ E¹(= $\mathbb{R}$) defined as

$f(x)=C_k$, $\forall x\in(x_{k-1},x_k)$, $k=1,2,3,.....n$

where $P=\{a=x_0,x_1,x_2,.....,x_{k-1},x_k.....x_n\}$ be the partition of closed interval [a, b], then f(x) is called step function.

Integration Of Step Function: -

Let f:[a, b] $\to$ E¹(= $\mathbb{R}$) be a step function defined as $f(x)=C_k$, $\forall x\in$

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$(x_k - x_k)$ where $P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}$ be the partition of $[a,b]$ integration of $f(x)$ is defined as

$$\int_a^b f(x)dx = \sum_{k=1}^{n} C_k (x_k - x_{k-1})$$

$$= C_1(x_1 - x_0) + C_2(x_2 - x_1) + \ldots + C_n(x_n - x_{n-1})$$

Proof - We first prove that if $\phi : [a, b] \to E^1$ is integrable $\phi(x) \geq 0, \forall x \in [a,b]$ then

$$\int_a^b \phi(x)dx \geq 0$$

Define a function $t : [a, b] \to E^1$ as -

$$t(x) \geq \phi(x), \qquad \forall x \in [a,b]$$

$\Rightarrow$ $t(x) \in T_\phi$

$\Rightarrow$ $t(x) \geq 0$

$\Rightarrow$ $\int_a^b t(x)dx \geq \int_a^b 0dx$ (By proerty of step function)

$\Rightarrow$ $\int_a^b t(x)dx \geq 0$ .....(i)

Also $\int_a^{\bar{b}} \phi(x)dx = g.l.b\left\{\int_a^b t(x)dx : t(x) \in T_\phi\right\}$

$\Rightarrow$ $\int_a^{\bar{b}} \phi(x)dx \geq \phi$

$\Rightarrow$ $\int_a^b \phi(x)dx \geq 0$

$\because$ $f(x) \leq g(x), \qquad \forall x \in [a,b]$

$\Rightarrow$ $g(x) - f(x) \geq 0, \qquad \forall x \in [a,b]$

$\Rightarrow$ $\int_a^b [g(x) - f(x)]dx \geq 0$

$\Rightarrow$ $\int_a^b g(x)dx - \int_a^b f(x)dx \geq 0$

$\Rightarrow$ $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Q.15. State and prove second mean value theorem for integrals. (2015)

Other Related Question -

Q. If $\int_a^b f(x)dx$ and $\int_a^b \phi(x)dx$ both exist and f is monotonic in [a,b] then prove that there exist $\xi \in [a,b]$ such that

$$\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^\xi f(x)dx + \phi(b)\int_\xi^b f(x)dx$$ (2013)

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Ans. Second Mean Value Theorem For Integrable: -

If $f:[a, b] \to E^1$ and $g:[a, b] \to E^1$ are integrable function and g is monotonic in [a, b] then $\exists \zeta \in [a,b]$ such that

$$\int_a^b (gf)(x)dx = g(a)\int_a^\zeta f(x)dx + g(b)\int_\zeta^b f(x)dx$$

Proof - Case 1 : If g[a, b] $\to$ E¹ is monotonically decreasing.

Define function $\phi:[a, b] \to E^1$ as

$$\phi(x) = g(x) - g(b), \qquad \forall x \in (a,b]$$

Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$

$g(x_1) \geq g(x_2)$ ($\because$ g is monotonically decreasing)

$\Rightarrow$ $g(x_1) - g(b) \geq g(x_2) - g(b)$ ($\because$ substract $g(b)$ both side)

$\Rightarrow$ $\phi(x_1) \geq \phi(x_2)$

$\Rightarrow$ $\phi(x)$ is monotonically decreasing and non negative

$\Rightarrow$ by Bonnet's form of 2nd MVT

$\therefore$ $\exists \zeta [a,b]$ such that

$$\int_a^b (\phi.f)(x)dx = \phi(a)\int_a^\zeta f(x)dx$$

$\Rightarrow$ $\int_a^b \{g(x) - g(b)\}f(x)dx = [g(a) - g(b)]\int_a^\zeta f(x)dx$

$\Rightarrow$ $\int_a^b (gf)(x)dx = g(a)\int_a^\zeta f(x)dx - g(b)\int_a^\zeta f(x)dx + \int_a^b g(b)f(x)dx$

$$= g(a)\int_a^\zeta f(x)dx + g(b)\left[\int_\zeta^a f(x)dx + \int_a^b f(x)dx\right]$$

$\Rightarrow$ $\int_a^b (gf)(x)dx = g(a)\int_a^\zeta f(x)dx + g(b)\int_\zeta^b f(x)dx$

Case 2 : If g:[a, b] $\to$ E¹ is monotonically increasing -

Define a function $\phi : [a, b] \to E^1$ as

$$\phi(x) = g(b) - g(x) \geq 0 \quad \forall x \in [a,b]$$

Let $x_1, x_2 \in [a, b]$

and $x_1 < x_2$

$\Rightarrow$ $g(x_1) \leq g(x_2)$ ($\because$ g is monotonically increasing)

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$\Rightarrow$ $g(x_1) \le g(x_2)$

$\Rightarrow$ $g(b) - g(x_1) \ge g(b) - g(x_2)$

$\Rightarrow$ $\phi(x_1) \ge \phi(x_2)$

$\Rightarrow$ $\phi(x)$ is monotonically decreasing and non negative

By Bonnet's form of 2nd MVT

$\exists\ x \in [a, b]$ such that

$$\int_a^b \phi(x) f(x)\,dx = \phi(a)\int_a^x f(x)\,dx$$

$$\Rightarrow \int_a^b (g(b) - g(x)) f(x)\,dx = (g(b) - g(a))\int_a^x f(x)\,dx$$

$$\Rightarrow \int_a^b (f \cdot g)(x)\,dx = g(a)\int_a^x f(x)\,dx + g(b)\int_x^b f(x)\,dx$$

Q.16. Let $f: [a, b] \to \mathbb{R}$ be a Riemann integrable function. If F is an antiderivative of f on $[a, b]$, prove that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Other Related Question :- (2015)

Q. If $f: [a, b] \to \mathbb{R}$ is a continuous map on $[a, b]$ $G(x) = \int_a^x f(t)\,dt$ then prove

$F(b) - F(a) = \int_a^b f(x)\,dx$, where $f:[a, b] \to \mathbb{R}$ is a primitive of f. (2012)

Ans. Since $f:[a, b] \to \mathbb{R}$ be a Riemann integrable function and hence $f:[a, b] \to \mathbb{R}$ continuous, therefore $\int_a^b f(x)\,dx$ exists.

Let $G:[a, b] \to \mathbb{R}$ be a function defined as

$$G(x) = \int_a^x f(t)\,dt \qquad \forall x \in [a, b]$$

$\therefore$ By fundamental theorem of integral calculus

$$\frac{d}{dx}[G(x)] = f(x)$$

$\Rightarrow$ $G(x)$ is primitive of $f(x)$.

$\therefore$ By Lagrange's Mean Value Theorem (LMVT) -

$F(x) - G(x) = K$ .....(i)

Put $x = a$ in eq. (i) we have

$F(a) - G(a) = K$

$K = F(a)$

$\Rightarrow$ Putting $K = F(a)$ and $x = b$ in eq. (i) we get $\left[\because G(x) = \int_a^x f(t)\,dt = 0\right]$

$F(b) - G(b) = F(a)$

$G(b) = F(b) - F(a)$

$\Rightarrow$ $\int_a^b f(t)\,dt = F(b) - F(a)$

$\Rightarrow$ $\int_a^b f(x)\,dx = F(b) - F(a)$

Q.17. Define a partition of an interval $[a, b]$. Let $h: [a, b] \to \mathbb{R}$ be a step function associated with a partition P of $[a, b]$. Define $\int_a^b h(x)\,dx$ and show that the value of $\int_a^b h(x)\,dx$ does not change if P is replaced by its refinement.

Ans. Partition of an interval $[a, b]$:- (2016)

Let $[a, b]$ be a closed interval. Consider "n+1" points $x_0, x_1, x_2, \ldots, x_n$ such that

$a = x_0 < x_1 < x_2 < x_3 < \ldots\ldots\ldots < x_{n-1} < x_n = b$ then

$P = \{x_0, x_1, x_2, \ldots\ldots\ldots x_{k-1}, x_k \ldots\ldots x_n\}$ is called partition of $[a, b]$ and $[x_{k-1}, x_k]$ $(k = 1, 2, \ldots\ldots\ldots n)$ is called subinterval of closed interval $[a, b]$.

Proof -

Let $P = \{x_0, x_1, x_2, \ldots\ldots\ldots x_{k-1}, x_k \ldots\ldots\ldots x_n\}$ be the partition of $[a, b]$ with respect to $h(x)$ such that

$a = x_0 < x_1 < x_2 \ldots\ldots < x_{k-1} < x_k < \ldots\ldots < x_n = b$

$\because$ $h(x)$ is a step function

$$h(x) = C_k, \quad \forall x \in (x_{k-1}, x_k)$$
$$k = 1, 2, \ldots\ldots n$$

$$\therefore \int_a^b h(x)\,dx = \sum_{k=1}^{n} C_k (x_k - x_{k-1}) \quad \ldots\ldots(I)$$

Let x' be added to the partition "P" and

$$P_1 = \{x_0, x_1, x_2, \ldots\ldots, x_{k-1}, x', x_k, \ldots\ldots, x_n\}$$

$\Rightarrow$ $h(x) = C_k \qquad \forall\ x \in (x_{k-1}, x')$

$h(x) = C_k \qquad \forall\ x \in (x', x_k)$

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$$\int_a^b h(x)dx = C_1(x_1 - x_0) + C_2(x_2 - x_1) + \ldots\ldots\ldots + C_k(x' - x_{k-1})$$
$$+ C_k(x_k - x') + \ldots\ldots\ldots + C_n(x_n - x_{n-1})$$
$$= C_1(x_1 - x_0) + C_2(x_2 - x_1) + \ldots\ldots\ldots + C_k(x' - x_{k-1} + x_k - x') +$$
$$+ C_n(x_n - x_{n-1})$$
$$= C_1(x_1 - x_0) + C_2(x_2 - x_1) + \ldots\ldots\ldots + C_k(x_k - x_{k-1}) + \ldots\ldots$$
$$+ C_n(x_n - x_{n-1})$$

$$\int_a^b h(x)dx = \sum_{k=1}^{n} C_k(x_k - x_{k-1})$$

$\Rightarrow$ Refinement of a partition does not change the value of integration.

