WINSOME SERIES

ODE

Paper-3

उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े उसे त्याग दो।



KEY POINT AU



KEY POINT AU

Bsc-Ist Year

Maths : Paper - III

[Syllabus]

Differential Equations and Vector Calculus

Unit - I

Origin, Concept and formation of an ordinary differential equation and its solution, initial-value problems and statement of Existence and Uniqueness theorems, order and degree of differential equation, First order equations, separable equations, equations reductble to separable for, exact equations. Integrating factors, linear equation. Geometrical meaning of a different equation and its solution, isoclines and direction field.

Unit - II

Picard's iteration method, Curves under geometrical conditions, orthogonal trajectories, mechanical applications (Newton's law cooling growth and decay problems, motion in a resisting medium escape velocity). Equations solvable for p, x, or y Clairaut's equations, singular solutions.

Unit - III

Homogeneous linear differential equations, linearly dependent and independent solutions, fundamental system of solutions, equations with constant coefficients, Methods of undetermined coefficients, variation of parameters and operators for determining particular integrals, Euler-Cauchy equations.

Unit - IV

(a) Application to damped and forced oscillations, linear systems of differential equations with constant coefficients.

(b) Derivatives of a vector function of a single scalar variable, scalar and vector fields, gradient, divergence and curl, vector identities.

Unit - V

Orthogonal curvilinear coordinates, spherical polar and cylindrical polar coordinates double integrals. Line, surface and volume integrals, Green's, Stokes and Gauss' Theorems (statements only) and their simple applications

नोट : पाठ्यक्रम कालेज से अवश्य मिला लें। किसी प्रकार की त्रुटि, परिवर्तन हेतु प्रकाशक जिम्मेदार नहीं होगा।

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Q. 3. Find the general solution of equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$. (2014) – 2

Q. 4. Find the differential equation of the family of ellips with foci at (-1,0) & (1,0). (2014) – 3

Q. 5. Obtain the general solution of the differential equation - $dx + \left[4x - \frac{8}{x^3}\right] y\,dy = 0$ (2015) –3

Q. 6. Solve the initial value problem - $x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2$ $y(\pi) = e^{\pi}\pi^3 + 2\pi^2$. (2015) – 4

Q. 7. Define isoclines of a differential equation of the first order & explain its relation. (2015, 2016, 2017, 2019) – 5

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Q. 9. State an existence and uniqueness theorem for the initial value problem - $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y$ (2016) – 6

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Q. 12. Solve the differential euqation. $\left[x + y\sin^2\left(\frac{x}{y}\right)\right]dy - ydx = 0$ (2017) –8

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Q. 16. Find general solution of differential eq^n

$$\frac{dx}{dy} + P(x)x = Q(x)x^n$$

(2018)

Q. 17. Solve, $\frac{dy}{dx} = \frac{1}{xy^3 - x}$ (2018)

Q. 18. Solve the differential equation -
$xy' - (\ln x)y^2 + y = 0, x > 0$ (2009, 2019)

Q. 19. The growth of a population is described by the differential equation

$$\frac{dN}{dt} = aN - bN^2$$

Solve this equation subject to the initial condition that N = N0, when t = 0. (2019)

Chapter-2 Machanical Application

Q. 1. Suppose that in a chemical reaction, the rate of conversion of one of the substances is proportional to the square of the mass of the unconverted substance at that time. If m_0 was its original mass, then find the mass of it at time t (2014)

Q. 2. Solve $y = x^2 + 2px + \frac{p^2}{2}$ to find a general and a singular solution. (2014, 2019)

Q. 3. Explain how to solve a first order differential equation of the form

$$y = (x, p), P = \frac{dy}{dx}$$

(2015)

Q. 4. Obtain the solution of the differential equation
$xyp^2 + (x^2 + y^2)p + xy = 0$ (2015)

Q. 5. A certain amount of a radioactive substance was observed to be of mass at time t, is governed by differential equation.

$$\frac{dm}{dt} = -km$$

If at t = 0, m = m_0 then find such value of t, when $m = \frac{m_0}{2}$

(2015)

Q. 6 Define the orthogonal trajections of a family of curve (2016, 2018)

Q. 7 Find the general solution of the differential equation

$$xyp^2 + (x + y)p + 1 = 0, \left[p = \frac{dy}{dx}\right]$$

and obtain a particular solution valid for all x and passing through the point $\left[\frac{-1}{2}, 2\right]$ (2016)

Q. 8 Apply picards method to find first three approximate solutions to the initial value problem

$$\frac{dy}{dx} = 2x - 2x^2 - 3, y(0) = 2$$ (2009, 2016)

Q. 9 Find a general solution of equation (2016)

$$\frac{dy}{dx} = 2y - 2x^2 + 3, y(0) = 2$$ (2009, 2016)

Q. 10 The fish population in a pond is changing and its growth rate is found to satisfy the differential equation

$$\frac{dx}{dt} = \alpha x(k - x)$$

where, α, k are positive constant and x(t) represents the fish population at any time t. If it is given that initially the fish population was x_0, then obtain the fish population x (t) at any time t and established that ultimately, as $t \to \infty$ the fish population $x(t) \to k$ (2017)

Q. 11 Prove that the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y$$

is equivalent to the integral equation problem

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$ (2017)

Q. 12. Obtain Picards three interates $y_1(x)$, $y_2(x)$, $y_3(x)$ for the initial value problem,

$$\frac{dy}{dx} = x + 2y, y(0) = 0$$ (2017)

Q. 13. Apply Picard's interation method to obtain three iternates fo IVP

$y' = 2x(1 + y), y(0) = 0$ (2018)

Q. 14. Find the orthogonal trajectories of the family of curves $x^2 + y^2 = c$. (2018)

Q. 15. Find the general and singular solutions of $(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0$ (2018) – 2

Q. 16. In a radioactive decay, initially 50 mg the material is present and after two house, the material has lost 10% of its original mass. Find the mass at any time t and the half life of the material. (2018) – 3

Q. 17. Solve by Picards iteration method the following IVP $y' = x^2y,\ y(0) = 1$ (2019) – 3

Q. 18. The current I(t) at any time t, is given by the following differential equation – 3

$$E = RI + L\frac{dI}{dt}$$

Solve this equation for I(t) is terms of t. (2019)

Q. 19. Sole the clairaut's equation- – 33

$$y = px + \frac{a}{p}$$ (2019)

Chapter-3 Hom0genous linear Differential Equation

Q. 1. If $y = \phi_1(x)$ and $y = \phi_2(x)$ are solutions of $y'' + f(x)y' + g(x)y = 0$, where $f(x)$ and $g(x)$ are continous on an interval I. If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of this equation. Then show that their Wronskian $w(\phi_1, \phi_2)$ is either identically zero or nowwhere zero on I. (2013,2014,2017,2015,2016,2018) – 34

Q. 2. Solve - $(D^2 - 2D + 5)y = e^{2x}\sin x$. (2014) – 34

Q. 3. Using the method of variation of parameter obtain the general solution of the non-homogenous differential eqution - – 35

$$\frac{d^2y}{dx^2} + a^2y = \sin ax$$ (2015)

Q. 4. Define the inverse differential operator $\frac{1}{f(D)}$ where $f(D) =$ – 36

$D_n + a_1D^{n-1} + ...a_{n-1}D + a_n \left[D \equiv \frac{d}{dx}\right]$ and $a_1, a_2, a_3 ..... a_n$ are constants. (2015, 2017)

Q. 5. Establish that, – 37

$$\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{\phi(x)}\cdot\frac{x^p}{p!}$$

If $f(D) = (D - a)^p\phi(D)$. (2015)

Q. 6. Solve $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2\ (x > 0)$ (2016) – 37

– 38

Q. 7. Solve the equation - $y'' + 2y' + y = 4e^{-x}\ln x$. by using the method of variation of parameters. (2016, 2018) – 39

Q. 8. Solve $y'' + 4y = 4\tan 2x$. (2016) – 40

Q. 9. Obtain the general solutions of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$$ (2017) – 41

Q. 10. By the method of undetermined coefficient find the general solution of $y'' + a^2y = \cos ax$. (2018) – 41

Q. 11. Solve the differential equation -

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - \frac{3y}{x^2} = \ln x$$ (2018) – 42

Q. 12. Solve the non-homogenous linear differential equation. $y'' - 3y' + 2y = \cos(e^{-x})$. by the method of variation of parameter. (2019) – 43

Q. 13. Show that the particular solution of $y'' + 2ay' + b^2y = a\cos wx,\ (a, w > 0)$ is

given by $y = \frac{A\sin(\omega x - \alpha)}{\sqrt{(b^2 - \omega^2)^2 + 4\omega^2a^2}}$

where, $\alpha = \tan^{-1}\left(\frac{2a\omega}{b^2 - \omega^2}\right)\ (0 < \alpha < \pi)$. (2019)

Q. 14. Solve the following IVP. $y'' - 4y + 3y' = 4e^{3x},\ y(0) = -1,\ y'(0) = 3$ (2019) – 45

Chapter-4 Application of Oscillation

Q. 1. Discuss the motion of a body of mass m. Suspended by an elastic spring with spring constant k, when there is no resistive force and there as an additional sinuooidal imput force with angular frequency $\sqrt{\frac{k}{m}}$. (2014, 2015, 2016, 2017) – 46

Q. 2. Solve the system $3y_1' + 2y_1 + y_2' - 8y_2 = 5e^x$. – 47

$4y'_1 + 2y_1 + y'_2 - 8y_2 = 5e^x + 2x - 3$.
for $y_1(x)$ & $y_2(x)$. (2014)

Q. 3. Define scaler and vector field and also gradient of a scaler field u(x, y, z). (2014) – 48

Q. 4. Prove that grad f is along the normal vector to the surface f (x, y, z) = c. (2014, 2017) – 48

Q. 5. Prove the identities - – 49

$\text{div}\left(\vec{f}\times\vec{g}\right) = \vec{g}\,.\,\text{cur}\,\vec{f} - \vec{f}\,.\,\text{curl}\,\vec{g}$ (2014)

Q. 6. Prove that – 49

cur (f $\vec{v}$) = f curl v + (grad f) × v. (2014)

Q. 7. Solve the system of differential equations- – 50

$\frac{dI_1}{dt} + 3I_1 - 2I_2 = 6$, $\frac{dI_2}{dt} + 2I_2 - I_1 = 0$

and show that $\lim_{t\to\infty} I_1(t) = 3$ & $\lim_{t\to\infty} I_2(t) = \frac{3}{2}$. (2015)

Q. 8. If a, b are constant vectors, ω is a constant and r is a vector function of the scalar variable t given by – 51
r = cos ωt a + sin ωt b, show that

(i) $\frac{d^2r}{dt^2} + \omega^2 r = 0$ (ii) $r\times\frac{dr}{dt} = \omega a\times b$ (2015)

Q. 9. Prove that curl (ϕA) = grad (ϕ) × A + ϕ curl A. (2015) – 51

Q. 10. Solve the system of linear equations- – 52

$\frac{dy_1}{dx} = 2y_1 + 3y_2 + 2e^{2x}$. $\frac{dy_2}{dx} = y_1 + 4y_2 + 3e^{2x}$. (2016)

Q. 11. A particle moves along a curve whose parametric equations are - – 53
$x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$
where t is time, find the velocity & acceleration of particle in dir of (2i - 3j -k) at time, t = 1 second. (2016)

Q. 12. If is a constant vector, prove that – 54

$\text{Curl}\,\frac{\vec{a}\times\vec{r}}{r^3} = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\,.\,\vec{r})$ (2016)

Q. 13. Solve the system of differential equation – 55

$\frac{dx}{dt} + y = \text{cost} - \sin t + e^t$

$\frac{dy}{dt} + x = \text{cost}\,t + \sin t - e^t$ (2017)

Q. 14. Consider the oscillation of mass m suspended by a spring of – 56

neglegible mass & spring modulus k, in a medium which offered a resistance proportional to its velocity with damping constant C. Discuss the case when $C^2 < 4km$. (2018, 2019)

Q. 15. Solve the system of the differential equations for x. – 57

$\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x$ (2018)

Q. 16. If $\vec{r}$ is the position vector of a point P (x, y, z) with respect to origin O, then find the gradient of a scalar function f(r). – 57
(2018)

Q. 17. Prove that $\text{div}\{f(x).\hat{r}\} = \frac{1}{r^2}.\frac{d}{dr}\{r^2 f(r)\}$ (2018) – 58

Q. 18. For a vector function $\vec{a}$, show that . – 59

$\vec{a}\times\text{curl}\,\vec{a} = \text{grad}\left[\frac{a^2}{2}\right] - (\vec{a}-\nabla)\vec{a}$ Discuss if $\vec{a}$ is irrotational. (2018)

Q. 19. Show that a necessary & sufficient condition for the vector – 59
$\vec{a}$ (t) to have constant direction is $\vec{a}\times\frac{d\vec{a}}{dt} = 0$. (2019)

Q. 20. Show that the directional derivative of a scalar point function f at a point P(x, y, z) is direction of a unit vector $\vec{a}$ is given by - – 60

$\frac{df}{ds} = \nabla f.\hat{a}$ (2019)

Chapter-5 Vector Calculus

Q. 1. Evaluate the double integral- – 63

$\int_0^1 \int_{x^2}^{x} (1 - xy)\,dy\,dx$. (2014)

Q. 2. Evaluate the line integral. $\int_C [x^2y\,dx + (x - z)dy + xyzdz]$ – 63
where
where, C is the arc if the parabola $y = x^2$ in the plane z = 2 from (0, 0, 2) to (1, 1, 2). (2014)

Q. 3. State and prove Gauss's Divergence theorem. – 63
(2014, 2015, 2019)

Q. 4. State stokes theorem and prove it. (2014, 2016). – 65

Q. 5. Evaluate the line integral F. dr, where F = (x + y) + (2x - z) + (y + x) and C is the boundary of the triangle with vertices (2,0,0), (0, 3, 0) & (0, 0, 6) (2014) – 67

Q. 6. Define cylindrical polar coordinates (p, ϕ, z) of a point P. – 68
(2015)

Q. 7. Find the work done by the force $\bar{F}(x, y, z) = 4xy + 8y + 2$ – 69
in displacing a unit mass, along a circle once, $x^2 + y^2 = 4, z = 0$.
(2015)

Q. 8. State & proof Greens theorem. (2015, 2016) – 70

Q. 9. Define the spherical polar co-ordinates (r, θ, ϕ) and express – 70
the unit base vector $\hat{r}, \hat{\phi}, \hat{\theta}$ along the co-ordinate curve in terms of $\hat{i}, \hat{j}, \hat{k}$, the unit base vectors along the x, y, z axes respectively.
(2016)

Q. 10. Verify the stokes theorem for the function – 71
$\bar{F} = -y\hat{i} + x\hat{j} + z\hat{k}$, where S is the circular disc $x^2 + y^2 \le 1, z = 0$.
(2016)

Q. 11. Evaluate $\iint_S \bar{F}.\hat{n}\, dS$ where $\bar{F} = 4x\,\hat{i} - 3y^2\,\hat{j} + z^2\,\hat{k}$, and S is – 72
the entire surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ & $z = 3$.
(2016, 2019)

Q. 12. Establish the Green's theorem is a particular case of stokes – 73
theorem. If R is plane region enclosed by the Curve C, then using Green's theorem for plane regions, establish that the area A(R) of the region R can be expressed as a line integral. Hence obtain P(x, y) and Q(x, y) such that,

$$A(R) = \int_c (P\,dx + Q\,dy)$$
(2017)

Q. 13. Verify the stokes theorem for following details- – 74

$\bar{F} = -y^3\,\hat{i} + x^2\,\hat{j}$

S: circular disc $x^2 + y^2 \le 1, z = 0$
(2017)

Chapter - 1
Basic Concept

Q. 1 ***Solve xy dx - (x² + 2y²) dy = 0 by finding a suitable integrating factors.*** ***(2014)***

Ans. Given $xydx - (x^2 + 2y^2)\,dy = 0$

$\Rightarrow xydx = (x^2 + 2y^2)\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 + 2y^2}$$

function of x only

$$\Rightarrow \frac{dy}{dx} - \frac{x}{(x^2 + 2y^2)}.y = 0$$

Comparing with general equation of linear differential eqⁿ -

$$\frac{dy}{dx} + p(x).y = Q(x)$$

Thus given equation is not exact.
Now, we have,
$M = xy \quad N = -(x^2 + 2y^2)\,dy$

$$IF = \frac{1}{Mx + ny} = \frac{1}{x^2y - x^2y - 2y^3} = \frac{-1}{2y^3}$$

Multiplying by IF we get

$$\Rightarrow \frac{-1}{2y^3}(xy)dx - \left(\frac{-1}{2y^3}\right)(x^2 + 2y^2)dy = 0$$

$$\Rightarrow \frac{-x}{2y^2}dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0$$

in this equation -

$$\frac{-x}{2y^2} = M \qquad \& \left(\frac{x^2}{2y^3} + \frac{1}{y}\right) = N$$

$$\frac{\partial M}{\partial y} = \frac{+2x}{2y^3} = \frac{+x}{y^3}$$

$$\frac{\partial N}{\partial x} = \frac{2x}{2y^3} = \frac{x}{y^3}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ {Exact Equation}