Q. 18. Let f and g be bounded real valued functions defined on [a, b]. Prove that $\int_a^b (f+g)(x)dx \le \int_a^b f(x)dx + \int_a^b g(x)dx$. (2016)

Ans. Let $P = \{a = x_0, x_1, \ldots\ldots x_{r-1}, \ldots\ldots x_n = b\}$ be any partition [a, b]. Let M_r', m_r'; M_r'', m_r''; M_r, m_r be the bounds of f, g and f + g in $I_r = [x_{r-1}, x_r]$. If α_1, α_2 be any two points of I_r. Then, we have

$$|[f(\alpha_2) + g(\alpha_2)] - [f(\alpha_1) + g(\alpha_1)]| \le |f(\alpha_2) - f(\alpha_1)| + |g(\alpha_2) - g(\alpha_1)|$$
$$\le (M_r' - m_r') + (M_r'' - m_r'')$$

$$\Rightarrow \quad M_r - m_r \le (M_r' - m_r') + (M_r'' - m_r'') \qquad \ldots\ldots\ldots(1)$$

Let $\varepsilon > 0$ be any given number.

Since f, g are integrable, there exist $\delta > 0$ such that for every partition of norm $\le \delta$, the corresponding oscillatory sum of "f" and "g" are both less than $\varepsilon/2$.

Now for any partition with norm $\le \delta$, we have, as implied by eq. (1).

$$\sum (M_r - m_r)\delta \le \sum (M_r' - m_r')\delta_r + \sum (M_r'' - m_r'')\delta_r$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus f + g is integrable in [a, b].

Now we prove that -

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Let "ε" be any integrable, there exists $\delta > 0$ such that for every partition of norm

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$\le \delta$ and for every $\varepsilon_r \in I_r$

$$\left|\sum f(\varepsilon_r)\delta_r - \int_a^b f(x)dx\right| < \frac{\varepsilon}{2} \text{ and } \left|\sum g(\varepsilon_r)\delta_r - \int_a^b g(x)dx\right| < \frac{\varepsilon}{2}$$

$$\left|\sum [f(\varepsilon_r) + g(\varepsilon_r)]\delta_r - \left[\int_a^b f(x) + \int_a^b g(x)dx\right]\right| < \varepsilon$$

$\Rightarrow$

Thus we obtain

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)$$

Q.19. If a function f is Riemann integrable over [a, b] then show that |f| is also Riemann integrable over [a, b] and $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$. (2016)

Ans. For a bounded function f(x), there exist a positive number "k" such that

$$|f(x)| \le k, \quad x \in [a, b]$$

$\Rightarrow$ the function | f | is bounded.

Let "ε" be any positive number. Now "f" is integrable implies that there exist a partition $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n = b\}$ such that the corresponding oscillatory sum for "f" is $< \varepsilon$.

Let M_r', m_r'; M_r, m_r be the bounds of "f" and |f| in $I_r = [x_{r-1}, x_r]$

Now, $\forall \alpha_1, \alpha_2 \in I$ we have

$$[|f(\alpha_2)| - |f(\alpha_1)|] \le f(\alpha_2) - f(\alpha_1) \le M_r' - m_r'$$

$$\Rightarrow \quad M_r - m_r \le M_r' - m_r'$$

$$\Rightarrow \quad \sum (M_r - m_r)\delta_r \le (M_r' - m_r')\delta_r < \varepsilon$$

Hence |f| is integrable in [a, b]

Since $|f| = \max\{f, -f\}$

$$\therefore \quad f(x) \le |f(x)| = |f|(x) \text{ and } -f(x) \le |f(x)| = |f|(x) \quad \forall x \in [a, b]$$

$$\Rightarrow \quad \int_a^b f(x) \le \int_a^b |f|(x)dx \quad \text{and} \quad \int_a^b \{-f(x)\}dx \le \int_a^b |f|(x)dx$$

$$\Rightarrow \quad \int_a^b f(x)dx \le \int_a^b |f|(x)dx \quad \text{and} \quad -\int_a^b f(x)dx \le \int_a^b |f|(x)dx$$

$$\Rightarrow \quad \left|\int_a^b f(x)dx\right| \le \int_a^b |f|(x)dx$$

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$\Rightarrow \quad \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

Q.20. State and prove a necessary and sufficient condition for the Riemann integrability of a bounded function f:[a, b] → ℝ. (2016,17)

Ans. Statement: -

The necessary and sufficient condition for a bounded function $f:[a,b] \to \mathbb{R}$ is Riemann integrable is that : $\forall\, \varepsilon > 0, \exists$ step function $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that

$$\int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$$

Proof - Necessary Part: -

let bounded function $f : [a, b] \to \mathbb{R}$ be Riemann integrable.

By definition -

$$\int_{\underline{a}}^{b} f(x)dx = \int_a^{\overline{b}} f(x)dx = \int_a^b f(x)dx \qquad \text{.....(i)}$$

Since $f : [a, b] \to \mathbb{R}$ is bounded therefore $\forall\, \varepsilon > 0, \exists$ step function $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that

$$\int_{\underline{a}}^{b} f(x)dx - \int_a^{\overline{b}} s_0(x)dx < \frac{\varepsilon}{2} \qquad \text{.....(ii)}$$

$$\int_a^b t_0(x)dx - \int_a^{\overline{b}} f(x)dx < \frac{\varepsilon}{2} \qquad \text{.....(iii)}$$

Adding eq.(ii) and (iii) we have

$$\int_{\underline{a}}^{b} f(x)dx - \int_a^b s_0(x)dx + \int_a^b t_0(x)dx - \int_a^{\overline{b}} f(x)dx < \varepsilon$$

$$\int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$$

Sufficient Part: -

Let $\forall\, \varepsilon > 0, \exists$ step function $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that

$$\int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon \qquad \text{.....(iv)}$$

Now we will show that bounded function $f:[a,b] \to \mathbb{R}$ is Riemann integrable.

$S_f = \{\text{Step function } s : [a, b] \to \mathbb{R} : s(x) \le f(x), \forall x \in [a,b]\}$

$$S_1 = \left\{\int_a^b s(x)dx : s(x) \in S_f\right\}$$

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$\int_a^b f(x)dx$ = least upper bound $\left\{\int_a^b s(x)dx : s(x) \in S_f\right\}$

and $s_0(x) \in S_f$

since $\int_a^b s_0(x)dx \in S_1 \Rightarrow \int_a^b s_0(x)dx \le \int_a^b f(x)dx$ .....(v)

$\Rightarrow$ $T_f = \{\text{step function } t : [a, b] \to \mathbb{R} : t(x) \le f(x), \forall x \in [a,b]\}$

$$S_2 = \left\{\int_a^b t(x)dx : t(x) \in T_f\right\}$$

$\int_a^b f(x)dx$ = greatest lower bound $\left\{\int_a^b t(x)dx : t(x) \in T_f\right\}$

Since $t_0(x) \in T_f$

$\int_a^b t_0(x)dx \in S_2$

$\Rightarrow \quad \int_a^b t_0(x)dx \ge \int_a^{\overline{b}} f(x)dx$

$\Rightarrow \quad \int_a^{\overline{b}} f(x)dx \le \int_a^b t_0(x)dx$ .....(vi)

or

Adding eq.(v) and (vi) we get

$$\int_a^{\overline{b}} f(x)dx + \int_a^b s_0(x)dx \le \int_a^b t_0(x)dx + \int_{\underline{a}}^{b} f(x)dx$$

$$\Rightarrow \quad \int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx \le \int_a^b t_0(x)dx - \int_a^b s_0(x)dx$$

$$\Rightarrow \quad \int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx \le \varepsilon \cdot \qquad \text{(using eq.(iv))}$$

$$\Rightarrow \quad \int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx \le 0 \qquad \left\{\begin{array}{l}\text{Since if } A \le \varepsilon, \forall \varepsilon > 0\\ \Rightarrow\ A \le 0\end{array}\right\}$$

$$\Rightarrow \quad \int_a^{\overline{b}} f(x)dx \le \int_{\underline{a}}^{b} f(x)dx \qquad \text{.....(vii)}$$

Since $\int_{\underline{a}}^{b} f(x)dx \le \int_a^{\overline{b}} f(x)dx$ .....(viii)

From eq.(vii) and (viii) we have

$$\int_a^{\overline{b}} f(x)dx = \int_{\underline{a}}^{b} f(x)dx$$

Therefore f(x) is Riemann integrable.

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Q.21. If $f:[a,b]\to\mathbb{R}$ is monotonic in [a, b] then prove that it is Riemann integrable over [a, b] (2017)

Ans. If $f:[a,b]\to\mathbb{R}$ is monotonic in [a, b] then we have to prove that $f:[a,b]\to\mathbb{R}$ is Riemann integrable.

Case (I): -

Let $f:[a,b]\to\mathbb{R}$ be monotonically increasing. Let $\varepsilon>0$ be any arbitrary choosen real number, then consider the partition -

$P=\{a=x_0, x_1, \ldots\ldots x_{k-1}, x_k, x_{k+1}, \ldots\ldots x_n=b\}$

Such that

$$x_k - x_{k-1} \le \frac{\varepsilon}{f(b)-f(a)+1} \qquad \ldots\ldots(i)$$

Define a step function $s_0(x)$ and $t_0(x)$ on [a, b] as

$s_0(x)=f(x_{k-1}), \quad \forall x\in(x_{k-1},x_k), \quad k=1,2,\ldots\ldots n$

and $t_0(x)=f(x_k), \quad \forall x\in(x_{k-1},x_k), \quad k=1,2,\ldots\ldots n$

Consider $\int_a^b t_0(x)dx-\int_a^b s_0(x)dx=\int_a^b\left[t_0(x)-s_0(x)\right]dx$

$$=\int_a^b\left[f(x_k)-f(x_{k-1})\right]dx$$

$$=\sum_{k=1}^{n}\left[f(x_k)-f(x_{k-1})\right](x_k-x_{k-1})$$

$$\le\frac{\varepsilon}{f(b)-f(a)+1}\sum_{k=1}^{n}\left[f(x_k)-f(x_{k-1})\right] \quad \left\{\begin{matrix}\text{since using}\\ \text{eq.(i)}\end{matrix}\right\}$$

$$\int_a^b t_0(x)dx-\int_a^b s_0(x)dx\le\frac{\varepsilon}{f(b)-f(a)+1}\left[f(x_1)-f(x_0)+f(x_2)-f(x_1)+\ldots\ldots+f(x_n)-f(x_{n-1})\right]$$

$$<\frac{\varepsilon}{f(b)-f(a)}\left[f(b)-f(a)\right]<\varepsilon$$

Therefore f is Riemann integrable.

Case (II): -

If $f:[a,b]\to\mathbb{R}$ is monotonically decreasing then $\forall\varepsilon>0$ choose a partition.

$P=\{a=x_0, x_1, \ldots\ldots x_{k-1}, x_k, x_{k+1}, \ldots\ldots x_n=b\}$

Such that

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$$x_k - x_{k-1} \le \frac{\varepsilon}{f(a)-f(b)+1} \qquad \ldots\ldots(ii)$$

Define step function $s_0(x)$ and $t_0(x)$ such that

$\forall x\in(x_{k-1},x_k),\ k=1,2,\ldots\ldots n$

$s_0(x)=f(x_k)$

$t_0(x)=f(x_{k-1})$

$s_0(x)\le f(x)\le t_0(x), \quad \forall x\in[x_{k-1},x_k]$

Therefore $s_0(x)\in S_f$ and $t_0(x)\in T_f$

Consider $\int_a^b t_0(x)dx-\int_a^b s_0(x)dx=\int_a^b\left[t_0(x)-s_0(x)\right]dx$

$$=\int_a^b\left[f(x_{k-1})-f(x_k)\right]dx=\sum_{k=1}^{n}\left[f(x_{k-1})-f(x_k)\right]\left[x_k-x_{k-1}\right]$$

$$\le\frac{\varepsilon}{f(a)-f(b)+1}\sum_{k=1}^{n}\left(f(x_{k-1})-f(x_k)\right) \quad \left\{\begin{matrix}\text{since using}\\ \text{eq.(ii)}\end{matrix}\right\}$$

$$\int_a^b t_0(x)dx-\int_a^b s_0(x)dx\le\frac{\varepsilon}{f(a)-f(b)+1}\left[f(x_0)-f(x_1)+f(x_1)-f(x_2)+\ldots\ldots+f(x_{n-1})-f(x_n)\right]$$

$$\le\frac{\varepsilon}{f(a)-f(b)+1}\left[f(a)-f(b)\right]$$

$<\varepsilon$

Therefore f is Riemann integrable.

Q.22. State and prove 'Bonnett's form of second Mean Value Theorem' of Integral Calculus. (2017)

Ans. Bonnet's form of second mean value theorem: -

If $\int_a^b f(x)dx$ and $\int_a^b\psi(x)dx$ both exist, ψ is monotonically decreasing and positive in [a, b] then there exist a point $\xi\in[a,b]$ such that

$$\int_a^b f(x)\psi(x)dx=\psi(a)\int_a^{\xi}f(x)dx$$

Proof: -

Let $P=\{a=x_0, x_1, \ldots\ldots x_{r-1}, x_r, \ldots\ldots, x_n=b\}$ be any partition of [a, b]. let M_r and m_r be bounds of f in $\ell_r=[x_{r-1},x_r]$.

Let $\xi_1=a$, and ξ_r, where $r\ne 1$, be any point of ℓ_r.

We have

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$$m_r\delta_r \le \int_{x_{r-1}}^{x_r} f(x)dx \le M_r\delta_r,\ m_r\delta_r \le f(\xi_r)\delta_r \le M_r\delta_r$$

Putting $r = 1, 2, 3, \ldots\ldots p$ where $p \le n$, and adding we obtain

$$\sum_{r=1}^{p} m_r\delta_r \le \int_a^{x_p} f(x)dx \le \sum_{r=1}^{p} M_r\delta_r$$

and
$$\sum_{r=1}^{p} m_r\delta_r \le \sum_{r=1}^{p} f(\xi_r)\delta_r \le \sum_{r=1}^{p} M_r\delta_r$$

Thus we have

$$\left|\int_a^{x_p} f(x)dx - \sum_{r=1}^{p} f(\xi_r)\delta_r\right| \le \sum_{r=1}^{p}(M_r - m_r)\delta_r \le \sum_{r=1}^{n}(M_r - m_r)\delta_r$$

$$\Rightarrow \quad \int_a^{x_p} f(x)dx - \sum_{r=1}^{n} O_r\delta_r \le \sum_{r=1}^{p} f(\xi_r)\delta_r \le \int_a^{x_p} f(x)dx + \sum_{r=1}^{n} O_r\delta_r$$

Where $O_r = (M_r - m_r)$ is the oscillation of f in ℓ_r. Now, $\int_a^t f(x)dx$, being a continuous function with t as variable, is bounded. Let C and D be its bounds.

Therefore we have -

$$C - \sum_{r=1}^{n} O_r\delta_r \le \sum_{r=1}^{p} f(\xi_r)\delta_r \le D + \sum_{r=1}^{n} O_r\delta_r$$

Using the Abel's lemma, we put as is justifiable

$$V_r = f(\xi_r)\delta_r,\ a_r = \psi(\xi_r);\ K = C - \sum O_r\delta_r,\ K = D + \sum O_r\delta_r$$

and obtain

$$\psi(a)\left[C - \sum_{r=1}^{n} O_r\delta_r\right] \le \sum_{r=1}^{n} f(\xi_r)\psi(\xi_r)\delta_r \le \psi(a)\left[D + \sum_{r=1}^{n} O_r\delta_r\right]$$

Let the norm of the partition tend to 0. We then obtain, in the limit,

$$C\psi(a) \le \int_a^b f(x)\psi(x)dx \le D\psi(a)$$

$$\Rightarrow \quad \int_a^b f(x)\psi(x)dx = \mu\psi(a)$$

wher μ is some number between C and D. The continuous function $\int_a^t f(x)dx$ must assume, for some $\xi \in [a, b]$ the value μ which lies between its bounds C,

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$$= 1 - \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{3^2}\right) + \frac{1}{3^2}\left(\frac{1}{3^2} - \frac{1}{3^3}\right) + \ldots\ldots + \frac{1}{3^{r-2}}\left(\frac{1}{3^{r-2}} - \frac{1}{3^{r-1}}\right)$$

$$= \frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3^2} + \frac{1}{3^2}\cdot\frac{2}{3^3} + \ldots\ldots + \frac{1}{3^{r-2}} + \frac{2}{3^{r-1}}$$

$$= \frac{2}{3}\left[1 + \left(\frac{1}{3^2}\right)^1 + \left(\frac{1}{3^2}\right)^2 + \left(\frac{1}{3^2}\right)^3 + \ldots\ldots + \left(\frac{1}{3^2}\right)^{r-2}\right]$$

$$= \frac{2}{3}\left[\frac{1 - \left(\frac{1}{3^2}\right)^{r-1}}{1 - \frac{1}{3^2}}\right] = \frac{2}{3}\left[\frac{1 - \frac{1}{3^{2r-2}}}{\frac{3^2 - 1}{3^2}}\right]$$

$$= \frac{2}{3}\left[\frac{1 - \frac{1}{3^{2r-2}}}{8}\right] \times 3^2 = \frac{2\times 3}{8}\left(1 - \frac{1}{3^{2r-2}}\right) = \frac{4}{3}\left[1 - \frac{1}{3^{2r-2}}\right]$$

Proceeding to the limit when $r \to \infty$ we get

$$\int_0^1 f(x)\,dx = \frac{4}{3}$$

Q.24. Prove that every step fuction f: [a,b] $\to \mathbb{R}$ is bounded and Rieman integrable on [a,b]. Is the converse true? Support your answer. (201

Ans. Let $p = \{x_0, x_1, x_2, < \ldots . x_{k-1} < x_k, \ldots . x_n\}$ be the partition of [a,b] wit respect to function f(x) such that.

$$a = x_0 < x_1 < x_2 < \ldots . x_{k-1} < x_k < \ldots . < x_n = b$$

$\because$ f(x) is step fuction.

$\because$ $f(x) = c_k \ \forall\, x \in (x_{k-1}, x_k)$, $k = 1, 2, \ldots, n$

$f(x) = c_1, \ \forall\, x \in (x_0, x_1)$

$f(x) = c_2, \ \forall\, x \in (x_1, x_2)$

.

.

.

$f(x) = c_n, \ \forall\, x \in (x_{n-1}, x_n)$

Take $H = \min(c_1, c_2, \ldots . c_n)$ .....(1)

$\Rightarrow$ $H \le f(x), \ \forall\, x \in [a,b]$

and take $k = \max(c_1, c_2, \ldots . c_n)$ .....(2)

$\Rightarrow$ $f(x) \le k, \ \forall\, x \in [a,b]$

From eq.(1) and (2)

$$H \le f(x) \le k, \ \forall\, x \in [a,b].$$

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in [a,b]. Show that f is identically zero on [a,b].

Sol. If possible let, for some $c \in [a,b]$, $f(c) > 0$

Let $\varepsilon = \left(\frac{1}{2}\right) \times f(c) > 0$. Since f is continuous at c, so for given $\varepsilon > 0$ th

such that.

$|f(x) - f(c)| < \varepsilon, \quad \forall x \in (c-\delta, c+\delta)$

$\Rightarrow$ $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ if $c - \delta < x < c + \delta$

$\Rightarrow$ $f(x) > f(c) - \varepsilon$ if $c - \varepsilon < x < c + \delta$

$\Rightarrow$ $f(x) > \left(\frac{1}{2}\right) \times f(c)$, it $c - \delta < x < c + \delta$

Now f is continous $[a,b] \Rightarrow$ f is integrable on [a,b]

$$\because \int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^b f(x)dx$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)dx, \text{ as } f(x) \geq 0, \forall x \in [a,b].$$

$$> \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} f(x)dx = \delta . f(c) > 0$$

Thus $\int_a^b f(x)dx > 0$, which contradicts the given hypothesis $\int_a^b f(x)dx = 0$

Hence $f(c) > 0$ can not hold. Similarly, we can show that $f(c) < 0$ can not hold

Hence $f(x) = 0 \ \forall x \in [a,b]$.