Its solution is $\int xy\,dx + \int (-2y^2)dy = 0$

(y = cost)

$$\Rightarrow \frac{x^2 y}{2} + \left(\frac{-2y^3}{3}\right) = 0$$

$$\Rightarrow \frac{x^2 y}{2} - \frac{2y^3}{3} = 0$$

Q. 2. Solve $x^4y^1 = 1 - 2x^3y$, which type of equation is it? (2014)

Ans. Given equation is

$$\Rightarrow x^4 \frac{dy}{dx} = 1 - 2x^3 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^4} - \frac{2y}{x}, \quad \Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^4}$$

It is the type of linear differential Equation, where $P(x) = \frac{2}{x}$,

$$Q(x) = \frac{1}{x^4}$$

Now, IF = $e^{\int p dx}$

$$\text{IF} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Solution:

$$\Rightarrow \text{y. (IF)} = \int Q(IF)dx + c$$

$$\Rightarrow \text{y. } x^2 = \int \frac{1}{x^4}.x^2 dx + c$$

$$\Rightarrow x^2 y = \frac{-1}{x} + c$$

Q. 3. Find the general solution of equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$. (2014)

Ans. Given equation is -

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$

Separating the variables -

$e^y\,dy = (e^x + x^2)\,dx$

Integrating the above eqn is both sides we get

$$\Rightarrow \int e^y dy = \int (e^x + x^2)dx$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + c$$

$$\Rightarrow (e^y - e^x) = \frac{x^3}{3} + c$$

Q. 4. Find the differential equation of the family of ellips with foci at (-1,0) & (1,0). (2014)

Ans. Consider the family of all ellipses with foci at (-1,0) and (1,0). We know that the sum of the focal distances of a point on the ellipse is a constant and is equal to the length of its major axis, so that

$$\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 2a, a > 1$$

for any point (x, y) on the ellipse. Transposing one of the items to the right and squaring, we get

$$(x+1)^2 + y^2 = 4a^2 + (x-1)^2 + y^2 - 4a\sqrt{(x-1)^2 + y^2}$$

or $$a\sqrt{(x-1)^2 + y^2} = a^2 - x.$$

On squaring and simplifying, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 1} = 1, a > 1 \qquad \ldots(1)$$

which is a one-parameter family with parameter a. A differential equation of this family can be seen to be (after differentiating and some algebraic manipulations)

$$(x + yy')(xy' - y) = y'. \qquad \ldots(2)$$

Q. 5. Obtain the general solution of the differential equation -

$$dx + \left[4x - \frac{8}{x^3}\right] y\,dy = 0$$

(2015)

Ans.

$$\frac{dx}{\left[4x - \frac{8}{x^3}\right]} = -y dy$$

$$\Rightarrow \frac{x^3 dx}{4x^4 - 8} = y dy \quad \ldots(1)$$

Let $4x^4 - 8 = z$

$\Rightarrow 16x^3 dx = dz$

$$\Rightarrow x^3 dx = \frac{dz}{16}$$

Putting above in equation (1), we get

$$\frac{dz}{16z} = -y dy$$

Integrating above, we get

$$\Rightarrow \int \frac{dz}{16z} = \int -y dy$$

$$\Rightarrow \frac{1}{16}\log z = \frac{-y^2}{2} + c$$

$$\Rightarrow \frac{1}{16}\log(4x^4 - 8) = \frac{-y^2}{2} + c$$

Q. 6. *Solve the initial value problem -*

$$x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2$$

$y(\pi) = e^{\pi}\pi^3 + 2\pi^2$. ***(2015)***

Ans. Given that

$$\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x$$

By comparing, with partial differential equation -

$$\frac{dy}{dx} + p(x).y = Q(x)$$

where, $P(x) = \left(\frac{-3}{x}\right)$, $Q(x) = x^3(ex + \cos x) - 2x$

$$IF = e^{\int dx} = e^{\int \left(\frac{-3}{x}\right) dx} = e^{\ln x^{-3}} = \frac{1}{x^3}$$

Solution:

$$\Rightarrow y.\ IF = \int Q(x).IF\, dx + c$$

$$\Rightarrow y.\frac{1}{x^3} = \int (x^3(e^x + \cos x) - 2x)\frac{1}{x^3}.dx + c$$

$$\Rightarrow \frac{y}{x^3} = \int \left(e^x + \cos x - \frac{2}{x^2}\right) dx + c$$

$$\Rightarrow \frac{y}{x^3} = e^x + \sin x - \frac{1}{x} + c$$

$\Rightarrow y = x^3 e^x + x^3 \sin x - x^2 + Cx^3$.

at $x = \pi$,

$y(\pi) = e^{\pi} x^3 + \pi^3(0) - \pi^2 + C\pi^3$

$\Rightarrow e^{\pi}\pi^3 + 2\pi^2 = e^{\pi}\pi^3 - \pi^2 + C\pi^3$

$\Rightarrow 3\pi^2 = C\pi^3$

$$\Rightarrow c = \frac{3}{\pi}$$

$$\therefore y = x^3 e^x + x^3 \sin x - x^2 + + \frac{3x^3}{\pi}$$

Q. 7. *Define isoclines of a differential equation of the first order & explain its relation.* ***(2015, 2016, 2017, 2019)***

Ans. Let the given differential equation is

$y' = f(x, y)$

for solving this differential equation we find isoclines by putting $f(x, y) = k$ (constant).

For different values of K we find different isoclines then we draw linear element along these isoclines and this obtain the direction field.

Q. 8. *Find the isoclines of the differential equation $(2x + y)\,dx + (y - x)\,dy = 0$* ***(2015)***

Ans. $(2x + y)\,dx = (x - y)\,dy$.

$$\Rightarrow \frac{dy}{dx} = \frac{(2x + y)}{(x - y)}$$

Now, putting $\frac{dy}{dx} = k$

$$\Rightarrow \frac{(2x + y)}{(x - y)} = k.$$

$\Rightarrow 2x + y = kx - ky$.

$\Rightarrow (k+1)y = (k-2)x$

$\Rightarrow y = \frac{(k-2)}{(x+1)}x$

which is a straight line.

Q. 9. State an existence and uniqueness theorem for the initial value problem -

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y$ **(2016)**

Ans. Existence theorem -

The initial value problem

$$\frac{dy}{dx} = f(x,y),\ y(x) = y \qquad ...(1)$$

has at least one solution y (x) provided the function f (x, y) is continuous and bounded for all values of x, in a domain D and there exists positive constant M and K such that,

$|f(x, y)| \le M$ ...(2)

and satisfies the Lischitz condition.

$|f(x, y_2) - f(x, y_1)| \le k |y_2 - y_1|$

for all points in domain D.

Uniqueness Theorem -

This initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y$$

has a unique solution provided the function f (x, y) is continuous and bounded for all values of x in a domain D and these exists positive contents M and k such that

$|f(x, y)| \le M$

and satisfies the Lipschitz condition.

$|f(x, y_2) - f(x, y_1)| \le k |y_2 - y_1|$

for all points in domain D.

Q. 10. Find the solutions of the initial value problem:

$\frac{dy}{dx} = y^{2/3}$ **y (0) = 0** **(2016)**

Ans. Given, $\frac{dy}{dx} = y^{2/3}$

$\Rightarrow \frac{dy}{x^{2/3}} = dx$

$y^{2/3}\, dy = dx$

Integrating both sides, we get

$\Rightarrow \int y^{-2/3} dy = \int dx$

$\Rightarrow \frac{y^{-2/3+1}}{\frac{-2}{3}+1} = x + c$

$\Rightarrow 3y^{1/3} = x + c$

$\Rightarrow y^{1/3} = \frac{x+c}{3}$

$\Rightarrow 27\, y = (x + c)^3$

Now, $y(0) = 0$

$\Rightarrow 0 = (0 + C)^3$

$\Rightarrow C = 0$

$\therefore y = \frac{1}{27}(x)^3$

Q. 11. Solve $ydx + (2x - ye^y)\, dy = 0$ **(2016)**

Ans. $ydx = (ye^y - 2x)\, dy$

$\Rightarrow \frac{dy}{dx} = \left(e^y - \frac{2x}{y}\right)$

$\Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = e^y$

Comparing with standard form of L.D.E.

$\frac{dx}{dy} + P(y).x = Q(y)$

$IF = e^{\int P(y)dy} = e^{\int \frac{2}{y} dy} = e^{\ln y^2} = y^2$

Solution:

x. IF = $\int Q(y)\, IF\, dy$.

$\Rightarrow x.y^2 = \int e^y . y^2 dy$

$y^2 \rightarrow$ first function

$\Rightarrow xy^2 = y^2.e^y - \int 2y.e^y\,dy + c$

$\Rightarrow xy^2 = y^2e^y - 2ye^y + \int 2e^y\,dy + c$

$\Rightarrow xy^2 = y^2e^y - 2ye^y + 2e^y + C$

$\Rightarrow xy^2 = e^y(y^2 - 2y + 2) + C.$

Q. 12. Solve the differential euqation.

$$\left[x + y\sin^2\left(\frac{x}{y}\right)\right]dy - ydx = 0$$ **(2017)**

Ans. Given, $\left[x + y\sin^2\left(\frac{x}{y}\right)\right]dy = ydx$

$$\left(\frac{dx}{dy}\right)y = x + y\sin^2\left(\frac{x}{y}\right)$$

Put, x = vy.

differentiating both sides -

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow y\left(v + y\frac{dv}{dy}\right) = vy + y\sin^2 v$$

$$\Rightarrow y\frac{dv}{dy} = \sin^2 v.$$

$$\Rightarrow \frac{dy}{y} = (\text{cosec}^2\, v)\, dv$$

Integrating both sides -

$\Rightarrow \ln y = -\cot v + c$

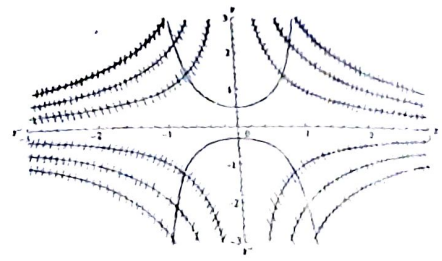
$$\Rightarrow \ln y = -\cot\left(\frac{x}{y}\right) + c$$

Q. 13. What do you mean by direction field in the process of solving $\frac{dy}{dx} = f(x, y)$ graphically? Also, by the method of isoclines, sketch solution curves of $\frac{dy}{dx} = xy, y(1) = -1$ **(2018)**

Ans. We consider, the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

Before discussing various standard methods utilized in solving the above equation, we shall give a useful graphical method for obtaining a rough sketch of the solution curves of equation (1) without actually solving the equation. We know that a solution of equation passing through a point (a, b) belonging to the domain of f, must have the slope f (a, b) there. This defines, locally, the direction of the solution curve passing through (a, b). We can thus speak of the direction field of the differential equation, and any solution of (1) must have a graph which at each point has the direction given by the above field. In order to be able to draw the solution curves we first draw some of the curves in the x -y plane along which f (x, y) is constant. These curves are called isoclines of (1). Along any such soclines given by



$$f(x, y) = \lambda, \qquad \ldots(2)$$

we can draw a number of parallel short line segments called linearl elements, with slope λ, which is the slope of the solution curves of (1) at any point of that isocline. This is so because at any point (a, b) on the isolcine (2), the slope of the solution curve passing through that point is f (a, b) by (1), and the slope of the lineal element at the point, which is λ, is also equal to f (a, b), because (a, b) is on the isocline (2).

Having drawn several such isoclines, with lineal elements lying close to each other, corresponding toneighbouring values of λ, we can draw approximate solution curves, joining contiguous lineal elements marked on neighbouring isoclines. The closer the isoclines, the better would be the approximation to the actual solution curves.

Q. 14. Find the general solution of $(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,ydy = 0$ **(2018)**

Ans. Let $M = (2x^2 + 3y^2 - 7)\,x$

$N = -(3x^2 + 2y^2 - 8)\,y$

$$\frac{\partial M}{\partial y} = 6xy \qquad \frac{\partial N}{\partial x} = -6xy$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore given eqⁿ is non exact D.E.

Now,

Let $t = x^2 + y^2$

then, $\frac{\partial t}{\partial x} = 2x \quad \& \frac{\partial t}{\partial y} = 2y$

$$\therefore \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{Nt_x - Mt_y} = \frac{12xy}{-(3x^2+2y^2-8)2xy-(2x^2+3y^2-7)2xy}$$

$$= \frac{6}{-5x^2-5y^2+15} = \frac{6}{5}\left[\frac{1}{3-t}\right]$$

$$\therefore \text{IF} = e^{\frac{6}{5}\int \frac{1}{3-t}dt} = e^{-\frac{6}{5}\ln(3-t)}$$

$$= \frac{1}{(3-t)^{6\backslash 5}} = \frac{1}{(3-x^2-y^2)^{6\backslash 5}}$$

Multiplying DE by IF we get

$$\frac{(2x^2+3y^2-7)xdx}{(3-x^2-y^2)^{6/5}} - \frac{(3x^2+2y^2-8)ydy}{(3-x^2-y^2)^{6/5}}$$

Now eqn (2) will be an exact differential equation -

Let $M_1 = \frac{(2x^2+3y^2-7)x}{(3-x^2-y^2)^{6/5}}$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{(3-x^2-y^2)^{6/5}.6xy - x(2x^2+3y^2-7)\frac{6}{5}(3-x^2-y^2)^{1/5}(-2y)}{(3-x^2-y^2)^{12/5}}$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{6xy\left[3-x^2-y^2+\frac{4}{5}x^2+\frac{6}{5}y^2-\frac{14}{5}\right]}{(3-x^2-y^2)^{12/5}}$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{6xy(1-x^2+y^2)}{5(3-x^2-y^2)^{11/5}}$$

Again let $N_1 = \frac{-(3x^2+2y^2-8)y}{(3-x^2-y^2)^{6/5}}$

$$\Rightarrow \frac{\partial N_1}{\partial y} = \frac{(-6xy)(3-x^2-y^2)^{6/5} + (3x^2+2y^2-8)y\frac{6}{5}(3-x^2-y^2)^{1/5}(-2x)}{(3-x^2-y^2)^{12/5}}$$

$$\Rightarrow \frac{\partial N_1}{\partial x} = \frac{-6xy\left[\frac{-1}{5}+\frac{x^2}{5}-\frac{y^2}{5}\right]}{(3-x^2-y^2)^{11/5}}$$

$$\Rightarrow \frac{\partial N_1}{\partial x} = \frac{6xy(1-x^2+y^2)}{5(3-x^2-y^2)^{11/5}}$$

Hence, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So, eqⁿ (2) is exact differential eqⁿ -

. 15. *State & Prove the necessary and sufficient condition for an equation M (x, y) dx + N (x, y)dy = 0 to be exact.* ***(2018)***

ns. **Definition :** The diffferential equation

$M(x, y)\,dx + N(x, y)\,dy = 0$ ...(1)

is said to be exact if the left hand side is an exact differential of some nction u(x, y), i.e.,

$du = Mdx + Ndy$ ...(2)

Since $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ ...(3)

thc equation (1) is exact if there exits some function u(x, y) such that

$\frac{\partial u}{\partial x} = M$ ...(4)

$\frac{\partial u}{\partial y} = N$ ...(5)

Then the exact equation (1) can be written as

$du = 0$

and integrating, we obtain its general solution as ...(6)

$u(x, y) = c$ ...(7)

(b) Necessary and sufficient condtion for an equation to be an exact uation:

We shall prove that if M (x, y) and N (x, y) are defined and continous d have continuous first partial derivatives in a region R in x - y plane, the the

equation (1) i.e.

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(i) Necessary part

If the equn. (1) is exact, then, from (4), (5), there exists some functi u (x, y) such that

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N$$

so that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$

Assuming that the mixed second partial derivatives of u exist and continous, we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

so that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

which is necessary condition for (1) to exact

(ii) Sufficient part:

If $\frac{\partial M}{\partial 6} = \frac{\partial N}{\partial x}$

then we show tht (1) is exact.

Let $\int M dx = V(x, y)$

then $M = \frac{\partial V}{\partial x}$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial y \partial x}$

or $\frac{\partial N}{\partial x} = \frac{\partial^2 V}{\partial y \partial x}$

Integrating partially w.r.t. x, we get

$$N = \frac{\partial V}{\partial y} + f(y)$$

$$\therefore\ M\,dx + N\,dy = \frac{\partial V}{\partial x}dx + \left\{\frac{\partial V}{\partial y} + f(y)\right\}dy$$

$$= = \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right) + f(y)dy$$

$$= dV(x, y) + d(f(y)\,dy)$$

$$= dV(x, y) + dF(y),\ F(y) = \int f(y)\,dy$$

$$= d\{V(x, y) + F(y)\}$$

$$= dW(x, y), \qquad W(x, y) = V(x, y) + F(y)$$

Thus, LHS of (1) is an exact differential of some function W(x, y). Hence eqn (1) is exact.

Q. 16. *Find general solution of differential eqⁿ.* Bernoulli's

$$\frac{dx}{dy} + P(x)y = Q(x)y^n$$

(2018)

Ans. A generalized form of the linear equation that can be solved, is the Bernouli's equation.

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \qquad ...(1)$$

This equation can be reduced to a linear form by using the transformation

$$v = y^{n+1}\ (n \neq 1)$$

The case n = 1 reduces (1) to a homogeneous linear equation. From (1) dividing throughout by y^n, one gets

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

or $$\frac{1}{(-n+1)}\frac{dv}{dx} + P(x)v = Q(x)$$

that is $$\frac{dv}{dx} + (-n+1)\,P(x)\,v = (-n+1)\,Q(x) \qquad ...(2)$$

which is a linear equation in v and thus can be solved by the method given for linear equations.

$$v(x) = e^{(n-1)\int P(x)dx}\left[(1-n)\int Q(x)^{(1-n)}\left[(1-n)\int\right]P(x)dx\,dx + c\right]$$

or $$y^{(1-n)} = e^{(n-1)\int P(x)dx}\left[(1-n)\int Q(x)^{(1-n)}\left[(1-n)\int\right]P(x)dx\,dx + c\right] \qquad ...(3)$$

which yields the general solution of equation (1).

Q. 17. Solve, $\frac{dy}{dx} = \frac{1}{yx^3 - x}$ **(2018)**

Ans. $\frac{dx}{dy} = yx^3 - x$ ...(1)

$\frac{dx}{dy} + x = yx^3$ ...(2)

dividing eq (2) by x^3.

$\frac{1}{x^3}\frac{dx}{dy} + \frac{1}{x^2} = y$ ...(3)

Put $\frac{1}{x^2} = v$

$\frac{dv}{dx} = \frac{-2}{x^3}\frac{dx}{dy}$

$\frac{-1}{2}\frac{dv}{dy} = \frac{1}{x^3}\left(\frac{dx}{dy}\right)$

Now, eq (3), becomes -

$\frac{-1}{2}\frac{dv}{dy} + v = y$

$\frac{dv}{dy} - 2v = -2y$ ...(4)

This is linear differential equation in v,

$IF = e^{-\int 2dy} = e^{-2y}$

Solution -

$ve^{-2y} = -2\int y.e^{-2y}dy$

$ve^{-2y} = -2\left[\frac{y}{2}e^{-2y} - \frac{1}{4}e^{-2y}\right] + c$

$v = y + \frac{1}{2} + Ce^{+2y}$ ...(5)

Now putting $v = \frac{1}{x^2}$, we get

$\frac{1}{x^2} = \frac{2y + Ce^{2y} + 1}{2}$

$(2y + ce^{2y} + 1)x^2 = 2.$

Q. 18. Solve the differential equation - $xy' - (\ln x)y^2 + y = 0, x > 0$ **(2009, 2019)**

Ans. Given differential equation is

$xy' - (\ln x)y^2 + y = 0$

or $y' + \frac{y}{x} = \frac{\log x}{x}.y^2$

dividing both sides by y^2

$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{\log x}{x}$

Let $y^{-1} = v \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

So, equation reduces to

$\Rightarrow \frac{-dv}{dx} + \frac{1}{x}v = \frac{\log x}{x} \Rightarrow \frac{dv}{dx} - \frac{1}{x}v = \frac{-\log x}{x}$

which is linear differential equation,

Now, Integrating factor is -

$\Rightarrow IF = e^{\int \frac{-1}{x}dx} = \frac{1}{x}$

Solution:

$\Rightarrow v.\frac{1}{x} = \int \frac{-\log x}{x^2}dx$

$\Rightarrow \frac{v}{x} = \frac{-1}{x}[1 + \log x] + c$

$\Rightarrow \frac{1}{y} = -[1 + \log x] + cx$

Q. 19. The growth of a population is described by the differential equation $\frac{dN}{dt} = aN - bN^2$ Solve this equation subject to the initial condition that N = N0, when t = 0. **(2019)**

Ans. The equation

$$\frac{dN}{dt} - aN = -bN^2 \text{ is bernoullis equation}$$

$$\Rightarrow \frac{1}{N^2}\frac{dN}{dt} - \frac{a}{N} = b \qquad \text{...(I)}$$

Put $z = N^{-1}$

$$\frac{dz}{dt} = \frac{-1}{N^2}\frac{dN}{dt}$$

Substituting these values in eq (1)

$$\frac{-dz}{dt} - az = b$$

$$\frac{dz}{dt} + az = -b \qquad \text{...(ii)}$$

Equation (ii) is linear equation

$IF = e^{\int adt} = e^{at}$

Solution:

$Z.e^{at} = \int (-b)\, e^{at}\, dt$

$z.e^{at} = -b\, e^{at} + c \qquad \text{...(iii)}$

Put $z = \frac{1}{N}$ in equation (iii)

Bernoulli's Solvable for y, z/y^n

$$\frac{1}{N}e^{at} = -be^{at} + c \qquad \text{...(iv)}$$

At t = 0, N = No

$$\Rightarrow \frac{1}{N_0}e^{(0)} = be^0 + c$$

$$\Rightarrow \frac{1}{N_0} = -b + c$$

$$\Rightarrow C = \frac{1}{N_0} + b$$

Chapter - 2
Machanical Application

Q. 1. Suppose that in a chemical reaction, the rate of conversion of one of the substances is proportional to the square of the of the unconverted substance at that time. If m_0 was its mass of the original mass, then find the mass of it at time t. (2014)

Ans. Let the differential equation is -

$$\Rightarrow \frac{dm}{dt} \alpha\, m^2$$

$$\Rightarrow \frac{dm}{dt} = km^2$$

where, k is constant of proportionality

$$\Rightarrow \frac{dm}{m^2} = kdt \qquad \text{...(1)}$$

Integrating equation (1), we get

$$\Rightarrow \int \frac{dm}{m^2} = \int kdt$$

$$\Rightarrow \frac{-1}{m} = kt + c \qquad \text{...(2)}$$

when $t = 0, m = m_0$

$$\frac{-1}{m_0} = 0 + c$$

$$\Rightarrow C = \frac{-1}{m_0}$$

Now, from equation (2)

$$\Rightarrow \frac{-1}{m} = kt - \frac{1}{m_0}$$

$$\Rightarrow \frac{1}{m_0} - \frac{1}{m} = kt$$

Q. 2. Solve $y = x^2 + 2px + \frac{p^2}{2}$ to find a general and a singular solution. (2014, 2019)

Ans. Given equation is,

$$y = x^2 + 2px + \frac{p^2}{2} \quad ...(1)$$

differentiating w.r.t. x

$$\Rightarrow p = 2x + 2p + 2x.\frac{dp}{dx} + p.\frac{dp}{dx}$$

$$\Rightarrow p = 2x + 2p + (2x + p)\frac{dp}{dx}$$

$$\Rightarrow (2x + p) + (2x + p)\frac{dp}{dx} = 0$$

$$\Rightarrow (2x + p)\left(\frac{dp}{dx} + 1\right) = 0 \quad ...(2)$$

First factor will give singular solution -

$\Rightarrow 2x + p = 0$

$$\Rightarrow 2x + \frac{dy}{dx} = 0$$

$\Rightarrow dy = -2x\,dx$

Integrating both sides

$\Rightarrow y = -x^2 + C,$

$\Rightarrow y + x^2 = C_1$

Second factor will give general solution -

$$\Rightarrow \frac{dp}{dx} = -1$$

$\Rightarrow dp = -dx$

Integrating both sides,

$\Rightarrow P = -x + C_2$

$$\Rightarrow \frac{dy}{dx} = -x + c_2 \quad ...(iii)$$

Again integrating eq^n (iii)

$$\Rightarrow y = -\frac{x^2}{2} + c_2 x + c_3$$

$$\Rightarrow y = c_2 x + c_3 - \frac{x^2}{2}$$

Q. 3. *Explain how to solve a first order differential equation of the form*

$$y = (x, p), P = \frac{dy}{dx}$$ ***(2015)***

Ans. Suppose the given differential equation is solvable for y. Then it can be put in the form ...(1)

$y = f(x, p)$

Differentiating (1) w.r.t x and denoting dy\dx by p, we obtain

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \quad ...(2)$$

which is differential equation in two variables x and p. Suppose it is possible to solve the differential equation (2). Let its solution be

$F(x, p, c) = 0$ ...(3)

where c is the arbitrary constant.

Eliminating p between (1) and (3), we get the required solution of (1) in the form

$\psi(x, y, c) = 0$

If is not easily practicable to eliminate p between (1) and (3), we may solve (1) and (3) to get x and y in terms of p and c in the form

$x = f_1(p, c).\ y = f_2(p, c)$ ...(4)

Special case: Equations that do not contain x : In this case the equation has the form $f(y, p) = 0$. If it is solvable for p, it will give

$p = \phi(y)$ i.e., $dy/dx = \phi(y)$.

which can be easily solved by separating the variables.

If it is solvable for y, it will give $y = \psi(p)$, which can be solved by the method just explain in article 3.

Q. 4. *Obtain the solution of the differential equation $xyp^2 + (x^2 + y^2)p + xy = 0$* ***(2015)***

Ans. Given eq^n is -

$\Rightarrow xyp^2 + (x^2 + y^2)p^2 + xy = 0$

$\Rightarrow (xyp^2 + p^2y^2) + (x^2p^2 + xy) = 0$

$\Rightarrow (xp + y)(yp + x) = 0$ ...(1)

The above equation has two factors.

From factor first -

$\Rightarrow xp + y = 0$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

WINSOME SERIES

ODE

Paper -3

उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े उसे त्याग दो।



KEY POINT AU



KEY POINT AU

$$\Rightarrow \frac{dy}{y} = \frac{-dx}{x}$$

Integrating both sides, we get

$\Rightarrow \ln y = -\ln x + \ln c_1$

$\Rightarrow \ln y + \ln x = \ln c_1$

$\Rightarrow \ln (y.x) = \ln c_1$

$\Rightarrow xy = C_1 \qquad \Rightarrow xy - C_1 = 0$

From factors second

$\Rightarrow yp + x = 0$

$\Rightarrow y.\frac{dy}{dx} + x = 0$

$\Rightarrow ydy = -xdx$

Integrating

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c$

$\Rightarrow y^2 + x^2 - 2C = 0$

$\Rightarrow y^2 + x^2 - C_2 = 0 \qquad \{\because C_2 = 2c\}$

The general equation is given by -

$\Rightarrow (xy - c_1)(y^2 + x^2 - c_2) = 0.$

Q. 5. ***A certain amount of a radioactive substance was observed to be of mass at time t, is governed by differential equation.***

$$\frac{dm}{dt} = -km$$

If at t = 0, m = m_0 then find such value of t, when $m = \frac{m_0}{2}$

(2015)

Ans. The given equation is

$$\Rightarrow \frac{dm}{dt} = -km$$

Integrating this, we get

$$\Rightarrow \int \frac{dm}{m} = -\int k dt$$

$\Rightarrow \ln m = -kt + c$ ...(1)

at $t = 0, m = m_0$.

$\therefore c = \ln m_0$.

Now, from equation (1),

$\Rightarrow \ln m = \ln m_0 - kt$

$\Rightarrow \ln m - \ln m_0 = -kt$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = -kt$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = +kt \qquad ...(2)$$

we have given that $m = \frac{m_0}{2}$

So, when

$$\Rightarrow t = \frac{1}{k}\ln(2)$$

Q. 6. ***Define the orthogonal trajections of a family of curve.*** ***(2016,2018)***

Ans. Consider a one-parameter family of curves

$F(x, y, c) = 0$, c = parameter ...(1)

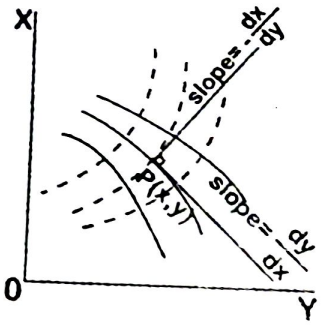
Differentiate it w.r.t. x . We get

$$\phi\left(x, y, \frac{dy}{dx}, c\right) = 0$$

Eliminate parameter c between above two equations. We obtain the differential equation of the family (1) as

$$f\left(x, y\frac{dy}{dx}\right) = 0 \qquad ...(2)$$

Now, at any point (x,y) on any curve (solid curve) of the family (1) or (2), the curve has the slope dy\dx, and the orthogonal trajectory (dotted curve) through the same point has its slope as - dx/dy, so that the product of two slopes is -1. So, if x, y, dy\dx characterize a curve of the family (2) then, x, y, -dx/dt characterize an orthogonal trajectory through the same point. Hence, the differential equation of the family of orthogonal trajectoris is given by replacing dy\dx by - dx/dy in (2) i.e.,



$$f\left(x, y, \frac{-dx}{dy}\right) = 0 \qquad ...(3)$$

Solving (3), we get the family of orthogonal trajectories of the fami of curves (1) as

$\psi(x, y, \lambda) = 0, \lambda =$ parameter ...(4

Q. 7. Find the general solution of the differential equation

$$xyp^2 + (x+y)p + 1 = 0, \left[p \equiv \frac{dy}{dx}\right]$$

and obtain a particular solution valid for all x and passin through the point $\left[\frac{-1}{2}, 2\right]$. **(2016)**

Ans. The given differential equation is -

$\Rightarrow xyp^2 + (x+y)p + 1 = 0$

$\Rightarrow xyp^2 + xp + yp + 1 = 0$

$\Rightarrow xp(yp+1) + 1.(yp+1) = 0$

$\Rightarrow (xp+1)(yp+1) = 0$

If, $xp + 1 = 0$

$$\Rightarrow p = \frac{-1}{x} \frac{\alpha+\beta}{2} \Rightarrow \frac{dy}{dx} = \frac{-1}{x}$$

$$\Rightarrow dy = \frac{-dx}{x}$$

$\Rightarrow y = -\ln x + \ln c$

$$\Rightarrow y = \ln\left(\frac{c}{x}\right)$$

If $yp + 1 = 0$

$$\Rightarrow p = \frac{-1}{y}, \quad \Rightarrow \frac{dy}{dx} = \frac{-1}{y}$$

$\Rightarrow ydy = -dx$

$$\Rightarrow \frac{-y^2}{2} = x + c$$

Hence, solution is -

$$\left[y - \ln\left(\frac{c}{x}\right)\right]\left[x + c + \frac{y^2}{2}\right] = 0$$

Q. 8. Apply picards method to find first three approximate solutions to the initial value problem

$$\frac{dy}{dx} = 2y - 2x^2 - 3, y(0) = 2$$ **(2009, 2016)**

Ans. Given

$$\frac{dy}{dx} = 2y - 2x^2 - 3.$$

$f(x, y) = 2y - 2x^2 - 3, x_0 = 0, y_0 = 2$

Now, n_m approximate solution is given by -

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$

For $n = 1$

$$y_1(x) = 2 + \int_0^x f(t, 2)\,dt$$

$$y_1(x) = 2 + \int_0^x (4 - 2t^2 - 3)\,dt$$

$$y_1(x) = 2 + x - \frac{2}{3}x^3$$

For $n = 2$

$$y_2(x) = 2 + \int_0^x f\left(2t + t - \frac{2}{3}t^3\right)$$

$$y_2(x) = 2 + \int_0^x \left(4 + 2t - \frac{4t^3}{3} - \frac{2}{3}t^3\right)dt$$

$$y_2(x) = 2 + x + x^2 - \frac{2}{3}x^3 - \frac{1}{3}x^4$$

Similarly,

for $n = 3$

$$y_1(n) = 2 + x + x^2 - \frac{1}{3}x^4 - \frac{2}{15}x^5$$

Q. 9. Find a general solution of equation.

$$\frac{dy}{dx} = 2y - 2x^2 - 3;\ y(0) = 2.$$ **(2009, 2016)**

Ans. Given equation

$$y = px + \sqrt{16 + 9p^2} \qquad \left[p = \frac{dy}{dx} \right]$$

is of the form

$y = px + f(p)$

hence, it is Clairaut's equation

General solution is given by

$$y = cx + \sqrt{16 + 9c^2}$$

Q. 10. *The fish population in a pond is changing and its growth rate is found to satisfy the differential equation.*

$$\frac{dx}{dt} = \alpha x(k - x)$$

where, a, k are positive constant and x(t) represents the fish population at any time t. If it is given that initially the fish population was x_0, then obtain the fish population x (t) at any time t and established that ultimately, as $t \to \infty$ the fish population $x(t) \to k$. ***(2017)***

Ans. The equation

$$\Rightarrow \frac{dx}{dt} = \alpha x(k - x)$$

$$\Rightarrow \frac{dx}{dt} - \alpha kx = -\alpha x^2$$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dt} - \frac{\alpha k}{x} = -\alpha \qquad \ldots(1)$$

Substituting $z = x^{-1}$

$$\frac{dz}{dx} = \frac{-1}{x}\frac{dx}{dt}$$

From equation (1)

$$\Rightarrow \frac{-dz}{dt} - \alpha kz = -\alpha$$

$$\Rightarrow \frac{dz}{dt} + (\alpha k)z = \alpha \qquad \ldots(2)$$

This is linear differential equation

$\therefore$ IF $= e^{\int \alpha k\, dt} = e^{\alpha kt}$

Solution:

$$\Rightarrow z e^{\alpha kt} = \alpha \int e^{\alpha kt}\, dt + c$$

$$\Rightarrow z e^{\alpha kt} = \frac{e^{\alpha kt}}{k} + c$$

$$\Rightarrow \frac{1}{x} = \frac{1}{x_0} + ce^{-\alpha kt}$$

at $t = 0$, $x(t) = x_0 \Rightarrow c = \frac{1}{x(0)} - \frac{1}{x_0}$

$$\therefore \frac{1}{x} = \frac{1}{x_0} + \left(\frac{1}{x(0)} - \frac{1}{x_0} \right) e^{-\alpha kt}$$

$$\Rightarrow \frac{x_0}{x} = 1 + \left[\frac{x_0}{x(0)} - 1 \right] e^{-\alpha kt}$$

$$\Rightarrow x = \frac{x_0}{1 + \left[\frac{x_0}{x(0)} - 1 \right] e^{-\alpha kt}} \qquad \ldots(3)$$

Since $x(0) < x_0$.