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Unit IV

Improper Integral: -

Integration $\int_a^b f(x)dx$ in which

(i) Range of integration is infinite, i.e. either or both of a, b are infinite and function is bounded in this range.

e.g. $\int_1^\infty \frac{1}{x}dx$

(ii) When limit a, b are finite and function f(x) is unbounded in closed interval [a, b]

e.g. $\int_0^1 \frac{1}{x}dx$

Integral discussed in first and second are called improper integral. In (I) case inegral is called **improper interal of first kind** and in case (II) it is called **improper integral of second kind.**

Note - Improper integral of first and second kind may exist together.

e.g. - $\int_0^\infty \frac{1}{x}dx = \int_0^a \frac{1}{x}dx + \int_a^\infty \frac{1}{x}dx$

Second kind First kind

Comparision test first form: -

If f and g be positive, bounded and integrable function defined in [a, t], for all $t > 0$, then

(i) If $g(x) \leq f(x), \ \forall x \geq a$ with $\int_a^\infty f(x)dx$ is convergent then $\int_a^\infty g(x)dx$ be also convergent.

(ii) If $g(x) \geq f(x), \forall x \geq a$ with $\int_a^\infty f(x)dx$ divergent then $\int_a^\infty g(x)dx$ is also divergent.

Comparision test second form: -

Let functions f, g be positive, bounded and integrable in [a, t], $t > a$

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell$ (finite and non zero) then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are either convergent or divergent.

Absolute convergence of integral: -

Let f(x) be a bounded function the $\int_a^\infty f(x)dx$ is called absolutely

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convergent if $\int_a^\infty |f(x)|dx$ is convergent

µ- Test (I) :-

Let function f be bounded and integrable in $[a, x]$ when $a > 0$ and μ be a real number where $\mu > 1$ such that $x^\mu f(x)$ is bounded $\forall x \geq a$ then $\int_a^\infty f(x)dx$ is absolutely convergent and hence convergent.

µ- Test (II) :-

If $f(x)$ is bounded and integrable in $[a, x]$, $a > 0$ and $x^\mu . f(x)$ has positive lower bound for all $x \geq a$ for some μ with $0 < \mu \leq 1$ then $\int_a^\infty f(x)dx$ is divergent

Note (i) :- If $\lim_{x \to \infty} f(x)$ is finite for some $\mu > 1$, then $\int_a^\infty f(x)dx$ is convergent.

(ii) If $\lim_{x \to \infty} x^\mu f(x)$ is finite and non zero (for $0 < \mu \leq 1$) then $\int_a^\infty f(x)dx$ is divergent.

Cauchy Test :-

Let $f \in R[a, x]$, $x \geq a$ then $\int_a^\infty f(x)dx$ is convergent iff $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon, \forall t_1, t_2 \geq \delta$.

Abel's Test :- If

(i) $\int_a^\infty f(x)dx$ is convergent.

(ii) ϕ is monotonic and bounded on $[a, \infty)$ then $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Dirichlet's Test :-

Let $\phi(x)$ be bounded and monotonic in $[a, \infty)$ and let $\phi(x) \to 0$ as $x \to \infty$ and also $\int_a^t f(x)dx$ be bounded $\forall t > a$ then $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Q.1. Test the convergence of the integral $\int_a^\infty \frac{x^3}{(a^2 + x^2)^2}dx$, where $a > 0$. (2011)

State the test used.

Ans. Let $I = \int_a^\infty \frac{x^3}{(a^2 + x^2)^2}dx$ .....(i)

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$$I = \int_a^1 \frac{x^3}{(a^2 + x^2)^2}dx + \int_1^\infty \frac{x^3}{(a^2 + x^2)^2}dx = I_1 + I_2 \quad \text{.....(ii)}$$

since $I_1 = \int_a^1 \frac{x^3}{(a^2 + x^2)^2}dx$ is proper integral and hence it is convergent.

take $I_2 = \int_1^\infty \frac{x^3}{(a^2 + x^2)^2}dx$

Therefore $\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} x^\mu \frac{x^3}{(a^2 + x^2)^2} = \lim_{x \to \infty} x^\mu \frac{x^3}{x^4\left(\frac{a^2}{x^2} + 1\right)^2}$

$$= \lim_{x \to \infty} \frac{x^{\mu - 1}}{\left(\frac{a^2}{x^2} + 1\right)^2} = 1, \quad \text{if } \mu - 1 = 0 \Rightarrow \mu = 1$$

Hence by µ-test I_2 is divergent. Hence given integral (1) is divergent.

µ-test :- (1) If $\lim_{x \to 0} x^\mu f(x)$ is finite for some $\mu < 1$ then $\int_a^x f(x)$ is convergent.

(2) If $\lim_{x \to 0} x^\mu f(x)$ is finite and nonzero (for $0 < \mu \leq 1$) then $\int_a^x f(x)dx$ divergent.

Q.2. Test the convergence of $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$. (2011)

Ans. We can write

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^1 \frac{dx}{\sqrt{x - x^2}}$$

Clearly 0 and 1 are the only points of infinite discontinuity of $f(x)$ in $[0, 1]$.

Substituting - $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}\sqrt{x}$

$x^2 = \sin^4\theta$

$(x - x^2) = \sin^2\theta(1 - \sin^2\theta) = \sin^2\theta\cos^2\theta$.

$\sqrt{x - x^2} = \sin\theta\cos\theta \Rightarrow 2\sqrt{x - x^2} = 2\sin\theta\cos\theta$

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$x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta\cos\theta} = \int 2\,d\theta$$

$$\int \frac{dx}{\sqrt{x(1-x)}} = 2\int d\theta = 2\theta = 2\sin^{-1}\sqrt{x}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x(1-x)}} = 2\sin^{-1}\sqrt{x} \qquad \text{.....(i)}$$

$$\Rightarrow \int_0^t \frac{dx}{\sqrt{x(1-x)}} = \int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^{t} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{t\to 0^+} \int_t^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \lim_{t\to 0^+} \int_{1/2}^{1-t} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{t\to 0^+} \left[2\sin^{-1}\sqrt{x}\right]_t^{1/2} + \lim_{t\to 0^+} \left[2\sin^{-1}\sqrt{x}\right]_{1/2}^{1-t}$$

(using eq.(i))

$$= \lim_{t\to 0^+} \left[2\sin^{-1}\sqrt{\frac{1}{2}} - 2\sin^{-1}\sqrt{t}\right] + \lim_{t\to 0^+} \left[2\sin^{-1}\sqrt{1-t} - 2\sin^{-1}\sqrt{\frac{1}{2}}\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi \text{ (finite)}$$

So $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ is convergent

Q.3. Show that $\int_a^\infty \sin x^2 dx$ is convergent. (2012)

Ans. We have $\int_a^\infty \sin x^2 dx = \int_a^1 \sin x^2 dx + \int_1^\infty \sin x^2 dx$ .....(i)

But $\int_a^1 \sin x^2 dx$ is a proper integral and so it is convergent.

We now test $\int_1^\infty \sin x^2 dx$ for convergence at ∞.

We have

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$$\int_1^\infty \sin x^2 dx = \int_1^\infty \left(2x\sin x^2\right)\frac{1}{2x}dx \qquad \text{.....(ii)}$$

Let $f(x) = 2x\sin x^2$ and $\phi(x) = 1/2x$.

Now, $\phi(x)$ is monotinic and $\to 0$ $x \to \infty$. Also

$$\left|\int_1^t f(x)dx\right| = \left|\int_1^t 2x\sin x^2 dx\right| = \left|\left[-\cos x^2\right]_1^t\right|$$

$$= \left|\cos 1 - \cos t^2\right| \le \left|\cos 1\right| + \left|\cos t^2\right| \le 2$$

Showing that $\int_1^t f(x)dx$ is bounded for all $t \ge 1$.

Therefore By Dirichlet test $\int_1^\infty f(x)\phi(x)dx$ i.e. $\int_1^\infty \sin x^2 dx$ is convergent.

Hence from eq.(i) the given integral $\int_a^\infty \sin x^2 dx$ is convergent.

Q.4. Show that $\int_0^{\pi/2} \sin x \log\sin x\, dx$ is convergent. (2012)

Ans. Here 0 is the only point of infinite discontinuity of the integrand in $[0, \pi/2]$. Here we have

$$\int_\varepsilon^{\pi/2} \sin x \log\sin x\, dx = \left[-\cos x \log\sin x\right]_\varepsilon^{\pi/2} + \int_\varepsilon^{\pi/2} \frac{\cos^2 x}{\sin x}dx$$

(Integrating By parts methods)

$$= \cos\varepsilon \log\sin\varepsilon + \int_\varepsilon^{\pi/2} (\operatorname{cosec} x - \sin x)dx$$

$$= \cos\varepsilon \log\sin\varepsilon + \left[\log\tan(x/2) + \cos x\right]_\varepsilon^{\pi/2}$$

$$= \cos\varepsilon \log\sin\varepsilon - \log\tan(\varepsilon/2) - \cos\varepsilon$$

$$= \cos\varepsilon \log\left\{2\sin(\varepsilon/2)\cos(\varepsilon/2)\right\} - \log\left\{\frac{\sin(\varepsilon/2)}{\cos(\varepsilon/2)}\right\} - \cos\varepsilon$$

$$= \cos\varepsilon \log(2\cos\varepsilon/2) + \cos\varepsilon \log\sin(\varepsilon/2) - \log\sin(\varepsilon/2) + \log\cos(\varepsilon/2) - \cos\varepsilon$$

$$= \cos\varepsilon \log(2\cos\varepsilon/2) + \log\cos(\varepsilon/2) - \cos\varepsilon - (1-\cos\varepsilon)\log\sin(\varepsilon/2)$$

$$\therefore \int_0^{\pi/2} \sin x \log\sin x\, dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\pi/2} \sin x \log\sin x\, dx$$

$$\lim_{\varepsilon\to 0^+} \left\{\cos\varepsilon \log(2\cos\varepsilon/2) + \log\cos(\varepsilon/2) - \cos\varepsilon - 2\sin^2(\varepsilon/2)\log(\varepsilon/2)\right\}$$

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$$= \log 2 - 0 - 1 - 0 = \log 2 - \log e = \log\left(\frac{2}{e}\right)$$

Ans. $\lim_{t\to 0} 2\sin^{-1}\left(\frac{t}{2}\right)\log\sin\left(\frac{t}{2}\right) = 1 \lim_{t\to 0} 2t^{-1}\log t$ if $t = \sin\theta/2$

$$= 2\lim_{t\to 0}\frac{\log t}{t^{-1}} \quad \left[\text{from } \frac{\infty}{\infty}\right]$$

$$= 2\lim_{t\to 0}\frac{(1/t)}{(-2)\times t^{-3}} = 0$$

Hence $\int_0^{\pi/2} \sin x \log\sin x dx$ is convergent with value $\log\left(\frac{2}{e}\right)$.