$\frac{x_0}{x(0)} - 1 \left(\frac{x_0}{x(0)} - 1 \right) e^{-\alpha kt}$ is positive for all t & tends tozero as $t \to \infty$

Hence from (3), $x < x_0$ for all t and $x \to x_0$ as $t \to \infty$. Therefore x never exceeds x_0.

Q. 11. *Prove that the initial value problem*

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

is equivalent to the integral equation problem

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$ ***(2017)***

Ans. $$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad \ldots(1)$$

where f (x, y) is a continuous function defined in some neighbourhood of the point (x_0, y_0).

Integrating, we can replace the I.V.P. (1) by the equivalent integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f\{t, y(t)\}dt \quad ...(2)$$

In fact (1) and (2) are equivalent, because if y (x) is a solution of (1), then y (x) is continous and f {x, y(x)} is continous. Integrating y'(x) = f {x, y (x)} from x_0 to x, we get $y(x)\Big|_{x_0}^{x} = \int_{x_0}^{x} f\{t, y(x)\}dx$ or y(x) – y(x_0) + $\int_{x_0}^{x}$ f {t, y(t)} dt or y(x) = y0 + $\int_{x_0}^{x}$ f{t, y(t)} dt by using the condition y(x_0) = y_0. Thus, we get (2) Conversely, if y(x) is a continuous solution of (2), then at x = x_0, integral vanishes and we get y(x_0) = y_0. Further, differentiating (2), we get y'(x) f {x, y(x)}. Thus from (2) we get (1)

Now, we find approximations to the solution y(x) of (2) by a process of iteration (successive approximations). A rough approximation of y(x) is y_0. Inser this approximation on the R.H.S of (2) and obtain a better approximation.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

Next, use $y_1(x)$ to obtain ever approximation:

$$y_2(x) = y_0 + \int_{x_0}^{x} f\{t, y_1(t)\}\, dt$$

and so on. The n^{th} step of the process, gives

$$y_n(x) = y_0 + \int_{x_0}^{x} f\{t, y_{n-1}(t)\}\, dt \quad ...(3)$$

In this manner, we get a sequence of approximations

$y_1(x), y_2(x), \ldots\ldots y_n(x), \ldots\ldots$

which under the conditions of existence and uniquences theorem converges to the solution y(x) of the problem (1).

Q. 12. *Obtain Picards three Interates $y_1(x)$, $y_2(x)$, $y_3(x)$ for the initial value problem,*

$$\frac{dy}{dx} = x + 2y, y(0) = 0 \quad \textbf{(2017)}$$

Ans. The equivalent integral equation to given problem is

$$y(x) = 1 + \int_{x_0}^{x} \{t + y(t)\}\, dt$$

and the n^{th} approximation is

$$y_n(x) = 1 + \int_{x_0}^{x} \{t + y_{n-1}(t)\}\, dt$$

$$y_1(x) = 1 + \int_{x_0}^{x} \{t + 1\}\, dt = 1 + x + \frac{x^2}{2!} \quad (1)$$

$$y_2(x) = 1 + \int_{x_0}^{x} \left\{t + 1 + t + \frac{t^2}{2}\right\} dt = 1 + x + 2\frac{x^2}{2!} + \frac{x^3}{3!} \quad (2)$$

$$y_3(x) = 1 + \int_{x_0}^{x} \left\{t + 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right\} dt = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \frac{x^4}{4!} \quad (3)$$

and in general, we get

$$y_n(x) = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!} \quad ...(4)$$

which convergest to

$$y(x) = 1 + x + 2(e^x - x - 1) + 0 = 2e^x - x - 1 \quad ...(5)$$

Now, we find exact solution of the given problem.

$$\frac{dy}{dx} = x + y, y(0) = 1$$

Put x + y = v, then $\frac{dv}{dx} = 1 + v$ or $\frac{dv}{1+v} = dx$

Integrating, we get ln (1 + v) = x + ln c

or $1 + v = ce^x$ or $x + y + 1 = ce^x$

$\therefore$ y(0) = 1 $\Rightarrow$ 0 1 + 1 = c.1 $\Rightarrow$ c = 2

$\therefore$ exact y = 2x - x - 1 ...(6)

or $$y = 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - x - 1$$

$$= 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) \quad (7)$$

We observe that first four terms of $y_3(x)$ as given by (3) are the same as in the series (7) of the exact solution. Further, the sequence $\{y_0(x)\}$ converges to (5) which is the exact tothe given problem as given by (6).

Q. 13. *Apply Picard's interation method to obtain three iternates fo IVP*

y' = 2x(1 + y), y (0) = 0 **(2018)**

Ans. Given problem is

$$\frac{dy}{dx} = 2x(1 + y) \text{ when } x = 0, y = 0 \quad ...(1)$$

We known that n[th] appromiation y_n of initial value problem

$$\frac{dy}{dx} = f(x, y), \text{ where } y = y_0 \text{ when } x = x_0 \quad ...(2)$$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \quad ...(3)$$

Comparing (1) & (2), we get

$$f(x, y) = 2x(1 + y),\ x_0 = 0,\ y_0 = 0 \quad ...(1)$$

Therefore from equation (3),

$$y_n = \int_0^x 2x(1 + y_{n-1})\, dx \quad ...(5)$$

First approximation -

Putting n = 1 is equation (5), we get

$$y_1 = \int_0^x 2x(1 + 0)\, dx = x^2$$

Second approximation -

Putting n = 2 in equation (5), we get

$$y_2 = \int_0^x 2x(1 + y_1)\, dx = \int_0^x 2x(1 + x^2)\, dx$$

$$y_2 = x^2 + \frac{1}{2!}(x^2)^2$$

This approximation -

Putting n = 3 in equation (5), we get

$$y_3 = \int_0^x 2x(1 + y_2)\, dx = \int_0^x 2x\left(1 + x^2 + \frac{x^4}{2}\right) dx$$

$$y_3 = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6$$

$$y_3 = x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!}$$

Now,

n[th] approximation -

$$y_n = \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots\ldots + \frac{(x^2)^n}{n!}$$

Q. 14. *Find the orthogonal trajectories of the family of curves $x^2 - y^2 = c$.* (2018)

Ans. Given curve -

$$x^2 - y^2 = c \quad ...(1)$$

Differentiating equation (2), with respect to x, we get

$$2x - \frac{dy}{dx} 2y = 0 \quad \text{or} \quad x - y\frac{dy}{dx} = 0 \quad ...(2)$$

For orthogonal trajectory put

$$\frac{dy}{dx} = \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-dx}{dy}$$

in equation (2), we get

$$x + y\frac{dx}{dy} = 0 \quad \text{or} \quad \frac{dy}{y} = \frac{-dx}{x}$$

Hence, Integrating both sides

$\Rightarrow \log y = -\log x + c$

$\Rightarrow \log y = \log \frac{c}{x}$

$\Rightarrow xy = c.$

Q. 15. *Find the general and singular solutions of $(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0$ (2018)*

Ans. Given equations is -

$$(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0 \quad ...(1)$$

$\Rightarrow y^2 - 2xyp + p^2x^2 = p^2 + 1$

$\Rightarrow (y - px)^2 = p^2 + 1$

$$\Rightarrow y = px \pm \sqrt{p^2 + 1} \quad ...(2)$$

which is clairauts form.

Hence general solution of equation (1), is given as -

$$(x^2 - 1)c^2 - 2xyc + y^2 - 1 = 0 \quad ...(3)$$

Now, the required singular solution will be included in the envelope of this equation. Since equation (3) is a quadratic equation in c, its envelop is $(B^2 - 4AC = 0)$

$\Rightarrow 4x^2y^2 - 4(x^2 - 1)(y^2 - 1) = 0$

$\Rightarrow x^2 + y^2 = 1 \quad ...(4)$

Differential equation (4) with respect to x, this gives

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Using it in the given eq(2), we have

$$\Rightarrow px \pm \sqrt{p^2+1} = \frac{-x^2}{y} \pm \sqrt{\frac{x^2}{y^2}+1}$$

$$\Rightarrow \frac{-x^2 \pm \sqrt{x^2+y^2}}{y}, \qquad \Rightarrow \frac{-x^2 \pm 1}{y} = y$$

Thus equation (4) satisfies the given differential equation and hence equation (3) is the required singular solution.

Q. 16. ***In a radioactive decay, initially 50 mg the material is present and after two house, the material has lost 10% of its original mass. Find the mass at any time t and the half life of the material.*** ***(2018)***

Ans. Let N be the amount of the material present at time t, then dN\dt = kN. Solving its, we get

$N = ce^{kt}$

At $t = 0, N = 50 \Rightarrow c = 50 \Rightarrow N = 50\, e^{kt}$

At $t = 2, N = 45 \Rightarrow 45 = 50\, e^{2k} \Rightarrow k = -\frac{1}{2} \text{In } \frac{9}{10} = .053$

$\therefore$ The mass at time t : $N = 50\, e^{.053t}$

If $t_{1/2}$ be the half life, then at $t = t_{1/2}$, $N = 25$

$\therefore 25 = 50\, e^{-0.053(t_{1/2})} = \frac{1}{.053} \text{In } 2 = \text{hours}$

Q. 17. ***Solve by Picards iteration method the following IVP $y' = x^2y$, $y(0) = 1$*** ***(2019)***

Ans. The integral equation is -

$y(x) = 1 + \int_0^x t^2 y(t)\, dt$

Picards iterations are -

$y_0(x) = 1$

$$y_1(x) = 1 + \int_0^x t^2.1.\, dt = 1 + \frac{x^3}{3}$$

$$y_2(x) = 1 + \int_0^x t^2 \left[1 + \frac{t^3}{3}\right] dt$$

$$y_2(x) = 1 + \frac{x^3}{3} + \frac{1}{2!}\left[\frac{x^3}{3}\right]^2$$

$$y_3(x) = 1 + \int_0^x t^2 \left[1 + \frac{x^3}{3} + \frac{1}{2!}\left[\frac{x^3}{3}\right]^2\right] dt$$

$$y_3(x) = 1 + \frac{x^3}{3} + \frac{1}{2!}\left[\frac{x^3}{3}\right]^2 + \frac{2}{3!}\left[\frac{x^3}{3}\right]^3$$

Now $y_n(x) = 1 + \int_0^x t^2 y_{n-1}(t).\, dt$

$$y_n(x) = 1 + \frac{x^3}{3} + \frac{1}{2!}\left[\frac{x^3}{3}\right]^2 + \ldots\ldots \frac{1}{n!}\left[\frac{x^3}{3}\right]^2$$

Q. 18. ***The current I(t) at any time t, is given by the following differential equation***

$$\mathbf{E = RI + L\frac{dI}{dt}}$$

Solve this equation for I(t) is terms of t. ***(2019)***

Ans. The current I(t), at any time t, is given by the differential equation

$$E = Rt + L\frac{dt}{dt}$$

or $$\frac{dt}{dt} + \left(\frac{R}{L}\right) I = \frac{E(t)}{L}$$

The above is a first order linear differential equation in I(t). The general solution of this differential

$$I(t) = e^{\frac{R}{L}t}\left[\int \frac{E(t)}{L} e^{\frac{Ru}{L}}\, dt + c\right] \qquad \ldots(2)$$

where the constant c is to be determined using the initial conditions. If

$I(t_0) = I_0$

then from (2)

$$I_0 = e^{\frac{Rt_0}{L}}\left[\int \frac{E(t)}{L} e^{\frac{Ru}{L}}\, dt + c\right]_{t=t_0} \qquad \ldots(3)$$

From (2) we get (3)

$$I(t)e^{\frac{Rt}{L}} - I_0 e^{\frac{Rt_0}{L}} = \int \frac{E(t)}{L} e^{\frac{Rt}{L}}\,dt - \left[\int \frac{E(t)}{L} e^{\frac{Rt}{L}}\,dt\right]_{u=t_0}$$

or $$I(t)e^{\frac{Rt}{L}} = I_0 e^{\frac{Rt_0}{L}} + \int_{t_0}^{t} \frac{E(u)}{L} e^{\frac{Ru}{L}}\,du$$

or $$I(t) = I_0 e^{\frac{R(t-t_0)}{L}} + e^{\frac{Ru}{L}} \int_{t_0}^{t} \frac{E(u)}{L} e^{\frac{Ru}{L}}\,du \quad ...(4)$$

We shall analyze the solution (4) in the cses of a constant electromotive force $E(t) = E_0$ (D.C) and a periodic e.m.f (A.C).

Case I. $E(t) = E_0$, a constant

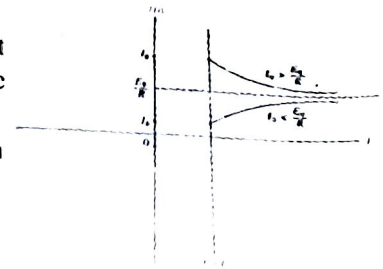
In this case (4) reduces to

$$I(t) = I_0 e^{\frac{R(t-t_0)}{L}} + \frac{E_0}{L} e^{\frac{-Rt}{L}} \left[\frac{L}{R} e^{\frac{Rt}{L}}\right]_{u=t_0}^{u=t}$$

$$= I_0 e^{\frac{R(t-t_0)}{L}} + \frac{E_0}{R}\left[1 - e^{\frac{R(t-t_0)}{L}}\right]$$

$$= \frac{E_0}{R} + \left[I_0 - \frac{E_0}{R}\right] e^{\frac{R(t-t_0)}{L}}$$

We note that as $t \to \infty$, the current tends to its stablized value, which would be its value $\frac{E_0}{R}$, if no inductor were present in the circuit.



Case II. $E(t) = E_0 \sin \omega t$

In this case the relation (4) assumes the form

$$I(t) = I_0 e^{\frac{-R(t-t_0)}{L}} + \frac{E_0}{L} e^{\frac{-Rt}{L}} \int_{t_0}^{t} e^{\frac{Rt}{L}} \sin(\omega t)\,dt$$

$$= I_0 e^{\frac{-R(t-t_0)}{L}} + \frac{E_0}{L} e^{\frac{-Rt}{L}} \cdot \frac{1}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \left[e^{\frac{Rt}{L}} \sin\left(\omega t - \phi \tan^{-1}\frac{\omega L}{R}\right)\right]$$

$$= I_0 e^{\frac{-R(t-t_0)}{L}} + \frac{E_0 e^{\frac{-Rt}{L}}}{\sqrt{R^2 + \omega^2 L^2}} \left[e^{\frac{Rt}{L}} \sin(\omega t - \phi) - e^{\frac{-Rt_0}{L}} \sin(\omega t_0 - \phi)\right]$$

$$= \frac{E_0 \sin(\omega t - \phi)}{\sqrt{R^2 + \omega^2 L^2}} + e^{\frac{-R(t-t_0)}{L}} \left[t_0 - \frac{E_0 \sin(\omega t_0 - \phi)}{\sqrt{R^2 + \omega^2 L^2}}\right]$$

where $\tan\phi = \frac{\omega L}{R}$

We observe that the solution (5) consists of two terms, one denoting its steady state and the other its transient behaviour. As t increases, the second term decreases exponentially and the first terms indicates that eventually the current would oscillate with the same frequency, as that of the impressed voltage but with a phase difference of ϕ



The steady state current of the circuits is given by

$$I(t) = \frac{E_0 \sin(\omega t - \phi)}{\sqrt{R^2 + \omega^2 L^2}}$$

Q. 19. Sole the clairaut's equation-

$$y = px + \frac{a}{p}$$

(2019)

Ans. The given differential equation -

$$y = px + \frac{a}{p}$$

Solution:

$$y = cx + \frac{a}{c}$$

Chapter - 3

Homogenous linear Differential Equation

Q. 1. If $y = f_1(x)$ and $y = f_2(x)$ are solutions of $y'' + f(x)\,y' + g(x)\,y = 0$, where $f(x)$ and $g(x)$ are continous on an interval I. If $f_1(x)$ and $f_2(x)$ are any two solutions of this equation. Then show that their Wronskian $w(f_1, f_2)$ is either identically zero or nowwhere zero on I. (2013,14,17,15,16,18)

Ans. Since ϕ_1, ϕ_2 are solution of

$$y'' + f(x)\,y' + g(x)\,y = 0 \quad ...(1)$$

$$\Rightarrow \phi''_1(x) + f(x)\,\phi'_1(x) + g(x)\,\phi_1(x) = 0 \quad ...(2)$$

$$\text{and } \phi''_2(x) + f(x)\,\phi'_2(x) + g(x)\,\phi_2(x) = 0 \quad ...(3)$$

Multiplying (2) & (3) by $\phi_2(x)$ and $\phi_1(x)$ respectively and subtracting one from other, we get-

$$[\phi_1(x)\,\phi''_2(x) - \phi''_1(x)\,\phi_2(x)] + a(x)\,[\phi_1(x)\,\phi'_2(x) - \phi_1'(x).\,\phi_2(x)] = 0$$

$$\text{or } \frac{dW(x)}{dx} + a(x)\,W(x) = 0 \quad ...(4)$$

where,

$$W(x) = \phi_1(x)\,\phi'_2(x) - \phi'_1(x)\,\phi_2(x).$$

Equation (2) is a linear differential equation of first order is W, which has a general solution of the form.

$$W(x) = C\text{-}e^{-\int a(x)\,dx}.$$

Where C is a constant.