Q.5. Show that the improper integral $\int_0^1 x^{n-1}\log x dx$ converges if and only if $n > 0$. (2013)

Ans. Let $\int_0^1 x^{n-1}\log x dx$

The integrand is negative in the interval (0, 1] and we therefore, consider

$$\int_0^1 -x^{n-1}\log x dx = \int_0^1 x^{n-1}\log\left(\frac{1}{x}\right)dx$$

The integrand is proper if $(n-1) > 0$, in as much as the integrand, in the case $\to 0$ as $x \to 0$ and accordingly, 0, is not a point of infinite discontinuity in this case.

Let $(n-1) \le 0$ so that we have now to examine the convergence at 0.

Let m be a positive number such that $(m+n-1) > 0$.

We have $\lim_{x\to 0}\left[-x^{m+n-1}\log x\right] = 0$

so that ε being a given positive number.

$-x^{m+n-1}\log x < \varepsilon \quad \Rightarrow \quad -x^{n-1}\log x < \varepsilon/x^m$.

For values of x, sufficiently near 0.

Now the integral of ε/x^m converges at 0 if and only if $m < 1$.

It is possible to choose a number $m < 1$ such that

$(m+n-1) = [(m-1)+n] > 0$

if and only if $n > 0$. Thus the integral converges if $n > 0$.

When $n = 0$, the integrand become $x^{-1}\log x$. We have

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$$\int_\varepsilon^1 x^{n-1}\log x dx = -\frac{(\log\varepsilon)^2}{2} \text{ which } \to -\infty \text{ as } \to 0$$

when $n < 0$, we have $x^{n-1} < x^{n-1} \quad \forall x \in [0, 1]$

So that in this case also the integral does not converges.

Thus the given integral converges if and only if $n > 0$.

Q.6. Discuss the convergence of the following integral

$$I_1 = \int_0^\infty \frac{x^m\left(1+x^n\right)}{1+x^p}dx$$, **where m, n and p are positive real numbers.**

Ans. Let (2013)

$$\int_0^\infty \frac{x^m\left(1+x^n\right)}{1+x^p}dx = \int_0^1 \frac{x^m\left(1+x^n\right)}{1+x^p}dx + \int_1^\infty \frac{x^m\left(1+x^n\right)}{1+x^p}dx \quad m, n > 0$$

Let $I_1 = \int_0^1 \frac{x^m\left(1+x^n\right)}{1+x^p}dx$

which is an proper integral. So I_1 is convergent.

$$I_2 = \int_1^\infty \frac{x^m\left(1+x^n\right)}{1+x^p}dx$$

Let $f(x) = \frac{x^m\left(1+x^n\right)}{1+x^p}$

Consider $\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^\mu \frac{x^m\left(1+x^n\right)}{1+x^p} = \lim_{x\to\infty}\frac{x^{\mu+m}\left(1+x^n\right)}{1+x^p}$

$$= \lim_{x\to\infty}\frac{x^{\mu+m+n-p}\left(\frac{1}{x^n}+1\right)}{\left(\frac{1}{x^p}+1\right)}$$

$= 1$ if $\mu + m + n - p = 0$ or $N = p - n - m$

By μ - test of Ist type of improper integral I_2 is convergent if $p - n - m > 1$. and I_2 is divergent if $p - n - m \le 1$.

Thus $\int_0^\infty \frac{x^m\left(1+x^n\right)}{1+x^p}dx$ is convergent if $p - n - m > 1$.

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and $\int_0^\infty \frac{x^m(1+x^n)}{1+x^p}dx$ is divergent if $p-n-m \le 1$.

Q.7. Discuss convergence of $\int_0^\infty e^{-x}x^{n-1}dx$, x is real. (2014)

Ans. Let $I = \int_0^\infty e^{-x}x^{n-1}dx$

If $n = 1$, then the given integral I (say) is a proper integral because its integrand $f(x) = e^{-x}.x^{n-1}$ is bounded in $[0, 1]$.

So I is convergent when $n \ge 1$.

If $0 < n < 1$, then integral of I is unbounded at $x = 0$.

Take $\phi(x) = x^{n-1}$. Then, we have

$$\lim_{x\to 0}\frac{f(x)}{\phi(x)} = \lim_{x\to 0} e^{-x} = 1, \text{ which is finite and non-zero.}$$

So by comparision test, $\int_0^1 f(x)dx$ and $\int_0^1 \phi(x)dx$ either both converges or both diverges.

Now $$\int_0^1 \phi(x)dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{n-1}dx = \lim_{\varepsilon\to 0}\left[\frac{x^n}{x_n}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0}\left(\frac{1}{n}-\frac{\varepsilon^n}{n}\right) = \frac{1}{n}$$

which is finite number.

$\therefore$ $\int_0^1 \phi(x)dx$ is convergent and hence $\int_0^1 x^{n-1}.e^{-x}dx$ is also convergent when $n > 1$.

Q.8. Show that $\int_0^\infty \frac{\sin x}{x}dx$ is not absolutely convergent. (2014)

Other Related Questions -

Q. Test the convergence of the $\int_0^\infty \frac{\sin x}{x}dx$. (2011)

Q. Show that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent. (2018)

Sol. $$\int_0^\infty \frac{\sin x}{x} = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx \quad \text{.....(i)}$$

Since $\int_0^1 \frac{\sin x}{x}dx$ is proper integral and hence it is convergent.

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Now we will check the convergence of $\int_1^\infty \frac{\sin x}{x}dx$.

Let $\varepsilon > 0$ be any arbitrary number.

Let us choose a positive number $x_1, x_2 > \frac{2}{\varepsilon} = m$, Now we have

(using integration by parts method)

$$\int_{x_1}^{x_2}\frac{\sin x}{x}dx = \left[\frac{-\cos x}{x}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{\cos x}{x^2}dx \quad \text{.....(ii)}$$

So that $$\left|\int_{x_1}^{x_2}\frac{\sin x}{x}dx\right| \le \left|\frac{\cos x_1}{x_1}-\frac{\cos x_2}{x_2}\right| + \left|\int_{x_1}^{x_2}\frac{\cos x}{x^2}dx\right| \quad (\because |x-y| = |x|+|y|)$$

$$\left|\int_{x_1}^{x_2}\frac{\sin x}{x}dx\right| \le \frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x_2}+\frac{1}{x_1} \le \frac{2}{x_1}$$

$\Rightarrow$ $< \frac{2}{2/\varepsilon}$

$\Rightarrow$ $= 0$

$\Rightarrow$ $\left|\int_{x_1}^{x_2}\frac{\sin x}{x}dx\right| < \varepsilon, \quad \forall\ x_1x_2 > m$

So by Cauchy general test.

$\int_1^\infty \frac{\sin x}{x}dx$ is convergent

So $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

In order to show that the given integral not absolutelyconvergent, we must show that $\int_0^\infty \frac{|\sin x|}{x}dx$ is not convergent.

Consider the proper integral $\int_0^{n\pi}\frac{|\sin x|}{x}dx$, where n is a positive integer. we have

$$\int_0^{n\pi}\frac{|\sin x|}{x}dx = \sum_{r=1}^{n}\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx$$

We put $x = (r-1)\pi + y$, so that y varies on $[0, \pi]$. We have

$$\left|\sin\left[(r-1)x+y\right]\right| = \left|(-1)^{r-1}\sin y\right| = \sin y$$

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$$\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx = \int_0^{\pi}\frac{\sin y}{(r-1)\pi+y}dy$$

Since, $r\pi$ is the maximum value of $[(r-1)\pi + y]$ when $y \in [0,\pi]$, we have

$$\int_0^{\pi}\frac{\sin y}{(r-1)\pi+y}dy \geq \frac{1}{r\pi}\int_0^{\pi}\sin y\,dy = \frac{2}{r\pi}$$

$$\therefore \quad \int_0^{n\pi}\frac{|\sin x|}{x}dx \geq \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_{r=1}^n \frac{1}{r}$$

Since $\sum_{r=1}^n \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi}\frac{|\sin x|}{x}dx \to \infty \qquad \text{as} \qquad n \to \infty$$

Let, now, t be real number. There exists a positive integer n such that $n\pi \leq t < (n+1)\pi$, we have

$$\int_0^t \frac{|\sin x|}{x}dx \geq \int_0^{n\pi}\frac{|\sin x|}{x}dx$$

$t \to \infty$ so that n also $\to \infty$. Thus we see that

$$\int_0^t \frac{|\sin x|}{x}dx \to \infty \Rightarrow \int_0^{\infty}\frac{|\sin x|}{x}dx \text{ does not converges.}$$

Hence $\int_0^{\infty}\frac{\sin x}{x}dx$ is not absolutely convergent.

Q.9. If a real-valued function ϕ is bounded and monotonic in $[1,\infty]$ and $\int_1^{\infty} f(x)dx$ is convergent then prove that $\int_1^{\infty} f(x)\phi(x)dx$ is convergent.

(2013)

Other Related Questions -

Q. Let ϕ be bounded and monotonic in $[a,\infty]$ and let

$$\int_a^{\infty} f(x)dx$$

be convergent. Then prove that

$$\int_a^{\infty} f(x)\phi(x)dx \text{ is convergent.}$$

(2018, 19)

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Ans. Since $\int_a^{\infty} f(x)dx$ is convergent.

$\Rightarrow$ $f \in \mathbb{R}[a,s]$ and $\forall \varepsilon > 0, \exists \delta > 0$, such that

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \frac{\varepsilon}{2k}, \quad \forall t_1, t_2 > \delta \qquad \text{.....(i)}$$

Also $\phi(x)$ is bounded and monotonic in $[a,\infty)$

$\Rightarrow$ $|\phi(x)| \leq K, \ \forall x \in [a,\infty)$

for $t_1, t_2 > \delta$, i.e. $a < t_1 < t_2 < s$ .....(ii)

$\therefore$ By second mean value theorem $\exists \zeta \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} f(x)\phi(x)dx = \phi(t_1)\int_{t_1}^{\zeta} f(x)dx + \phi(t_2)\int_{\zeta}^{t_2} f(x)dx$$

$$\left|\int_{t_1}^{t_2} f(x)\phi(x)dx\right| = \left|\phi(t_1)\int_{t_1}^{\zeta} f(x)dx + \phi(t_2)\int_{\zeta}^{t_2} f(x)dx\right|$$

$$\leq |\phi(t_1)|\cdot\left|\int_{t_1}^{\zeta} f(x)dx\right| + |\phi(t_2)|\cdot\left|\int_{\zeta}^{t_2} f(x)dx\right|$$

$$< k.\frac{\varepsilon}{2k} + k.\frac{\varepsilon}{2k}, \quad \forall t_1, t_2 > \delta \qquad (\because \text{using eq.(i) and (ii)})$$

$\Rightarrow$ $\left|\int_{t_1}^{t_2} f(x)\phi(x)dx\right| < \varepsilon, \quad \forall t_1, t_2 > \delta$

$\therefore$ By Cauchy test $\int_a^{\infty} f(x)\phi(x)dx$ is convergent.