$\Rightarrow$ This function vanishes either identically or no where on I according as the constant is zero or not.

Q. 2. Solve - $(D^2 - 2D + 5)y = e^{2x}\sin x$. (2014)

Ans. The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$\therefore\ m = \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$m = \pm\, 2i$$

Hence, $C.F = e^x(C_1 \cos 2x + C_2 \sin 2x)$

$$P.I = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x = e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x = \frac{1}{2} e^{2x} \frac{1}{D+2} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{(D-2)}{(D-2)(D+2)} \sin x = \frac{1}{2} e^{2x} \left(\frac{D-2}{D^2 - 4}\right) \frac{1}{2} e^{2x} \frac{(D-2)}{-1-4} \sin x$$

$$= \frac{-1}{10} e^{2x} (D-2) \sin x = \frac{-1}{10} e^{2x} \{D(\sin x) - 2\sin x\}$$

$$= \frac{-1}{10} e^{2x} (\cos x - 2\sin x)$$

Hence, the complete solution is

$$y = CF + PI$$

$$y = e^x (C_1 \cos 2x + C_2 \sin 2x) - \frac{1}{10} e^{2x} (\cos x - 2\sin x).$$

Q. 3. Using the method of variation of parameter obtain the general solution of the non-homogenous differential equtionn-

$$\frac{d^2 y}{dx^2} + a^2 y = \sin ax \quad \textbf{(2015)}$$

Ans. The corresponding homogenous equation is

$$y'' + a^2 y = 0 \quad ...(1)$$

Which has the general solution

$$\phi(x) = C_1 \cos ax + C_2 \sin ax$$

$$\Rightarrow y_p(x) = \alpha_1(x) \cos ax + \alpha_2(x) \sin ax \quad ...(2)$$

For a particular solution -

From (2), we get

$$y'_p(x) = -\alpha_1(x)\,a \sin ax + \alpha_2(x)\,a \cos ax + \alpha'_1(x) \cos ax + \alpha'_2(x) \sin ax$$

Let, $\alpha'_1(x) \cos ax + \alpha'_2(x) \sin ax = 0 \quad ...(3)$

then, $y'_p(x) = -a\alpha_1(x) \sin ax + a\alpha_2(x) \cos ax \quad ...(4)$

Differentiating w.r.t. x,

$$y''_p(x) = -a^2\alpha_1(x) \cos ax - a^2\alpha_2(x) \sin ax - a\alpha'_1(x) \sin ax + a\alpha'_2 \cos ax \quad ...(5)$$

Now substituting $y_p(x)$ & $y''_p(x)$ is given differential equation, we get

$$-a\alpha'_1(x) \sin ax + a\alpha'_2(x) \cos ax = \sin ax \quad ...(6)$$

Solving for $\alpha'_1(x)$ & $\alpha'_2(x)$, we get

$$\alpha'_1(x) = -\frac{\sin^2 ax}{a}$$

$$\alpha'_2(x) = \frac{\sin ax \cos ax}{a}$$

Integrating these w.r.t. x.

$$\alpha_1(x) = \frac{-x}{2a} + \frac{1}{4a^2}\sin 2ax$$

$$\alpha_2(x) = \frac{-1}{4a^2}\cos 2ax$$

Hence $y_p(x) = \left[\frac{-x}{2a} + \frac{1}{4a^2}\sin 2ax\right]\cos ax - \frac{1}{4a^2}\cos 2ax \sin ax.$

$$y_p(x) = \frac{-x}{2a}\cos ax + \frac{1}{4a^2}\sin(2ax - ax)$$

$$y_p(x) = \frac{-x}{2a}\cos ax + \frac{\sin ax}{4a^2}$$

The general solution becomes -

$y = \phi(x) + y_p(x)$

$$y = C_1 \cos ax + C_2 \sin ax \frac{-x}{2a}\cos ax + \frac{\sin ax}{4a^2}$$

Q. 4. Define the inverse differential operator $\frac{1}{f(D)}$ where $f(D) = D_n + a_1D^{n-1} + \ldots\ldots a_{n-1}D + a_n \left[D \equiv \frac{d}{dx}\right]$ and a_1, a_2 ... constants. (2015, 2017)

Ans. Let $\frac{1}{f(D)}Q$ denote some function of x which when operated upon by f(D) gives Q. This function of x is a particular solution of differential equation.

$f(D)\, y = Q$

As f(D) and $f(D)^{-1}$ are inverse operations therefore,

D $\{D^{-1}(Q)\} = Q$

or $\frac{d}{dx}\{D^{-1}(Q)\} = Q$

or $D^{-1}(Q) = \int Q\, dx.$

Q. 5. Establish that,

$$\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{\phi(x)}.\frac{x^p}{p!}$$

If $f(D) = (D-a)^p\, \phi(D)$. (2015)

Ans. In this case, if we calculate, P.I. by previous method, then

$$\text{P.I} = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} = \frac{e^{ax}}{0}$$

hence, this method fails.

Now, since a is a root of f(D), (D- a) must be a factor of f(D) which can be written in the form $(D - a)\,\phi(D)$.

$$\therefore \text{P.I} = \frac{1}{(D-a)\phi(D)}e^{ax} = \frac{1}{(D-a)\psi(a)}e^{ax}$$

where $\phi(a)$ is now a constant only.

$$= \frac{1}{\phi(a)}.\frac{e^{ax}.1}{(D-a)} = \frac{1}{\phi(a)}e^{ax}.\frac{1}{(D+a-a)}.1$$

$$= \frac{1}{\phi(a)}e^{ax}\left(\frac{1}{D}\right)1 = \frac{1}{\phi(a)}e^{ax}.x$$

Similarly, if $f(D) = (D - a)^p\, \phi(D)$ then P.I. of e^{ax} is

$$\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)^p\,\phi(D)} = \frac{e^{ax}.1}{\phi(a)(D-a)^p}$$

$$= \frac{e^{ax}}{\phi(a)}.\left[\frac{1}{(D-a-a)^p}\right]1 = \frac{e^{ax}}{\phi(a)}\left(\frac{1}{D^p}\right)1 = \frac{e^{ax}}{\phi(a)}.\frac{x^p}{p!}$$

Q. 6. Solve $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2$ $(x > 0)$ (2016)

Ans. Given equation -

$$\Rightarrow y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2$$

$$\Rightarrow x^2y'' - 4xy' + 6y = x^4$$

Using operator D, the equation is written as -

$\Rightarrow [D(D-1) - 4D + 6]y = e^{4z}.$

{Let $z = \ln x$ or $x = e^z$}

$\Rightarrow (D^2 - D - 4D + 6)\, y = e^{4z}$

$\Rightarrow (D^2 - 5D + 6)\, y = e^{4z}$ ....(1)

Auxiliary equation -

$\Rightarrow m2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2, 3$

$\Rightarrow C.F. = C1e2z + C2e3z$

$$PI = \frac{1}{(D^2 - 5D + 6)} \cdot \left(e^{4z}\right) = \frac{1}{16 - 20 + 6} e^{4z} \quad = \frac{e^{4z}}{2}$$

Solution:

$$y = C_1 x^2 + C_2 x^3 + \frac{x^4}{2}.$$

Q. 7. Solve the equation -
$y'' + 2y' + y = 4e^{-x} \ln x$.
by using the method of variation of parameters. (2016, 2018)

Ans. Given differential equation is -

$y'' + 2y' + y = 4e^{-x} \log x$

The auxiliary equation is -

$\Rightarrow (m^2 + 2m + 1) = 0$

$\Rightarrow m = -1, -1.$

$CF \Rightarrow y = (C_1 + C_2 x)e^{-x}.$

Now by variation of parameters method

Let, PI is $y_p(x) = ue^{-x} + ve^{-x}$

where, $u = -\phi_2(x) \int \frac{r(x)\phi_1(x)}{w(x)}$

and $v = \phi_1(x) \int \frac{r(x)\phi_2(x)}{w(x)}$

Here $\phi_1(x) = e^{-x}$, $\phi_2(x) = xe^{-x}$.

$$W(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (-xe^{-x} + e^{-x} \end{vmatrix}$$

$= -xe^{-2x} + e^{-2x} + xe^{-2x} = e^{-2x}$

and $r(x) = 4e^{-x} \log x.$

$$\Rightarrow u = -xe^{-x} \int \frac{4e^{-x} \log x \times e^{-x}}{e^{-2x}} dx$$

$\Rightarrow u = -xe^{-x} \int 4 \log x\, dx$

$\Rightarrow u = -4x^2 e^{-x} [1 - \log x]$

and $v = e^{-x} \int \frac{4e^{-x} \log x \times xe^{-x}}{e^{-2x}} dx$

$\Rightarrow v = +e^{-x} \int x^4 \log x\, dx$

$$\Rightarrow v = 4e^{-x} \left[\frac{x^2}{2} \log x - \frac{1}{4} x^2\right]$$

Hence complete sol^n is -

$y = C_1 e^{-x} + C_2 xe^{-x} + 4e^{-x} + ve^{-x} x.$

Q. 8. Solve
$y'' + 4y = 4 \tan 2x$. ***(2016)***

Ans. And $P.I. = \frac{1}{D^2 + 2^2} 4 \tan 2x = \frac{1}{(D + 2i)(D - 2)} 4 \tan 2x$

$$= \frac{1}{i}\left[\frac{1}{D - 2i} - \frac{1}{D + 2i}\right] \tan 2x = \frac{1}{i}\left[\frac{1}{D - 2i} \tan x - \frac{1}{D + 2i} \tan 2x\right]$$

Now, $\frac{1}{D - 2t} \tan 2x = e^{2ix} \int e^{-2ix} \tan 2x dx \int (\cos 2x - i \sin 2x) \frac{\sin 2x}{\cos 2x} dt$

$$= e^{2ix} \int \left(\sin 2x - \frac{i \sin^2 2x}{\cos 2x}\right) dx = e^{2ix} \int \left(\sin 2x - \frac{i(1 - \cos^2 2x)}{\cos 2x}\right) dx$$

$= e^{2ix} \int (\sin 2x - i \sec 2x + i \cos 2x) dx = e^{2ix} i \int (e^{-2ix} - \sec 2x) dx$

$$= ie^{2ix} \left[\frac{e^{-2ix}}{-2i} - \frac{1}{2} \log(\sec 2x + \tan 2x)\right]$$

$$= -\frac{1}{2} - \frac{ie^{2ix}}{2} \log(\sec 2x + \tan 2x)$$

Similarly, $\frac{1}{D + 2i} \tan 2x = \frac{-1}{2} + \frac{ie^{2n}}{2} \log(\sec 2x + \tan 2x)$

$$P.I. = \frac{1}{i}\left[-i\left(\frac{e^{2ix} + e^{-2ix}}{2}\right) \log(\sec 2x + \tan 2x)\right]$$

$= -\cos 2x \log(\sec 2x + \tan 2x)$

Hence the complete solution is

$y = C.F. + P.I.$

$= c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$

Q. 9. *Obtain the general solutions of*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$ **(2017)**

Ans. Given equation is -

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$

Using operator D, the equation becomes -

$\Rightarrow [D(D-1)+D-1]y=ze^{2z}$.

$\Rightarrow [D^2-D+D-1]y=ze^{2z}$

$\Rightarrow [D^2-1]y=ze^{2z}$

Auxiliary Equation -

$\Rightarrow (m^2-1)=0$

$\Rightarrow m=\pm 1$

$\therefore CF=C_1e^x+C_2e^{-x}$.

Now, for particular Integral -

Let $$y=Ax+\frac{B}{x}$$

be the primitie of the given equation, where A & B are function of x,

Then $$\frac{dy}{dx}=A-\frac{B}{x^2}+\frac{dA}{dx}x+\frac{1}{x}\frac{dB}{dx}$$

Let us choose A & B such that

$$x\frac{dA}{dx}+\frac{1}{x}\frac{dB}{dx}=0 \quad ...(1)$$

$$\Rightarrow \frac{dy}{dx}=A-\frac{B}{x^2}$$

and $$\frac{d^2y}{dx^2}=\frac{dA}{dx}-\frac{1}{x^2}\frac{dB}{dx}+\frac{2}{x^3}B$$

Putting these values,

$$\frac{dA}{dx}-\frac{1}{x^2}\frac{dB}{dx}=e^x \quad ...(2)$$

Solving equation (1) & (2)

$$\frac{dA}{dx}=\frac{1}{2}e^x \;\&\; \frac{dB}{dx}=\frac{-x^2}{2}e^x$$

$$\therefore A=\frac{1}{2}e^x+C_3 \text{ and } B=\frac{-1}{2}x^2e^x+xe^x-e^x+C_4$$

Thus complete solution of given equation is

$$y=c_1x+\frac{c_2}{x}-\frac{1}{x}e^x$$

Q. 10. *By the method of undetermined coefficient find the general solution of*
$y''+a^2y=\cos ax$. **(2018)**

Ans. Let $y''+a^2y=\cos ax$. ...(1)

The homogenous equation of equation (1) is

$(D^2+a^2)y=0$.

Auxiliary equation is -

$\Rightarrow m^2+a^2=0$

$\Rightarrow m=\pm ia$

Therefore complementry function is

$y=C_1\cos ax+C_2\sin ax$

Let particular integral is given by -

$y_p(x)=x(k_1\cos ax+k_2\sin ax)$ ...(2)

$y'_p(x)=x(-ak_1\sin ax+ak_2\cos ax)+(k_1\cos ax+k_2\sin ax)$.

$y''(x)=x(-a^2k_1\cos ax-a^2k_2\sin a)+(-ak_1\sin ax+ak_2\cos ax)$
$+(-k_1a\sin ax)+k_2a\cos ax$.

Putting these values in equation (1), we get

$y''_p(x)+a^2y_p(x)=\cos ax$.

$\Rightarrow -k_1a^2x\cos ax-k_2a^2x\sin ax-k_1a\sin ax+k_2a\cos ax-k_1a\sin ax+k_2a\cos ax+k_1a^2x\cos ax+k_2a^2x\sin ax=\cos ax$

$\Rightarrow -2k_1a\sin ax+2k_2a\cos ax=\cos ax$. ...(2)

Comparing corresponding coefficient,

$$k_1=0,\ k_2=\frac{1}{2a}$$

Now, from equation (2).

$$y_p(x)=\frac{x}{2a}\sin ax \quad ...(3)$$

Hence complete solution is -

$$y=c_1\cos ax+c_2\sin ax+\frac{x}{2a}\sin ax.$$

Q. 11. *Solve the differential equation -*

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-\frac{3y}{x^2}=\ln x$$ **(2018)**

Ans. Given differential equation -

$$x^2\frac{d^2y}{dx^2} - \frac{xdy}{dx} - 3y = x^2\log x$$

Put, $x = e^z \Rightarrow z = \log x$

$\Rightarrow [D(D-1) - D - 3]y = e^{2z}.z$

$\Rightarrow (D^2 - 2D - 3)y = e^{2z}.z$

Its Auxiliary equation is

$(m^2 - 2m - 3)y = 0$

$(m+1)(m-3) = 0.$

$m = 1, 3$

$\therefore CF = C_1e^z + C_2e^{3z}.$

$$\text{And PI} = \frac{1}{D^2 - 2D - 3}e^{2z}.z$$

$$\Rightarrow \text{PI} = e^{2z}.\frac{1}{(D+2)^2 + 2(D+2) - 3}.z$$

$$\Rightarrow \text{PI} = e^{2z}\frac{1}{D^2 + 2D - 3}z = \frac{-1}{3}e^{2z}\left\{1 + \left[\frac{2D + D^2}{3}\right]\right\}z$$

$$= \frac{-1}{3}e^{2z}\left[z + \frac{2}{3}\right] = -\frac{1}{9}e^{2z}(3z+2)$$

Hence general solution is -

$$y = c_1e^z + c_2e^{3z} - \frac{1}{9}e^{2z}(3z+2).$$

$$y = \frac{c_1}{x} + c_2x^3 - \frac{1}{9}x^2(3\log x + 2)$$

Q. 12. *Solve the non-homogenous linear differential equation.*
$y'' - 3y' + 2y = \cos(e^{-x})$.
by the method of variation of parameter. *(201*

Ans. The homogeneous equation corresponding to the above equation

$y'' - 3y' + 2y = 0$

which has a general solution

$\phi_h = c_1e^x + c_2e^{2x}.$

We use the method of variation of parameters and set

$y_p = \alpha_1(x)e^x + \alpha_2(x)e^{2x}$

giving $y'_p(x) = \alpha_1(x)e^x + 2\alpha_2(x)e^{2x} + \alpha'_1(x)e^x + \alpha'_2(x)e^{2x}$. Setting

$\alpha'_1(x)e^x + \alpha'_2(x)e^{2x} = 0$

we get $y'_p(x) = \alpha_1(x)e^x + 2\alpha_2(x)e^{2x}$. Therefore

$y''_p(x)[\alpha_1(x)e^x + 4\alpha_2(x)e^{2x}] + [\alpha'_1(x)e^x + 2\alpha'_2(x)e^{2x}]$.