Q.10. Test the convergence of integrals $\int_1^{\infty} x^2 e^{-3x}dx$ and $\int_0^{\pi/2}\frac{\cos x}{x}dx$.

(2015)

Sol. $\int_1^{\infty} x^2 e^{-3x}dx$

Let $\phi(x) = \frac{1}{e^{3x}} \Rightarrow \lim_{x\to\infty}\phi(x) = 0$

Also ϕ is monotonically decreasing.

Let $f(x) = x^2$ and $a \geq 0$ if $a \neq 0$

consider $\int_1^a x^2 dx = \left[\frac{x^3}{3}\right]_1^a = \frac{a^3}{3} - \frac{1}{3}$ (finite)

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$\Rightarrow \quad \left|\int_1^a x^2 dx\right| < \frac{a^3}{3}$

$\Rightarrow \quad \int_1^a x^2 dx$ is bounded

So by Dirichlet test -

$\int_1^\infty f(x)\phi(x)dx$ is convergent

$\Rightarrow \quad \int_1^\infty x^2 e^{-3x} dx$ is convergent.

(ii) $\int_0^{\pi/2} \frac{\cos x}{x} dx$

Let $f(x) = \frac{\cos x}{x}$

which has infinite discontinuity at $x = 0$

consider $\lim_{x\to 0}(x-0)^\mu f(x) = \lim_{x\to 0}(x)^\mu \frac{\cos x}{x} = \lim_{x\to 0} x^{\mu-1}\cos x = 1$

if $\mu - 1 = 0 \quad \Rightarrow \quad \mu = 1$

So by μ - test of second kind of improper integral

$\int_0^{\pi/2} \frac{\cos x}{x} dx$ is divergent.

Q.11. State Cauchy's criterion for convergence of improper integrals of first kind. Test the convergence of $\int_0^\infty e^{-x^2} dx$. (2015)

Ans. Cauchy's criterior for convergence of improper integral: -

Let $f \in \mathbb{R}[a, s]$, $s \geq a$ then $\int_a^\infty f(x)dx$ is convergent

iff $\forall \varepsilon > 0, \exists \delta > 0$ such that

$\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon, \quad \forall t_1, t_2 > \delta$

Let $\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$

$I_1 = \int_0^1 e^{-x^2} dx$ is a proper integral. So it is convergent.

Consider $I_2 = \int_1^\infty e^{-x^2} dx$

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Since we know that $e^{x^2} > x^2, \quad \forall x \in \mathbb{R}$

$\Rightarrow \quad \frac{1}{e^{-x^2}} > x^2 \quad \Rightarrow \quad e^{-x^2} < \frac{1}{x^2}$

$\Rightarrow \quad \int_1^\infty e^{-x^2} dx < \int_1^\infty \frac{1}{x^2} dx$

But $\int_1^\infty \frac{1}{x^2} dx$ is converges

$\therefore$ So by comparision test -

$\int_1^\infty e^{-x^2} dx$ is also convergent.

Q.12. State and prove Dirichlet's test for convergence of improper integrals. (2015)

Other Related Questions -

Q. State and prove 'Dirichlet's Theroem' for the convergence of the improper integral $\int_a^\infty f(x)\phi(x)dx$. (2017)

Q. If f(x) is bounded and monotonic for $x \geq a$ and $\lim_{x\to\infty}\phi(x) = 0$ and $\int_a^x f(x)dx$ is bounded for $x \geq$ then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent. (2014)

Q. If ϕ be bounded and monotonic in $[a, \infty]$ and tends to '0' as x tends to ∞. Also let $\int_a^b f(x)dx$ is bounded for $\forall x \geq a$, then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent. (2012)

Ans. Dirichlet Theorem: -

Let $\phi(x)$ be bounded and monotonic decreasing $\forall x \geq a$ and $\lim_{x\to\infty}\phi(x) = 0$ and let $\int_a^x f(x)dx$ be bounded for $x \geq a$ then $\int_0^\infty f(x)\phi(x)dx$ is convergent.

Proof - By second mean value theorem $\exists \alpha \in [x_1, x_2]$ where $\forall x_1, x_2 \in [a, \infty)$ we have

$\int_{x_1}^{x_2} f(x)\phi(x)dx = \phi(x_1)\int_{x_1}^{\alpha} f(x)dx + \phi(x_2)\int_{\alpha}^{x_1} f(x)dx$

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Therefore $\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| \le |\phi(x_1)|\left|\int_{x_1}^{\alpha} f(x)dx\right| + |\phi(x_2)|\left|\int_{\alpha}^{x_2} f(x)dx\right|$

Since $\phi(x)$ is bounded and $\lim_{x\to\infty}\phi(x) = 0$

Therefore $|\phi(x) - 0| < \varepsilon'$ or $|\phi(x)| < \varepsilon'$

Therefore $\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| \le \varepsilon'\left|\int_{x_1}^{\alpha} f(x)dx\right| + \varepsilon'\left|\int_{\alpha}^{x_2} f(x)dx\right|$ .....(i)

Now consider

$$\left|\int_{x_1}^{\alpha} f(x)dx\right| = \left|\int_{x_1}^{a} f(x)dx + \int_{a}^{\alpha} f(x)dx\right|$$

$$\left|\int_{x_1}^{\alpha} f(x)dx\right| = \left|-\int_{a}^{x_1} f(x)dx + \int_{a}^{\alpha} f(x)dx\right|$$

$$\left|\int_{x_1}^{\alpha} f(x)dx\right| \le \left|-\int_{a}^{x_1} f(x)dx\right| + \left|\int_{a}^{\alpha} f(x)dx\right|$$

Since $\int_a^x f(x)dx$ is bounded

$$\therefore \quad \left|\int_{x_1}^{\alpha} f(x)dx\right| \le k + k\,(= 2k) \le 2k$$

Similarly we can show that

$$\left|\int_{\alpha}^{x_2} f(x)dx\right| \le 2k_1$$

Therefore from eq.(i) we have

$$\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| < \varepsilon'(2k) + \varepsilon'(2k_1)$$

$$\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| < \varepsilon'(2k + 2k_1)$$

We take $\varepsilon' = \dfrac{\varepsilon}{2k + 2k_1}$

Then we get $\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| < \varepsilon \quad \forall x_1, x_2 \in [a, \infty)$

Therefore $\int_a^{\infty} f(x)\phi(x)dx$ is convergent.

Q.13. Let f and g be positive and Riemann integrable functions on [a, x] for

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all $x > a$ such that $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = \ell$, $0 < \ell < \infty$. Show that $\int_a^{\infty} f(x)dx$ is convergent if and only if $\int_a^{\infty} g(x)dx$ is convergent.

Other Related Question -

Q. If f, g are positive bounded and integrable functions on [a+ε, b] and 'a' is the only point of discontinuity such that $\lim_{x\to a+0}\dfrac{f(x)}{g(x)} = \ell$ where ℓ is finite and non-zero then prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both either converge or diverge together. (2016)

Q. If f and g are bounded and integrable function on [a+ε, b] and 'a' is the only point of discontinuity such that $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \ell$, where ℓ is non-zero finite, then prove that and $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both either converge or diverge together. (2014)

(2012, 19)

Sol. Let $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = \ell$

Choose $\varepsilon > 0$ such that $\ell - \varepsilon > 0$

For $\varepsilon > 0$, $\exists$ a real no. $k > a$ such that

$$x > k \Rightarrow \left|\frac{f(x)}{g(x)} < \ell\right| < \varepsilon$$

$$\Rightarrow \quad \ell - \varepsilon < \frac{f(x)}{g(x)} < \ell + \varepsilon \qquad \forall x > k > a \qquad \text{.....(1)}$$

Case(I) - Take $\ell - \varepsilon < \dfrac{f(x)}{g(x)}$

$$\Rightarrow \quad g(x)(\ell - \varepsilon) < f(x), \qquad \forall x > k > a$$

Let $\int_a^{\infty} f(x)dx$ is convergent

$$\Rightarrow \quad \int_k^{\infty} f(x)dx \text{ is also convergent}$$

$\therefore$ By comparision test of Ist form

$$\int_k^{\infty} g(x)dx \text{ is also convergent.}$$

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$\Rightarrow \int_a^x g(x)dx = \int_a^k g(x)dx + \int_k^x g(x)dx$ is also convergent.

Case II:-

Take $\frac{f(x)}{g(x)} < \ell + \varepsilon$ $\forall\ x > k > a$

$\Rightarrow g(x)(\ell+\varepsilon) > f(x)$.

Let $\int_a^x f(x)dx$ is divergent.

$\Rightarrow \int_k^x f(x)dx$ is also divergent.

$\int_k^x g(x)dx$ is also divergent.

$\Rightarrow \int_a^x g(x)dx = \int_a^k g(x)dx + \int_k^x g(x)dx$ is also divergent.

Q.14. Prove that the improper integral $\int_a^\infty f(x)dx$ is convergent if and only if for any $\varepsilon > 0$ there exist $\delta > a$ such that $\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon$, when ever $t_1, t_2 > \delta$. (2016)

Other Related Questions -

Q. State and establish the test for convergence of an imporper integral $\int_a^\infty f(x)dx$. (2013)

Sol. Let $\int_a^\infty f(x)dx$ is convergent and $f \in \mathbb{R}[a, s]$, $s \geq a$

$\Leftrightarrow \lim_{S\to\infty} \int_a^s f(x)dx = s(\text{finite})$

iff $\lim_{S\to\infty} g(s) = \ell$ where $g(s) = \int_a^s f(x)dx$

$\Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0$

Such that $|g(t_1) - g(t_2)| < \varepsilon$ $\forall t_1, t_2 > \delta$

iff $\left|\int_a^{t_1} f(x)dx - \int_a^{t_2} f(x)dx\right| < \varepsilon$ $\forall t_1, t_2 > \delta$

(Because "g" is uniformly continuous)

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iff $\left|-\int_{t_2}^{t_1} f(x)dx\right| < \varepsilon$, $\forall t_1, t_2 > \delta$

iff $\left|\int_{t_2}^{t_1} f(x)dx\right| < \varepsilon$

Q.15. Determine values of m and n, for which $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent.

Other Related Questions -

Q. Show that the integral (2016)

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

is convergent if $m > 0$, $n > 0$.

Sol. The integral is proper if $m \geq 1$ and $n \geq 1$. (2018)

The number, 0 is a point of infinite discontinuity if $m < 1$ and the number 1 is a point of infinite discontinuity if $n < 1$.

Let $m < 1$ and $n < 1$.

We take any number, say 1/2 between 0 and 1 and examine the convergence of the two improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx$ at 0, 1 respectivelly.

Convergence at 0 - We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}}$$

and take $\phi(x) = \frac{1}{x^{1-m}}$, so that

$$\frac{f(x)}{\phi(x)} \to 1 \text{ as } x \to 0$$

As $\int_0^{1/2} \phi(x)dx = \int_0^{1/2} \frac{1}{x^{1-m}}dx$

is convergent at 0 if and only if $1 - m < 1 \Leftrightarrow 0 < m$, We deduced that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0 if and only if $m > 0$.

Convergence at 1 - We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}}$$

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and take $\phi(x) = \frac{1}{(1-x)^{n}}$

So that $\frac{f(x)}{\phi(x)} \to 1$ as $x \to 1$

As $\int_0^1 \phi(x)dx = \int_0^1 \frac{1}{(1-x)^{n}} dx$

is convergent, if and only if, $1 - n < 1 \Leftrightarrow 0 < n$.

We deduced that the integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ converges if and only if, n > 0.

Thus $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exist for positive values of m, n only.

Q.16. Examine the convergence of

(i) $\int_0^1 \frac{x^{p-1}}{1+x} dx$ and (ii) $\int_0^1 x^{n-1}e^{-x}dx, n \in N$ (2016)

Ans.(i) Let $f(x) = \frac{x^{p-1}}{1+x}$

if $p - 1 < 0 \Rightarrow p < 1$

then $\int_0^1 \frac{x^{p-1}}{1+x} dx$ is an improper integral otherwise it is proper integral.

$$f(x) = \frac{1}{x^{1-p}} \cdot \frac{1}{1+x} = \frac{1}{x^{1-p}(1+x)}$$

Let $g(x) = \frac{1}{x^{1-p}}$

then $\frac{f(x)}{g(x)} = \frac{1}{(1+x)}$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \text{ (finite)}$$

But the integral

$$\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{1-p}} dx$$

converges if $1 - p < 1 \Rightarrow p > 0$

Therefore by comparision test the given improper integral $\int_0^1 \frac{x^{p-1}}{1+x} dx$ is converges if $p > 0$.

(ii) Let $f(x) = x^{n-1}e^{-x}$

Convergence at 0 -

$f(x)$ has discontinuity at $x = 0$.

Let $g(x) = \frac{1}{x^{n-1}}$

then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^{n-1}e^{-x}}{(1/x^{1-n})} = \lim_{x \to 0} e^{-x} = 1$ $(\neq 0$ and finite$)$

But $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{1-n}} dx$

converges iff $(1 - n) < 1$

$\Rightarrow$ $n > 0$

Convergence at ∞ -

We known that for large value of "x" and fixed "n"

$e^x > x^{n+1}$

$\Rightarrow \frac{1}{e^{-x}} > x^{n-1}x^2 \Rightarrow x^{n-1}.e^{-x} < \frac{1}{x^2}$

$\Rightarrow \int_0^1 x^{n-1}.e^{-x}dx < \int_0^1 \frac{1}{x^2} dx$

But $\int_1^\infty \frac{1}{x^2} dx$ convergence.

Therefore by comparision test $\int_1^\infty x^{n-1}e^{-x}dx$ is converges for all "n".

Q.17. Let $0 < a < b < \infty$. Show that the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges if and only if $n < 1$.

Hence or otherwise, examine the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. (2017)

Ans. We write

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b (x-a)^{-n} dx$$

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$$= \lim_{\varepsilon\to 0}\left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+\varepsilon}^{b} = \lim_{\varepsilon\to 0}\frac{1}{1-n}\left[\frac{1}{(x-a)^{n-1}}\right]_{a+\varepsilon}^{b}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right] \quad \text{.....(i)}$$

Now $\lim_{\varepsilon\to 0}\varepsilon^{n-1} = \infty$ if $n-1<0$ i.e. $n<1$

Therefore $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n-1}} = 0$

Hence from eq(i) $\int_a^b \frac{dx}{(x-a)^n}$ is finite if $n<1$

Therefore the integral is convergent iff $n<1$.

Now we have to check the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. Here 0, is the only point of infinite discontinuity of integrand.

We have $f(x) = \frac{1}{x^{1/3}(1+x^2)}$

Take $\psi(x) = \frac{1}{x^{1/3}}$

Now $\lim_{\varepsilon\to 0}\frac{f(x)}{\psi(x)} = \lim_{x\to 0}\frac{1}{1+x^2} = 1$

Hence $\int_0^1 f(x)dx$ and $\int_0^1 \psi(x)dx$ have identical behaviours. But n = 1/3 being less than 1.

Hence the given integral are converge because we know that if $f(x)/\psi(x) \to \ell$ when $x \to a$, and ℓ is neither 0 nor infinite, then the two integrals $\int_a^b f(x)dx$ and $\int_a^b \psi(x)dx$ either both converges or both do not converge.

So $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

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Q.18. Show that $\int_0^\infty x^n e^{-x}dx$ is convergent if and only if n > -1. (2017)

Ans. We have to show that $\int_0^\infty x^n e^{-x}dx$ is convergent if and only if n > -1. For this

Let $f(x) = x^n.e^{-x} = x^{n+1-1}.e^{-x} = x^{k-1}e^{-x}$ where $k = n+1$

Now 0, is a point of infinite discontinuity of the function if $k<1$. Thus we have to examine the convergence at ∞ as well as at 0.

We consider any positive number > 0 say 1, and examine the convergence of two integrals $\int_0^1 f(x)dx$ and $\int_1^\infty f(x)dx$. at 0 and ∞ repectively.

(i) Convergence at 0: -Let $k<1$. We take

$$\phi(x) = \frac{1}{x^{1-k}} \quad \text{so that} \quad \lim_{x\to 0}\frac{f(x)}{\phi(x)} = 1$$

Also $\int_0^1 \frac{1}{x^{1-k}}dx$ converges iff $1-k \Leftrightarrow 0<k$

$\Leftrightarrow 0<n+1$ (∵ Putting value of $k = n+1$)

$\Leftrightarrow n>-1$

(ii) Converges at ∞: –

Now given k we have for sufficient large value of x.

$$e^x > x^{k+1} \Leftrightarrow x^{k-1}e^{-x} < \frac{1}{x^2}$$

But $\int_1^\infty \frac{1}{x^2}dx$ converges. Therefore $\int_1^\infty x^{k-1}e^{-x}dx$ converges or we can say $\int_1^\infty x^n e^{-x}dx$ converges (∵ putting value of k).

Thus $\int_0^\infty x^n e^{-x}dx$ converges iff $n>-1$.

Q.19. Show that $\int_0^\infty \sin x^3 dx$ is convergent. (2017)

Ans. We have $\int_0^\infty \sin x^3 dx = \int_0^1 \sin x^3 dx + \int_1^\infty \sin x^3 dx$ .....(i)

But $\int_0^1 \sin x^3 dx$ is proper integral and so it is convergent.

We now test $\int_1^\infty \sin x^3 dx$ for convergence at ∞. We have

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$$\int_1^\infty \sin x^3\,dx = \int_1^\infty \left(3x^2 \sin x^3\right)\frac{1}{3x^2}\,dx \qquad \text{.....(ii)}$$

Let $f(x) = 3x^2 \sin x^3$ and $\phi(x) = \frac{1}{3x^2}$ .....(iii)

Now $\phi(x)$ is monotonic and $\phi(x) \to 0$ as $x \to \infty$. Also

$$\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \left(3x^2 \sin x^3\right)dx\right| = \left|\left[-\cos x^3\right]_1^t\right|$$

$$= \left|\cos 1 - \cos t^3\right| \le \left|\cos 1\right| + \left|\cos t^3\right| \le 2$$

This showing that $\int_1^t f(x)\,dx$ is bounded for all $t \ge 1$. Therefore by Dirichlet's theorem $\int_1^\infty f(x)\phi(x)\,dx$ i.e. $\int_1^\infty \sin x^3 dx$ is convergent. Hence from eq.(i) the given integral is convergent.

Q.20. Determine real values of α and β for which integral $\int_0^{\pi/2} \sin^\alpha x.\cos^\beta x\,dx$ is convergent. **(2015)**

Sol. Let

$$f(x) = \sin^\alpha x \cos^\beta x = \left(\frac{\sin x}{\cos x}\right)^\alpha .\left(\cos^{\alpha+\beta} x\right) \quad \text{or} \quad \left(\frac{\cos x}{\sin x}\right)^\beta \sin^{\alpha+\beta} x$$

$$= (\tan x)^\alpha \left(\cos^{\alpha+\beta} x\right) \quad \text{or} \quad (\cot x)^\beta \sin^{\alpha+\beta} x$$

Clearly f has infinite discontinuity at $x = 0$ if $\alpha < 0$ and infinite discontinuous at $x = \pi/2$ if $\beta < 0$.

When $\alpha < 0, \beta < 0$ -

In this case 0, $\pi/2$ are point of infinite discontinuity.

So $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_{\pi/4}^{\pi/2} f(x)\,dx$ .....(i)

Test of $\int_0^{\pi/4} f(x)\,dx$ -

In this case 0 is only the point of infinite discontinuous.

Let $g(x) = \frac{1}{x-\alpha}$ where $0 < -\alpha < 1$ or $-\alpha < 1 \Rightarrow \alpha > -1$

clearly $\int_0^{\pi/4} g(x)\,dx$ is convergent at 0 if $\alpha > -1$.