Then after some simplification

$e^x\alpha'_1(x) + e^{2x}\alpha'_2(x) = 0$

$e^x\alpha'_1(x) + 2e^{2x}\alpha'_2(x) = \cos(e^{-x})$.

Upon solving for $\alpha'_1(x)$ and $\alpha'_2(x)$, we get

$$\alpha'_1(x) = \frac{-e^{2x}\cos(e^{-x})}{e^{3x}} = -e^{-x}\cos(e^{-x}) \quad ...(1)$$

and

$$\alpha'_2(x) = \frac{e^x\cos(e^{-x})}{e^{2x}} = e^{-2x}\cos(e^{-x}) \quad ...(2)$$

Integrating (1) and (2) and ignoring the constants of integration, we get

$\alpha_1(x) = -\int e^{-x}\cos(e^{-x})dx = \sin(e^{-x})$.

$\alpha_2(x) = \int e^{-2x}\cos(e^{-x})dx = -[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]$.

Hence $y_p(x) = \alpha_1(x)e^x + \alpha_2(x)e^{2x}$

$= e^x\sin(e^{-x}) - e^x\cos(e^{-x}) = -e^{2x}\cos(e^{-x})$.

Thus the general solution is given by

$y = c_1e^x + c_2e^x - e^{2x}\cos(e^{-x})$.

Q. 13. *Show that the particular solution of*
$y'' + 2ay' + b^2y = a\cos wx$, (a, w > 0) is

given by $$y = \frac{A\sin(\omega x - \alpha)}{\sqrt{(b^2 - \omega^2)^2 + 4\omega^2a^2}}$$

where, $a = \tan^{-1}\left(\frac{2a\omega}{b^2 - \omega^2}\right)$ ***(0 < a < p).*** ***(2019)***

Ans. Given differential equation is -

$y'' + 2ay' + b^2y = a\cos\omega x$ ...(1)

Auxiliary equation is -

$\Rightarrow m^2 + 2m + b^2 = 0$

$$\Rightarrow m = \frac{-2a \pm \sqrt{4a^2 - b^2}}{2}$$

$$\Rightarrow m = -a \pm \sqrt{a^2 - b^2}$$

$$\Rightarrow m = -a \pm p \qquad \left\{p = \sqrt{a^2 - b^2}\right\}$$

$\Rightarrow y_c(x) = c_1e^{(-a+p)x} + c_2e^{(-a-p)x}$.

Using undetermined coefficient method

WINSOME SERIES

ODE

Paper -3

उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े उसे त्याग दो।



KEY POINT AU



KEY POINT AU

Let, $y_p(x) = c_3 \cos \omega x + c_4 \sin \omega x$

$\Rightarrow y'_p(x) = -\omega c_3 \sin \omega x + c_4 \omega \cos \omega x.$

$\Rightarrow y''_p(x) = -c_3\omega^2 \cos \omega x - c_4 \omega^2 \sin \omega x$

from (1)

$-c_3 w^2 \cos \omega x = c_4 \omega x - c_4\omega^2 \sin \omega x - 2ac_2 \omega \sin \omega x + 2 ac_4 \cos \omega x + b^2 c_3 \cos \omega x + b^2 c_4 \sin \omega x = A \cos \omega x$

$\Rightarrow \cos \omega x (-c_3\omega^2 + 2ac_4\omega + b^2c_3) + \sin \omega x (-c_4 \omega^2 - 2ac_3\omega + b^2c_4)$

Comparing the coefficient both sides-

$c_3 (b^2 - \omega^2) + 2a\omega c_4 = A$ ...(2)

$-2a\omega c_3 + c_4(b^2 - \omega^2) = 0$ ...(3)

Multiply (2) by $2a\omega$ & (3) by $(b^2 - \omega^2)$

and adding gives

$c_4 [4a^2\omega^2 + (b^2 - \omega^2)^2] = 2a\omega A.$

$$\Rightarrow c_4 = \frac{2a\omega A}{4a^2\omega^2 + (b^2 - \omega^2)^2}$$

From (2), $(b^2 - \omega^2)c_3 = A - \dfrac{4a^2\omega^2 A}{4a^2\omega^2 (b^2 - \omega^2)^2}$

$$\Rightarrow c_3 = \frac{(b^2 - \omega^2)A}{4a^2\omega^2 + (b^2 - \omega^2)^2}$$

therefore, $y_p(x) = \dfrac{(b^2 - \omega^2)A}{4a^2\omega^2 + (b^2 - \omega^2)^2} \cos \omega x + \dfrac{2a\omega A}{4a^2\omega^2 + (b^2 - \omega^2)} \sin \omega x.$

Let $\cos \alpha = \dfrac{(b^2 - \omega^2)}{\sqrt{4a^2\omega^2 + (b^2 - \omega^2)^2}}$

& $\sin \alpha = \dfrac{2a\omega}{\sqrt{4a^2\omega^2 + (b^2 - \omega^2)^2}}$

$$\Rightarrow \tan \alpha = \frac{2a\omega}{b^2 - \omega^2}$$

$$\Rightarrow \alpha = \tan^{-1}\left[\frac{2a\omega}{b^2 - \omega^2}\right] \frac{A \cos(\omega x - \alpha)}{\sqrt{4a^2\omega^2 + (b^2 - \omega^2)^2}}$$

So, $y_p(x) = \dfrac{A \cos(\omega x - \alpha)}{\sqrt{4a^2\omega^2 + (b^2 - \omega^2)^2}}$

Q. 14. *Solve the following IVP.*

$y'' - 4y + 3y' = 4e^{3x}$, $y(0) = -1$

$y'(0) = 3$ ***(2019)***

Ans. Given differential equation is -

$y'' - 4y' + 3y = 4e^{3x}$ ...(1)

Auxiliary equation is -

$\Rightarrow (m^2 - 4m + 3) = 0.$

$\Rightarrow (m - 3)(m - 1) = 0$

$\Rightarrow m = 3, 1$

$\therefore$ C.F. $= C_1e^{2x} + C_2e^x$

Now,

$$PI = \frac{1}{(D^2 - 4D + 3)} 4e^{3x}$$

$$PI = 4x.\frac{1}{2D - 4} e^{3x}$$

$$PI = 4x.\frac{1}{2} e^{2x} = 2xe^{3x}$$

Solution is -

$y = C_1e^{3x} + C_2e^x + 2xe^{3x}$

at $x = 0, y = -1.$

$\Rightarrow -1 = C_1 + C_2 + 0.$

$\Rightarrow C_1 + C_2 = -1$ ...(2)

at $x = 0$, $\dfrac{dy}{dx} = 3$

$\Rightarrow = + C_2e^x + 2e^{3x} + 6xe^{3x}$

$\Rightarrow 3 = 3C_1 + C_2 + 2$

$3C_1 + C_2 = 1$ ...(3)

From (2) & (3)

$C_1 = 1, C_2 = -2.$

$\therefore y = e^{3x} - 2e^x + 2xe^{3x}$

Chapter - 4
Application of Oscillation

Q. 1. ***Discuss the motion of a body of mass m. Suspended by an elastic spring with spring constant k, when there is no resistive force and there as an additional sinuooidal imput force with angular frequency*** $\sqrt{\frac{k}{m}}$ ***. (2014, 2015, 2016, 2017)***

Ans. If an additional force r (t) is applied from outside, which is called input of driving force or external force, then the resulting motion is called a forced motion.

The most important case is that in which the impressed force is periodic and isof form $r(t) = F_0 \cos \omega t$, so that equation becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \quad ...(1)$$

where m, c, k, F_0, ω are positive constants and F_0 is the amplitude and ω the frequency of the impressed force.

The solution (1) comprises the C.F. $x_h(t)$ and P.I. $x_p(t)$, we determine here the particular integral $x_p(t)$ of the equation (1)

Using the method of undetermined coefficients, we assume that

$x_p(t) = k_1 \cos \omega t + k_2 \sin \omega t$ ...(2)

Substituting in (1), we get

$[(k - m\omega^2)k_1 + \omega c k_2] \cos \omega t + [-\omega c k_1 + (k - m\omega^2)k_2] \sin \omega t = F_0 \cos \omega t$

which gives two equations for k_1, k_2 as

$(k - m\omega^2)k_1 + \omega c k_2 = F_0$

$-\omega c k_1 + (k - m\omega^2)k_2 = 0$

Solving these equations, we get

$$k_1 = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + \omega^2 c^2}, k_2 = \frac{\omega c F_0}{(k - m\omega^2)^2 + \omega^2 c^2} \quad ...(3)$$

Inserting these values of k_1, k_2 in (2), we get the P.I. $x_p(t)$. Further setting $\omega_0 = \sqrt{k/m}$, we obtain the general solution of (1) as

or
$$x(t) = x_h(t) + \frac{m\left(\omega_0^2 - \omega^2\right)F_0}{m^2\left(\omega_0^2 - \omega^2\right) + \omega^2 c^2} \cos \omega t$$

$$+ \frac{\omega c F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \sin \omega t \quad ...(4)$$

Q. 2. ***Solve the system***

$3y'_1 + 2y_1 + y'_2 - 8y_2 = 5e^x$.

$4y'_1 + 2y_1 + y'_2 - 8y_2 = 5e^x + 2x - 3$.

for $y_1(x)$ ***&*** $y_2(x)$. ***(2014)***

Ans. denoting $\frac{d}{dx}$ by D,

$$3\frac{d}{dx}(y_1) + 2y_1 + \frac{d}{dx}(y_2) - 8y_2 = 5e^x$$

$\Rightarrow 3Dy_1 + 2y_1 + Dy_2 - 8y_2 = 5e^x$ ...(1)

$$\& \; 4\frac{d}{dx}(y_1) + 2y_1 + \frac{d}{dx}(y_2) - 8y_2 - 5e^x + 2x - 3$$

$\Rightarrow 4Dy_1 + 2y_1 + Dy_2 - 8y_2 = 5e^x + 2x - 3$ ...(2)

Further simplifications-

$(3D + 2)y_1 + (D - 8)y_2 = 5e^x$ ...(3)

$\& (4D + 2)y_1 + (D - 8)y_2 = 5e^x + 2x - 3$ ...(4)

from (2) & (4)

$\Rightarrow (3D + 2)y_1 - (4D + 2)y_1 = -2x + 3$

$\Rightarrow (-D)y_1 = -2x + 3$

$\Rightarrow Dy_1 = (2x - 3)$

$$\Rightarrow y_1 = \frac{1}{D}(2x - 3)$$

$\Rightarrow y_1 = x^2 - 3x$.

From equation (3),

$(3D + 2)\{x^2 - 3x\} + (D - 8)y_2 = 5e^x$

$3(2x - 3) + 2x^2 - 6x + (D - 8)y_2 = 5e^x$

$\Rightarrow 6x - 9 + 2x^2 - 6x + (D - 8)y_2 = 5e^x$

$\Rightarrow (D - 8)y_2 = 5e^x - 2x^2 + 9$

$\therefore CF = C_1 e^{+8x}$

$$\& \; PI = \frac{1}{D - 8}(5e^x - 2x^2 + 9)$$

$$\Rightarrow PI = \frac{5e^x}{-3} + \frac{1}{(D - 8)}(-2x^2 + 9)$$

$$\Rightarrow PI = \frac{-5}{3}e^x + \frac{1}{(8 - D)}(2x^2 - 9)$$

$$\Rightarrow PI = \frac{-5}{3}e^x + \frac{1}{8}(1-D/8)^{-1}\left(2x^2-9\right)$$

$$\Rightarrow PI = \frac{-5}{3}e^x + \frac{1}{8}\left[1+\frac{D}{8}\right](2x^2-9)$$

$$\Rightarrow PI = \frac{-5e^x}{3} + \frac{1}{8}\left[(2x^2-9)+\frac{x}{2}\right]$$

$$\therefore y_2 = C_1 e^{8x} - \frac{5e^x}{3} + \frac{1}{8}\left(2x^2-9+\frac{x}{2}\right)$$

Q. 3. Define scaler and vector field and also gradient of a scaler field u(x, y, z). (2014)

Ans. **Scaler point function -** If to each point p (x, y, z) of a region R in space there corresponds a unique scaler f (p), then f is called scalar point function & we can say that scaler field f has been defined in R.

Vector point function - If to each point P (x, y, z) of a region R in space there corresponds a unique vector f (P) then f is called a vector point function and we can say that a vector field f has been defined on R.

Gradient of scaler field -

Let u(x, y, z) be a defined and differntiable at each point (x, y, z) in a certain region of space. Then the gradient of u written as Δu at grad u defined as -

$$\Delta u = \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right]u$$

$$\Delta u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

Q. 4. Prove that grad f is along the normal vector to the surface f (x, y, z) = c. (2014, 2017)

Ans. **Proof :** We have grad

$$f = \Delta f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

Also grad f is a vector normal to the surface (x, y, z) = c. Since $\hat{n}$ is a unit vector normal to the surface f (x, y, z) = c, therefore let gra $f = A\,\hat{n}$, where A is some scalar to be determined.

Now, $\frac{df}{dn}$ = directional derivative of f in the direction of $\hat{n}$

$= \nabla f \cdot \hat{n} = A\,\hat{n} \cdot \hat{n} = A.$ $\quad [\because \nabla f = \text{grad} f = A\,\hat{n}]$

$\therefore$ grad f = ∇ f = $\frac{df}{dn}\,\hat{n}$.

Q. 5. Prove the identities -

div $\left(\vec{f}\times\vec{g}\right)$ = $\vec{g}$. cur $\vec{f}$ - $\vec{f}$. curl $\vec{g}$ (2014)

Ans. $\nabla.(\vec{f}\times\vec{g}) = \Sigma\left(i\frac{\partial}{\partial x}\right).(\vec{f}\times\vec{g})$

$$= \Sigma i\left[\vec{f}\times\frac{\partial\vec{g}}{\partial x} + \frac{\partial\vec{f}}{\partial x}\times\vec{g}\right].$$

$$\nabla.(\vec{f}\times\vec{g}) = \Sigma i\left[\vec{f}\times\frac{\partial\vec{g}}{\partial x}\right] + \Sigma i\left[\frac{\partial\vec{f}}{\partial x}\times\vec{g}\right]$$

$$= -\Sigma i\left[\frac{\partial\vec{g}}{\partial x}\times\vec{f}\right] + \Sigma\left[i\times\frac{\partial\vec{f}}{\partial x}\right].\vec{g} = -\Sigma\left[i\times\frac{\partial\vec{g}}{\partial x}\right]\vec{f} + \Sigma\left[i\times\frac{\partial\vec{f}}{\partial x}\right].\vec{g}$$

Since dot & cross are interchangable -

$$= -(\nabla\times\vec{g}).\vec{f} + \left(\nabla\times\vec{f}\right).\vec{g}$$

$$\therefore \nabla.\left(\vec{f}\times\vec{g}\right) = \left(\nabla\times\vec{f}\right).\vec{g} - (\nabla\times\vec{g}).\vec{f}$$

$$\Rightarrow \text{div}\left(\vec{f}\times\vec{g}\right) = \vec{g}.\text{curl}\,\vec{f} - \vec{f}\,\text{curl}\,\vec{g}$$

Q. 6. Prove that

curl (f $\vec{v}$) = f curl v + (grad f) × v. (2014)

Ans. $\text{Curl}\,(fv) = \Sigma i\frac{\partial}{\partial x}(f\vec{v})$

$$= \Sigma i\frac{\partial f}{\partial x}\times\vec{v} + \Sigma f\, i\times\frac{\partial\vec{v}}{\partial x}$$

$$\iint_S \left\{\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right\}dS = \iint_R \left\{\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\left(\frac{1}{2}\right)\right\}(a\,dx\,dy)$$

$$-\int_R \left(\frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right)dx\,dy \;\hat{n}\; \oint_C (F_1dx+F_2dy+F_3dz)$$

$= (\text{grad f})\,\vec{v} + f\,\text{curl}\,\vec{v}$.