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$$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^\alpha \cos^{\alpha+\beta} x = 1 \times 1 = 1 \quad \text{(non zero and finite)}$$

So by comparision test $\int_0^{\pi/4} f(x)\,dx$ is convergent at 0 if $\alpha > -1$.

Test of $\int_{\pi/4}^{\pi/2} f(x)\,dx$ - In this case $\pi/2$ is only the point of infinite discontinuous.

Let $h(x) = \frac{1}{\left(\frac{\pi}{2} - x\right)^{-\beta}}$ where $0 < -\beta < 1 \Rightarrow -\beta < 1 \Rightarrow \beta > -1$

Clearly $\int_{\pi/4}^{\pi/2} h(x)\,dx$ is convergent at $\pi/2$ if $\beta > -1$.

and $$\lim_{x\to\pi/2} \frac{f(x)}{h(x)} = \lim_{x\to\pi/2^-} \frac{\cot^\beta x}{\left(\frac{\pi}{2} - x\right)^\beta} \sin^{\alpha+\beta} x$$

Put $x = \pi/2 - h$

$$\Rightarrow \lim_{h\to\pi/2} \frac{f(x)}{h(x)} = \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)}{\frac{\pi}{2} - \frac{\pi}{2} + h} \sin^{\alpha+\beta}\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h\to 0} \frac{\tanh}{h} \cos^{\alpha+\beta} h$$

$$= \lim_{h\to 0} \frac{\tanh}{h} . \lim_{h\to 0} \cos^{\alpha+\beta} h = 1 \times 1 = 1 \qquad \text{(non zero \& finite)}$$

So by comparision test $\int_{\pi/4}^{\pi/2} f(x)\,dx$ is convergent at $\pi/2$ if $\beta > -1$.

So by eq.(i) $\int_0^{\pi/2} f(x)\,dx$ is convergent if $\alpha > -1$ and $\beta > -1$.

Q.21. Define the convergence of improper intergrals of second kind and and test the convergence of $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$. **(2019)**

Ans. Here 1 is only point of infinite discontinuity.

Let $f(x) = \frac{\sqrt{x}}{\log x}$ and $g(x) = \frac{1}{x-1}$, so that

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$$\int_1^\infty \sin x^3\,dx = \int_1^\infty \left(3x^2 \sin x^3\right)\frac{1}{3x^2}\,dx \qquad \text{.....(ii)}$$

$$\text{Let } f(x) = 3x^2\sin x^3 \text{ and } \phi(x) = \frac{1}{3x^2} \qquad \text{.....(iii)}$$

Now $\phi(x)$ is monotonic and $\phi(x) \to 0$ as $x \to \infty$. Also

$$\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \left(3x^2 \sin x^3\right)dx\right| = \left|\left[-\cos x^3\right]_1^t\right|$$

$$= \left|\cos 1 - \cos t^3\right| \le |\cos 1| + \left|\cos t^3\right| \le 2$$

This showing that $\int_1^t f(x)\,dx$ is bounded for all $t \ge 1$. Therefore by Dirichlet's theorem $\int_1^\infty f(x)\phi(x)\,dx$ i.e. $\int_1^\infty \sin x^3\,dx$ is convergent. Hence from eq.(i) the given integral is convergent.

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Sol. Let

$$f(x) = \sin^\alpha x \cos^\beta x = \left(\frac{\sin x}{\cos x}\right)^\alpha \cdot \left(\cos^{\alpha+\beta} x\right) \text{ or } \left(\frac{\cos x}{\sin x}\right)^\beta \sin^{\alpha+\beta} x$$

$$= (\tan x)^\alpha \left(\cos^{\alpha+\beta} x\right) \text{ or } (\cot x)^\beta \sin^{\alpha+\beta} x$$

Clearly f has infinite discontinuity at $x = 0$ if $\alpha < 0$ and infinite discontinuous at $x = \pi/2$ if $\beta < 0$.

When $\alpha < 0, \beta < 0$ -

In this case 0, $\pi/2$ are point of infinite discontinuity.

So $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_{\pi/4}^{\pi/2} f(x)\,dx$ .....(i)

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In this case 0 is only the point of infinite discontinuous.

Let $g(x) = \frac{1}{x-\alpha}$ where $0 < -\alpha < 1$ or $-\alpha < 1 \Rightarrow \alpha > -1$

clearly $\int_0^{\pi/4} g(x)\,dx$ is convergent at 0 if $\alpha > -1$.

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$$\therefore \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^\alpha \cos^{\alpha+\beta} x = 1 \times 1 = 1 \text{ (non zero and finite)}$$

So by comparision test $\int_0^{\pi/4} f(x)\,dx$ is convergent at 0 if $\alpha > -1$.

Test of $\int_{\pi/4}^{\pi/2} f(x)\,dx$ - In this case $\pi/2$ is only the point of infinite discontinuous.

Let $h(x) = \frac{1}{\left(\frac{\pi}{2} - x\right)^{-\beta}}$ where $0 < -\beta < 1 \Rightarrow -\beta < 1 \Rightarrow \beta > -1$

Clearly $\int_{\pi/4}^{\pi/2} h(x)\,dx$ is convergent at $\pi/2$ if $\beta > -1$.

and $$\lim_{x\to\pi/2} \frac{f(x)}{h(x)} = \lim_{x\to\pi/2} \frac{\cot^\beta x}{\left(\frac{\pi}{2} - x\right)^\beta} \sin^{\alpha+\beta} x$$

Put $x = \pi/2 - h$

$$\Rightarrow \lim_{h\to\pi/2} \frac{f(x)}{h(x)} = \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)}{\frac{\pi}{2} - \frac{\pi}{2} + h} \sin^{\alpha+\beta}\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h\to 0} \frac{\tanh}{h} \cos^{\alpha+\beta} h$$

$$= \lim_{h\to 0} \frac{\tanh}{h} . \lim_{h\to 0} \cos^{\alpha+\beta} h = 1 \times 1 = 1 \qquad \text{(non zero \& finite)}$$

So by comparision test $\int_{\pi/4}^{\pi/2} f(x)\,dx$ is convergent at $\pi/2$ if $\beta > -1$.

So by eq.(i) $\int_0^{\pi/2} f(x)\,dx$ is convergent if $\alpha > -1$ and $\beta > -1$.

Q.21. Define the convergence of improper intergrals of second kind and and test the convergence of $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$. **(2019)**

Ans. Here 1 is only point of infinite discontinuity.

Let $f(x) = \frac{\sqrt{x}}{\log x}$ and $g(x) = \frac{1}{x-1}$, so that

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$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1x-1x^{\frac{1}{2}}}{\log x}$$

$$=\lim_{x\to 0}\frac{x-x^{\frac{1}{2}}}{\log x}=1$$

The integral $\int_0^1 \frac{dx}{x-1}$ diverges and therefore $\int_0^1 \frac{\sqrt{x}}{\log x}dx$ is also diverges.

Q.22. Discuss the convergence and divergence of the integral $\int_0^1 x^m \operatorname{cosec}^n x\,dx$. (2019)

Sol. Let $\int_0^{\frac{\pi}{2}} x^m \operatorname{cosec}^n x\,dx$. .....(1)

Writting $f(x)=\frac{x^m}{(\sin x)^n}=\left(\frac{x}{\sin x}\right)^n x^{m-n}=\left(\frac{x}{\sin x}\right)^n\frac{1}{x^{n-m}}$

We see that $\lim_{x\to 0} f(x)=\begin{cases}0 \text{ if } m-n>0\\ 1 \text{ if } m-n=0\\ \infty \text{ if } m-n<0\end{cases}$

Thus (i) is proper integral if $(m-n)\geq 0$ and and improper if $(m-n)<0$; 0 being the only point of infinite discontinuity of the integrand in this case.

Let $(m-n)<0$ so that $(n-m)>0$

Take $\phi(x)=\frac{1}{x^{n-m}}$

Then $\frac{f(x)}{\phi(x)}\to 1$ as $x\to 0$

Also $\int_0^{\frac{\pi}{2}}\phi(x)=\int_0^{\frac{\pi}{2}}\frac{1}{x^{n-m}}dx$

converges, if and only if, $(n-m)<1 \Leftrightarrow n<(m+1)$.

Therefore the intergral converges if and only if $n<(m+1)$, which also includes the case $n<m$ when the integral is proper.

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Unit V

Definition: -

Let X be a non empty set then a map $d: X\times X\to \mathbb{R}$ is called metric on X if following conditions are satisfied.

(i) $d(x,y)\geq 0, \quad \forall x,y\in X$ **(Non-negative condition)**

(ii) $d(x,y)=0 \Leftrightarrow x=y$

(iii) $d(x,y)=d(y,x), \quad \forall x,y\in X$ **(symmetric condition)**

(iv) $d(x,y)\leq d(x,z)+d(z,y) \quad \forall x,y\in X$ **(Triangle inequality)**

then the pair (X, d), where X is non empty set and d is metric on X is called metric space.

Note - If X be a non empty set and d is metric on X then d(x, y) denotes the distance between x and y under the metric and also distance may be changed with metric.

Usual metric: -

Let $\mathbb{R}$ be set of real number then the map $d: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined as

$d(x,y)=|x-y| \quad \forall x,y\in\mathbb{R}$

then d is known as usual metric and $(\mathbb{R}, d)$ is known as usual metric space on real line.

Discrete Metric Space: -

Let X be a non-empty set. A map $d: X\times X\to\mathbb{R}$ is defined as -

$$d(x,y)=\begin{cases}0, & x=y\\ 1, & x\neq y\end{cases}$$

Then (X, d) is called discrete metric space.

Pseudo metric space: -

Let X be a non empty set then $d: X\times X\to\mathbb{R}$ is called pseudo metric if following condition are satisfied -

(i) $d(x,y)\geq 0, \quad \forall x,y\in X$

(ii) if $x=y \Rightarrow d(x,y)=0 \nRightarrow x=y$

(iii) $d(x,y)=d(y,x), \quad \forall x,y\in X$

(iv) $d(x,y)\leq d(x,z)+d(z,y), \quad \forall x,y,z\in X$

Then (X, d) is called pseudo metric space.

Bounded metric space: -

Any metric space (X, d) is called bounded if $\exists \ell, u\in\mathbb{R}$ such that

$\ell\leq d(x,y)\leq u, \quad \forall x,y\in X$

Neighbourhood of a point in metric space: -

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