Q. 7. ***Solve the system of differential equations-***

$$\frac{dI_1}{dt}+3I_1-2I_2=6$$

$$\frac{dI_2}{dt}+2I_2-I_1=0$$

and show that $\lim_{t\to\infty} I_1(t)=3$ **&**

$\lim_{t\to\infty} I_2(t) = \frac{3}{2}$. ***(2015)***

Ans. The equation can be written as -

$(D+3)I_1-2I_2=6$ ...(1)

$-I_1+(D+2)I_2=0$ ...(2)

{where $\frac{d}{dt}=D$}

Multiplying equation (2) by (D + 3) & adding with equation (1)

$[(D+3)(D+2)-2]I_2=6.$

$[D^2+5D-4]\,I_2=6.$

Auxiliary equation -

$\Rightarrow m^2+5m+4=0$

$m=-1,-4$

$\therefore CF=C_1e^{-t}+C_2e^{-4t}.$

Now, Particularl integral

$$PI=\frac{6}{D^2+5D+4}=\frac{3}{2}$$

$$\therefore I_2(t)=C_1e^{-t}+C_2e^{-4t}+\frac{3}{2}.$$

Putting values of $I_2(t)$ is equation (2),

$I_1(t)=C_1e^{-t}-2C_2e^{-4t}+3.$

$$\therefore \lim_{t\to\infty} I_1(t)=3 \;\&\; \lim_{t\to\infty} I_2(t)=\frac{3}{2}.$$

Q. 8. ***If a, b are constant vectors, w is a constant and r is a vector function of the scalar variable t given by***

r = cos ωt a + sin ωt b, show that

(i) $\frac{d^2r}{dt^2}+\omega^2 r=0$, ***(ii)*** $r\times\frac{dr}{dt}=\omega a\times b$ ***(2015)***

Ans. (i) $\frac{dr}{dt}=\frac{d}{dt}(\cos\omega t)\,a+\frac{d}{dt}(\sin\omega t)\,b$

$= -\omega\sin\omega x\,a+\omega\cos\omega t\,b.$

$$\therefore \frac{d^2r}{dt^2}=-\omega^2\cos\omega t\,a-\omega^2\sin\omega t\,b$$

$= -\omega^2(\cos\omega t\,a+\sin\omega t\,b)=-\omega^2 r.$

$$\therefore \frac{d^2r}{dt^2}+\omega^2 r=0.$$

(ii) $r\times\frac{dr}{dt}=(\cos\omega t\,a+\sin\omega t\,b)\times(-\omega\sin\omega t\,a+\omega\cos\omega t\,b)$

$= \omega\cos^2\omega t\,a\times b-\omega\sin^2\omega t\,b\times a$ $[\because a\times a=0,\ b\times b=0]$

$= \omega\cos^2\omega t\,a\times b+\omega\sin^2\omega t\,a\times b$

$= \omega(\cos^2\omega t+\sin^2\omega t)\,a\times b=\omega a\times b.$

Q. 9. ***Prove that***

curl (φA) = grad (φ) ´ A + φ curl A. ***(2015)***

Ans. We have

$\text{Curl}(\phi A)=\nabla\times(\phi A)$

$$=\left[i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right]\times(\phi A)$$

$$=\Sigma\left\{i\times\frac{\partial}{\partial x}(\phi A)\right\}$$

$$=\Sigma\left\{i\times\left(\frac{\partial\phi}{\partial x}A+\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$=\Sigma\left\{i\times\left(\frac{\partial\phi}{\partial x}A\right)\right\}+\Sigma\left\{i\times\left(\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$= \left\{\Sigma\left(\frac{\partial \phi}{\partial x} i\right)\right\} \times A + \Sigma\left\{\phi\left(i \times \frac{\partial A}{\partial x}\right)\right\}$$

$$= \left\{\Sigma\left(\frac{\partial \phi}{\partial x} i\right)\right\} \times A + \phi\Sigma\left[i \times \frac{\partial A}{\partial x}\right]$$

$$= (\nabla \phi) \times A + \phi(\nabla \times A)$$

Q. 10. Solve the system of linear equations-

$$\frac{dy_1}{dx} = 2y_1 + 3y_2 + 2e^{2x}.$$

$$\frac{dy_2}{dx} = y_1 + 4y_2 + 3e^{2x}.$$

(2016)

Ans. Substituting, $\frac{d}{dx} \equiv D$

$Dy_1 - 2y_1 - 3y_2 = 2e^{2x}$

$(D - 2)y_1 - 3y_2 = 2e^{2x}$

& $Dy_2 - y_1 - 4y_2 = 3e^{2x}$ ...(1)

$(D - 4)y_2 - y_1 = 3e^{2x}$

Multiplying equation (2) by (D -2) & adding to equation (2)- ...(2)

$(D - 4)(D - 2)y_2 - 3y_2 = (D - 2)3e^{2x} + 2e^{2x}$

$\Rightarrow (D^2 - 6D + 8)y_2 - 3y_2 = 6e^{2x} - 6e^{2x} + 2e^{2x}$

$\Rightarrow (D^2 - 6D + 5)y_2 = 2e^{2x}.$

Its Auxiliary equation is

$\Rightarrow m^2 - 6m + 5 = 0$

$\Rightarrow m = 5, 1$

$\therefore CF = C_1e^x + C_2e^{5x}.$

$$\& PI = \frac{1}{(D^2 - 6D + 5)} 2e^{2x} = \frac{2e^{2x}}{4 + 5 - 12}$$

$$PI = \frac{-2e^{2x}}{3}, \quad \therefore y_2 = C_1e^x + C_2e^{5x} - \frac{2}{3}e^{2x}$$

...(3)

Now, $\frac{dy_2}{dx} = = C_1e^x + 5C_2e^{5x} - \frac{4}{3}e^{2x}$ ...(4)

Putting values of y_2 & $\frac{dy_2}{dx}$ in equation (2).

$$\Rightarrow C_1e^x + 5C_2e^{5x} - \frac{4}{3}e^{2x} = y_1 + 4\left(C_1e^x + C_2e^{5x} - \frac{2}{3}e^{2x}\right) + 3e^{2x}$$

$$\Rightarrow C_1e^x + 5C_2e^{5x} - \frac{4}{3}e^{2x} = y_1 + 4C_1e^x + 4C_2e^{5x} - \frac{8}{3}e^{2x} + 3e^{2x}.$$

$$\Rightarrow y_1 = 3C_1e^x - C_2e^{5x} - e^{2x}\left(\frac{8}{3} - \frac{4}{3} - 3\right)$$

$$\Rightarrow y_1 = 3C_1e^x - C_2e^{5x} - \frac{e^{2x}}{3}(-5)$$

$$\Rightarrow y_1 = 3C_1e^x - C_2e^{5x} + \frac{5e^{2x}}{3}.$$

Q. 11. A particle moves along a curve whose parametric equations are -
$x = 2t^2, y = t^2 - 4t, z = 3t - 5$
where t is time, find the velocity & acceleration of particle in dir of (2i - 3j -k) at time, t = 1 second. **(2016)**

Ans. Let $\vec{r}$ be the position vector.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}.$$

Velocity, $= (4t)\hat{i} + (2t - 4)\hat{j} + (3)\hat{k}$

$\vec{v}$ (at t = 1) $= 4\hat{i} + (-2)\hat{j} + 3\hat{k}$:

acceleration,

$$\vec{a} = \frac{d\,r^2}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = 4\hat{i} + 2\hat{j} + 0\hat{k} = 4\hat{i} + 2\hat{j} + 0\hat{k}$$

$\vec{a}$ (at t = 1) $= 4\hat{i} + 2\hat{j} + 0\hat{k}$

Now, unit vector in given direction (2i - 3j - k) is -

$$\hat{n} = \frac{2i - 3j - k}{\sqrt{4 + 9 + 1}} = \frac{2i - 3j - k}{\sqrt{14}}$$

$\therefore$ Component of velocity in given dir^n

$$\Rightarrow \vec{v}\cdot\hat{n} = \frac{(4\vec{i}-2\vec{j}+3\vec{k})(2\vec{i}-3\vec{j}-\vec{k})}{\sqrt{14}}$$

$$\Rightarrow \frac{8+6-3}{\sqrt{14}} \Rightarrow \frac{11}{\sqrt{14}}$$

& component of acceleration in given direction

$$\vec{a}\cdot\hat{n} = \frac{(4\vec{i}+2\vec{j}+0\vec{k})(2\vec{i}-3\vec{j}-\vec{k})}{\sqrt{14}} = \frac{8-6}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

Q. 12. If $\vec{a}$ is a constant vector, prove that

$$\text{Curl}\ \frac{\vec{a}\times\vec{r}}{r^3} = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}.\vec{r})$$

Ans. We have

$$\text{curl}\frac{a\times r}{r^3} = \nabla\times\left(\frac{a\times r}{r^3}\right) = \Sigma\left\{i\times\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right)\right\}$$

Now $$\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x}(a\times r) + \frac{1}{r^3}\left(a\times\frac{\partial r}{\partial x}\right) + \frac{1}{r^3}\left(\frac{\partial a}{\partial x}\times r\right)$$

Now $\frac{da}{dt} = 0$ because a is a constant vector.

Also $r = xi + yj + zk$

$\frac{\partial r}{\partial x} = i$, Further $\frac{\partial r}{\partial x} = \frac{x}{r}$, (i) becomes

$$\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) = -\frac{3}{r^4}\frac{x}{r}(a\times r) + \frac{1}{r^3}(a\times i) = -\frac{3x}{r^5}(a\times r) + \frac{1}{r^3}(a\times i)$$

$$\therefore i\times\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) = -\frac{3x}{r^5}i\times(a\times r) + \frac{1}{r^3}i\times(a\times i)$$

$$= -\frac{3x}{r^5}[(i.r)a - (i.a)r] + \frac{1}{r^3}[(i\cdot i)a - (i\cdot a)i]$$

$$= -\frac{3x}{r^5}xa + \frac{3x}{r^5}a_1 r + \frac{1}{r^3}a - \frac{1}{r^3}a_1 i$$

[$\because i.r = x$ and $i.a = a_1$ if $a = a_1 i + a_2 j + a_3 k$]

$$= -\frac{3x^2}{r^5}a + \frac{3}{r^5}a_1 x r + \frac{1}{r^3}a - \frac{1}{r^5}a_1 i$$

$$\therefore \Sigma\left\{i\times\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right)\right\}$$

$$= \left\{-\frac{3}{r^5}\Sigma x^2\right\}a + \left\{\frac{3}{r^5}\Sigma a_1 x\right\}r + \frac{3}{r^5}a - \frac{1}{r^3}\Sigma a_1 i$$

$$= -\frac{3}{r^5}r^2 a + \frac{3}{r^5}(r\cdot a)r + \frac{3}{r^3}a - \frac{1}{r^3}a, \quad [\because \Sigma x_1 = r^2, \Sigma a_1 x = r.a, \Sigma a_1 i = a]$$

$$= -\frac{a}{r^3} + \frac{3}{r^5}(a\bullet r)r$$

Q. 13. Solve the system of differential equation

$$\frac{dx}{dt} + y = \cos t - \sin t + e^t \ , \ \frac{dy}{dt} + x = \cos t + \sin t - e^t \qquad (2017)$$

Ans. These equation can be as-

$Dx + y = \cos t - \sin t + e^t$ ...(1)

and $Dy + x = \cos t + \sin t - e^t$ ...(2)

Multiplying equation (1) by D & subtracting (2) from it, we have

$\Rightarrow (D^2 - 1)x = -\sin t - \cos t + e^t - \cos t - \sin t + e^t$

$\Rightarrow (D^2 - 1)x = -2(\sin t + \cos t - e^t)$

Its Auxiliary equation is-

$m^2 - 1 = 0$

$m = \pm 1$

$\therefore CF = C_1 e^t + C_2 e^{-t}$

Now, $$PI = \frac{-2}{(D)^2 - 1}(\sin t + \cos t - e^t)$$

$$= (\sin t + \cos t) - \frac{2}{D^2 - 1}e^t = \sin t + \cos t - \frac{2t}{2D}e^t = \sin t + \cos t + te^t$$

$\therefore x = C_1 e^t + C_2 e^{-t} + \sin t + \cos t - te^t$ ...(3)

Putting value of x in equation (2)

$$(-te^t)\frac{dy}{dt} + C_1 e^t + C_2 e^{-t} + \sin t + \cos t = \sin t + \cos t - e^t$$

$$\Rightarrow \frac{dy}{dt} = (te^t - e^t) - C_1 e^t - C_2 e^{-t}$$

$$\Rightarrow \frac{dy}{dt} = e^t(t-1) - C_1e^t - C_2e^{-t}$$

$$\Rightarrow y = te^t - e^t - e^t - C_1e^t + C_2e^{-t}$$

$$\Rightarrow y = e^t(t-2-C_1) + C_2e^{-t}$$

Q. 14. Consider the oscillation of mass m suspended by a spring of negligible mass & spring modulus k, in a medium which offered a resistance proportional to its velocity with damping constant C. Discuss the case when $C^2 < 4km$. (2018, 2019)

Ans. Consider the motion under a damping force (viscosity or resistance of the medium) Assuming that the damping force is proportional to the velocity (x), we have the equation governing the damped motion as

$$mx + cx + kx = 0 \quad ...(1)$$

where m, c, k are all positive, m denotes the mass of the body, k the stiffness of the spring and c the resistance per unit velocity.

With $c/m = 2\alpha$ and $k/m = \omega_0^2$, equation (1) takes the form

$$\ddot{x} + 2\alpha + \omega_0^2 x = 0, \; a = c/2m, \quad ...(2)$$

The auxiliary equation is

$$\lambda_2 + 2\alpha\lambda + \omega_0^2 = 0 \quad ...(3)$$

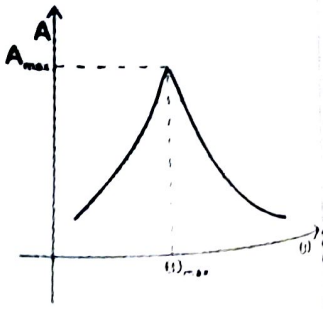
which gives the roots as

$$\lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The general solution (2) will depend upon the nature of roots of the equation (3) which, in turn, will depend upon the damping. Accordingly, we consider three cases.

Underdamping ($c^2 < 4mk$).

When damping force is further decreased so that $c^2 < 4mk$ or $\alpha^2 < \omega_0^2$, then the roots (4) are complex conjugate. Let $\omega_0^2 - \alpha^2 = \omega_1^2$, then $\lambda_1 = -\alpha + i\omega_1$, $\lambda_2 = -\alpha - i\omega_1$ (α, ω_1 real).



Consequently, the general solution of (2), in this case, is

$$x = e^{-\alpha t}\{c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\}$$

which represents damped oscillations, i.e., x oscillates with an amplitude which falls off exponentially and tends to zero as $t \to \infty$.

This motion is underdamped and transient.

The time period T of these damped oscillations is given by

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega_0^2 - \alpha^2}} = \frac{2\pi}{\sqrt{k/m - c^2/4m^2}} \quad ...(6)$$

and its frequency f is given by

$$f = \frac{1}{T}\sqrt{k/m - c^2/4m^2} \; 2\pi \quad ...(7)$$

The frequency f as given by (7) is called the natural frequency of the system. Also, evidently the frequency (7) is decreased by the damping.

Q. 15. Solve the system of the differential equations for x.

$$\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x$$

(2018)

Ans. Let $\frac{dx}{dt} = 2y$ ...(1)

$$\frac{dy}{dt} = 2z \quad ....(2)$$

$$\frac{dz}{dt} = 2x \quad ...(3)$$

Now, differentiating equation (1), w.r.t. t, we get

$$\frac{d^2x}{dt^2} = \frac{2dy}{dt} = 2(2z) = 4z \quad ...(4)$$

Again, differentialting equation (4), w.r.t. t,

$$\Rightarrow \frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 8x, \; \Rightarrow \frac{d^3x}{dt^3} = 8x \quad ...(5)$$

$$\Rightarrow (D^3 - 8)x = 0$$

Now, corresponding auxiliary equation is -

$$\Rightarrow m^3 - 8 = 0, \; \Rightarrow (m-2)(m^2 + 2m + 4) = 0$$

$$\Rightarrow m = 2, -1 \sqrt{3}.$$

Hence, general solution is

$$x = C_1e^{2t} + e^{-t}(C_1 \cos \sqrt{3}\, t + C_2 \sin \sqrt{3}\, t)$$

Q. 16. If $\vec{r}$ is the position vector of a point P (x, y, z) with respect to origin O, then find the gradient of a scalar function f (r).

(2018)

Ans. Let $\overrightarrow{OP} = r$ be the position vector of point P (x, y, z) w.r.t. origin O.

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Paper -3

उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े उसे त्याग दो।



KEY POINT AU



KEY POINT AU

$r = x\hat{i} + y\hat{j} + z\hat{k} = \Sigma x\hat{i}$

$r = \sqrt{x^2 + y^2 + z^2}$ ...(1)

$\frac{\partial r}{\partial x} = \hat{i}, \quad \frac{\partial r}{\partial y} = \hat{j}, \quad \frac{\partial r}{\partial z} = \hat{k}$

& $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

The gradient of a scalar function $\phi = f(r)$ is -

$\text{grad } f(r) = \nabla f(r) = \Sigma \hat{i} \frac{\partial}{\partial x} f(r)$

$= f'(r) \frac{\partial r}{\partial x}$

$= \Sigma \hat{i} f'(r) \frac{x}{r} = \frac{f'(r)}{r} \Sigma x\hat{i}$

$= \frac{f'(r)}{r} \vec{r} = f'(r) \cdot \hat{r}$

$\therefore \text{grad } f(r) = f'(r) \cdot \hat{r}$.

Q. 17. Prove that

$$\text{div}\{f(x)\,\hat{r}\} = \frac{1}{r^2} \cdot \frac{d}{dr}\{r^2 f(r)\}$$ **(201**

Ans. $\text{div}(f(r)\,\hat{r}) = \text{div}\left[\frac{f(r)}{r}\vec{r}\right]$

$\therefore \text{div}(f(r)\,\hat{r}) = \frac{f(r)}{r} \text{ div } \hat{r} + \hat{r} \text{ grad}\left[\frac{f(r)}{r}\right]$

$\frac{3f(r)}{r} + \vec{r} \cdot \hat{r} \frac{\partial}{\partial r}\left[\frac{f(r)}{r}\right] = \frac{3f(r)}{r} + \vec{r} \cdot \frac{\partial}{\partial r}\left[\frac{f(r)}{r}\right]$

$= \frac{3f(r)}{r} + \left[-\frac{f(r)}{r} + \frac{1}{r} f'\right]$

$= \frac{f(r)}{r} + f' = \frac{1}{r^2} \frac{\partial}{\partial r}\{r^2 f(r)\}$

Q. 18. For a vector function $\vec{a}$, show that

$$\vec{a} \times \text{curl } \vec{a} = \text{grad}\left[\frac{a^2}{2}\right] - (\vec{a} \cdot \nabla)\vec{a}$$

Discuss if $\vec{a}$ is irrotational. **(2018)**

Ans. $\text{grad}\left[\frac{a^2}{2}\right] - (\vec{a}.\nabla)\vec{a}$

$= \text{grad}\left[\frac{\vec{a}.\vec{a}}{2}\right] - (\vec{a}.\nabla)\vec{a}, \quad = \frac{1}{2}\text{grad}(\vec{a}.\vec{a}) - (\vec{a}.\nabla)\vec{a}$

$\frac{1}{2}\text{grad}(\vec{a}.\vec{a}) - (\vec{a}.\nabla)\vec{a}$.

$= \frac{1}{2}\left[2(\vec{a} \times \text{curl}\,\vec{a} + \vec{a}\,\text{curl}\,\vec{a} + (\vec{a}.\nabla)\vec{a})\right] - (\vec{a}.\nabla)\vec{a}$

$= \vec{a} \times \text{curl } \vec{a}$

Hence, $(\vec{a} \times \text{curl}\,\vec{a}) = \text{grad}\left[\frac{a^2}{2}\right] - (\vec{a} \cdot \nabla)\vec{a}$

If $\vec{a}$ is irrotational, then

Curl $\vec{a} = 0$

$\therefore 0 = \text{grad grad}\left[\frac{a^2}{2}\right] - (\vec{a} - \nabla)\vec{a}$

Q. 19. Show that a necessary & sufficient condition for the vector $\vec{a}(t)$ to have constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = 0$. **(2019)**

Ans. Let a be a vector function of the scalar variable t. Let A be a unit vector in the direction of a. If a be the magnitude of a, then a = aA.

$\therefore \frac{da}{dt} = a\frac{dA}{dt} + \frac{da}{dt}A$

Hence $a \times \frac{da}{dt} = (aA) \times \left(a\frac{dA}{dt} + \frac{da}{dt}A\right)$

$$= a^2 A \times \frac{dA}{dt} + a\frac{da}{dt} A \times A$$

$$= a^2 A \times \frac{dA}{dt} \quad [\because A \times A = 0] \qquad ...(1)$$

The condition is necessary . Suppose a has a constant direction. Then A is a constant vector because it has constant direction as well as constant magnitude.

Therefore $\frac{dA}{dt} = 0$

$\therefore$ From (1), we get $a \times \frac{da}{dt} = a^2 A \times 0 = 0$

Therefore the condition is necessary.

The condition is sufficient. Suppose that $a \times \frac{da}{dt} = 0$

Then from (1), we get

$$a^2 A \times \frac{dA}{dt} = 0 \text{ or } A \times \frac{d A}{dt} - 0 \qquad ...(2)$$

Since A is of constant length, therefore $A \bullet \frac{dA}{dt} = 0$

From (2) and (3), we get $\frac{dA}{dt} = 0$

Hence A is a constant vecotor i.e., the direction of a is constant.

Q. 20. *Show that the directional derivative of a scalar point functio f at a point P(x, y, z) is direction of a unit vector* $\vec{a}$ *is give by -*

$$\frac{df}{ds} = \nabla f . \hat{a}$$ **(201**

Ans. Let f (x, y, z) define a scalar field in the region R. Let x = xi + yj + z denote the position vector of any point P (x, y, z) in this region. If s denotes th distance of P from some fixed point A in the direction of $\vec{a}$,. Therefore $\frac{d r}{ds}$ is unit vector at P in this direction i.e., $\frac{d r}{ds} = \hat{a}$

But r = xi + yj + zk.

$$\therefore \frac{d r}{dx} = \frac{dx}{ds} i + \frac{dy}{ds} j + \frac{dz}{ds} z = \hat{a}.$$

$$\nabla f = \left(i\frac{d f}{dx} + j\frac{d f}{dy} + k\frac{d f}{dz} \right).\left(\frac{d x}{d s} i + \frac{d y}{d s} j + \frac{d z}{d s} \right)$$

$$= \frac{d f}{d x}\frac{d x}{d s} + \frac{d f}{d y}\frac{d y}{d s} + \frac{d f}{d z}\frac{d z}{d s} = \frac{d f}{d s}$$

Now

= directional derivative of f at P in the direction of $\vec{a}$.

Alternative Proof: Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\vec{a}$. If *l*, m, n are the direction cosines of the line PQ, then *l*i +*m*j + *n*k = the unit vector in the direction of PQ = $\vec{a}$. Further if PQ = b, then the co-ordinates of Q are (x, + *l*δs, y + mδs, z + nδs). Now the directions derivative of f at P in the direction of $\vec{a}$ is

$$= \lim_{Q \to P} \frac{f(Q) - f(P)}{PQ} = \lim_{\delta s \to 0} \frac{f(x + l\delta s, y + m\delta s, z + n\delta s) - f(x, y, z)}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{f(x, y, z) + \left(l\delta s \frac{\partial f}{\partial x} + m\delta s \frac{\partial f}{\partial y} + n\delta s \frac{\partial f}{\partial z} \right) + ... - f(x, y, z)}{\delta s}$$

on expanding by Taylor's theorem

$$= l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right).(li + mj + nk) = \nabla f . \vec{a} .$$

Chapter - 5
Vector Calculus

Q. 1. Evaluate the double integral-

$\int_0^1 \int_{x^2}^{x} (1-xy)\, dy\, dx$. **(2014)**

Ans. $\int_0^1 [\int_{x^2}^{x} (1-xy)\, dy]\, dx$

$$\Rightarrow \int_0^1 \left[y - \frac{xy^2}{2} \right]_{x^2}^{x} dx$$

$$\Rightarrow \int_0^1 \left(x - \frac{x^3}{3} - x^2 + \frac{x^5}{2} \right) dx$$

$$\Rightarrow \left[\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} + \frac{x^6}{12} \right]_0^1$$

$$\Rightarrow \frac{1}{2} - \frac{1}{12} - \frac{1}{3} + \frac{1}{12} = \frac{1}{6}$$

Q. 2. Evaluate the line integral. $\int_C [x^2y\, dx + (x-z)dy + xyzdz]$ where where, C is the arc if the parabola $y = x^2$ in the plane $z = 2$ from (0, 0, 2) to (1, 1, 2). **(2014)**

Ans. $\int_c x^2y\, dx + (x-z)dy + xyz\, dz$.

$$\Rightarrow \int_0^1 x^2 (x^2) dx + \int_0^1 \left(\sqrt{5} - 2\right) dy + \int 0$$

$$\Rightarrow \int_0^1 x^4\, dx + \int_0^1 y^{1/2} - 2\, dy + 0$$

$$\Rightarrow \left[\frac{x^5}{5} \right]_0^1 + \left[\frac{2y^{3/2}}{3} - 2y \right]_0^1$$

$$\Rightarrow \frac{1}{5} + \frac{2}{3} - 2 \qquad \Rightarrow \frac{3+10-30}{15} = \frac{-17}{15}$$

Q. 3. State and prove Gauss's Divergence theorem. **(2014, 2015, 2019)**

Ans. Theorem - If V be a closed bounded region in space whose boundary is a piecewise smooth orientable surfacer S and F (x, y, z) be a vector function which is continuous and has continuous first partial derivatives in some domain containin V, then the volume integral of the divergence of F taken over the volume V enclosed by the surface S is equal to the surface integral of the normal component of F taken over the surface S, i.e.

$$\iiint_V \text{div } F.\, dV = \iint_S F.\, \hat{n}\, dS \qquad ...(1)$$

where is the outward unit normal vector to S w.r.t V.

In terms of rectangulr components, if $F = F_1\, \hat{i}\ F_2\, \hat{j} + F_3\, \hat{k}$, then (1) assumes the form

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S \left(F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \right).\hat{n}.dS \qquad ...(2)$$

and further, if $\hat{n} = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}$ where α, β, γ are direction angles of $\hat{n}$, then $\hat{n} \cdot \hat{i} = \cos\alpha$, $\hat{n} \cdot \hat{j} = \cos\beta$, $= \hat{n} \cdot \hat{k} \cos\gamma$. . Consequently, (2) takes the form

$$\iiint_V \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS \qquad ..(3)$$

and since orthogonal projections of dS on yz, zx and xy planes are respectively $dy\, dz = dS \cos\alpha$, $dz\, dx = dS \cos\beta$ and $dx\, dy = dS \cos\gamma$ (3), may also be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \iint_S (F_1\, dydz + F_2\, dzdx + F_3 dxdy) dS \qquad ...(4)$$

Proof: First we prove that

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_S F_3 \cos\gamma\, dS = \iint_S F_3 \hat{k}.\hat{n}\, dS \qquad ...(i)$$

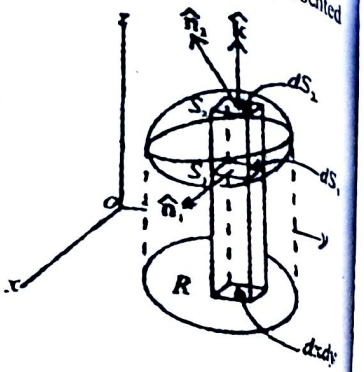
for a special region V bounded by a piecewise smooth orientable surface S with the property that any straight line parallel to any one of the coordinate axes and intersecting V has at most one segment (or a single point) in common with V.

Let R be the orthogonal projection of S on the xy-plane, let S be subdivided into three surfaces S_1, S_2 and S_3. Let the equation of the bottom S_1 of S be $z = f(x, y)$ and that of the top S_2 of S be $z = g(x,y)$. The vertical portion S_3

of S, if any, is omitted for the reasons stated below. The region V is represented by

$f(x, y) \le z \le g(x, y)$

where $(x, y) \in R$. Now, consider an elementary prism parallel to z-axis having the area of the normal section as $dx\,dy$. Let this prism cut a portion dS_1 of the bottom S_1 and a portion dS_2 of the top S_2, and let n_1 and n_2 be the outward unit normal to these protions. Then



$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R dx\,dy \int_f^g \frac{\partial F_3}{\partial z}\,dz$$

$$= \iint_R dx\,dy \left[F_3(x, y, z)\right]_f^g$$

$$= \iint_R dx\,dy\,[F_3(x, y, g) - F_3(x, y, f)]$$

$$= \iint_R F_3[x, y, g(x, y)]\,dx\,dy - \iint_R F_3[x, y, f(x, y)]\,dx\,dy \quad \text{(ii)}$$

Now,

$$\iint_S F_3 \cos\gamma\, dS = \iint_{S_1} F_3 \cos\gamma_1\, dS_1 + \iint_{S_2} F_3 \cos\gamma_2\, dS_2 + \iint_{S_3} F_3 \cos\gamma_3\, dS_3$$

On S_1, γ_1 is obtuse so that $\cos\gamma_1 < 0$ or $\hat{k}.\hat{n}_1 < 0$. Consequently, $dx\,dy = -(\hat{k}.\hat{n}_1)\,dS_1$. One S_2, γ_2 is acute so that $\cos\gamma_2 > 0$ or $\hat{k}.\hat{n}_2 > 0$.

There for $dx\,dy = (\hat{k}.\hat{n}_2)\,dS_2$. On S_3, $\gamma_3 = \pi/2$ so that $\cos\gamma_3 = 0$ or $\hat{k}.\hat{n}_3 = 0$

Hence the lateral surface S_3 does not contribute to (iii).

Now, (ii) takes the form

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_{S_2} F_3(\hat{k}.\hat{n}_2)\,dS_2 - \iint_{S_1} F_3(-\hat{k}.\hat{n}_2)\,dS_1$$

$$\iint_{S_1} F_3 \hat{k}.\hat{n}_1\,dS_1 + \iint_{S_2} F_3 \hat{k}.\hat{n}_2\,dS_2$$

$$\iint_S F_3 \hat{k}.\hat{n}\,dS \quad \text{...(iv)}$$

Similarly, projecting S on yz- plane and zx- plane, we get

$$\iiint_V \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \iint_S F_1 \hat{n}.dS \quad \text{...(v)}$$

$$\iiint_V \frac{\partial F_2}{\partial x}\,dx\,dy\,dz = \iint_S F_2 \hat{n}.dS \quad \text{(vi)}$$

Addigin (iv), (v) and (vi), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}).\hat{n}.dS$$

or $\iiint_V \text{div}\, F\, dV = \iint_S F.\hat{n}.dS$

This establishes Gauss's thorem for special regions.

Q. 4. State stokes theorem and prove it. (2014, 2016)

Ans. **Theorem** - If S be a piecewise smooth oriented surface whose boundary is a piecewise smooth simple closed curve C and if $F(x, y, z)$ be a vector function which is continuous and has continuous first partial derivatives in a region in space containing S, then the surface integral of the component of curl F normal to the surface S is equal to the line integral of F taken round the curve C, the normal to S being taken in the righthanded sense with respect to orientation of C, i.e.,

$$\iint_S (\text{curl F}).\hat{n}.dS = \oint_C F.dr \quad \text{...(1)}$$

In terms of rectangular components if F = , then takes the form

$$\iint_S \left\{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right\}.\hat{n}.dS \quad \text{...(2)}$$

$$= \oint (F_1\,dx + F_2\,dy + F_3\,dz)$$

Further, if $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$, where α, β, γ are the direction angles then becomes

$$\iint_S \left\{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right\} dS$$

$$= \oint_C (F_1\,dx + F_2\,dy + F_3\,dz) \quad \text{...(3)}$$

and taking the orthogonal projections of dS on the coordinate planes, we get

$$\iint_S \left\{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\,dz + \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial x}\right)dz\,dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy\right\}$$

$$= \oint_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

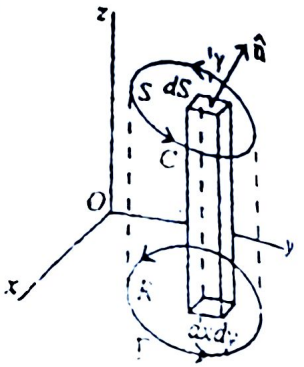
Proof: Consider a surface S bounded by a closed curve C, whose orthogonal projections on xy, yz and zx planes are regions bounded by simple closed curves. Further, assume that S can be represented by

$z = f(x, y)$ or $y = g(z, x)$ or $x = h(y, z)$ ...(i)

where f, g, h are continuous functions and have continuous first partial derivatives.

Now, consider Stoke's theorem in the form (3). First we prove that the integrals involving F_1 in (3) are equal, i.e.

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \oint_C F_1 dx \quad ..(ii)$$



First consider the RHS of (ii). Let R be the orthogonal projection of S on xy-plane and be bounded by the oriented simple curve . Using the representation $z = f(x, y)$ of S as given in (i), we can write RHS of (ii) as

$$\oint_C F_1(x, y, z)\, dx = \oint_\Gamma F_1[x, y, f(x, y)]\, dx \quad ...(iii)$$

Applying Green's theorem in the plane, to the functions $F_1 = F_1[x, y, f(x,y)]$ and $F_2 = 0$, we get

$$\oint_C F_1[x, y, f(x, y)]\, dx = -\iint_R \frac{\partial F_1[x, y, f(x, y)]}{\partial y} dx\, dy \quad ...(iv)$$

Further, since $F_1 = F_1(x, y, z)$ and $z = f(x, y)$, we have

$$\frac{\partial F_1[x, y, f(x, y)]}{\partial y} = \frac{\partial F(x, y, z)}{\partial y} + \frac{\partial F_1(x, y, z)}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} \quad ...(v)$$

Combining (iii), (iv) and (v), we find that RHS of (ii) reduces to

$$\oint_C F_1 dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial y}\frac{\partial f}{\partial y}\right) dx\, dy \quad (A)$$

Now, consider LHS of (ii). Let the surface S: $z = f(x, y)$ be written as $\phi(x, y, z) = z - f(x, y) = 0$. Then, with $|\nabla\phi| = a$,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{a}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{1}{a}\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

Also, $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k} = (\cos\alpha, \cos\beta, \cos\gamma)$

$$\cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x}, \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y}, \cos\gamma = \frac{1}{a} \quad ...(vi)$$

Further, since the projection $dx\,dy = dS\,|\,\hat{n}.\hat{k}\,| = dS\cos\gamma = dS/a$, the LHS of (ii) takes the form

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \iint_R \left\{\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\left(\frac{1}{2}\right)\right\}(a\,dx\,dy)$$

$$= -\int_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy \quad (B)$$

Now, combining (A) and (B), we get

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \oint_C F_1 dx \quad (C)$$

Similarly, taking representation of S as $y = g(z, x)$ and $x = h(y, z)$ and the orthogonal projections of S on zx-plane and yz-plane, respectively, we get

$$\iint_S \left(\frac{\partial F_1}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) dS = \oint_C F_2 dy \quad (D)$$

$$\iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) dS = \oint_C F_3 dz \quad (E)$$

Adding (C), (D) and (E), we get

$$\iint_S \left\{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_3}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right\} dS$$

$$\oint_C (F_1 dx + F_2 dy + F_3 dz)$$

or $\iint_S (\text{curl } F).\ \hat{n}\, dS = \oint_C F.\, dr$

Q. 5. *Evaluate the line integral*
$\int F.\, dr$, where $F = (x + y) + (2x - z) + (y + x)$
and C is the boundary of the triangle with vertices (2,0,0), (0, 3, 0) & (0, 0, 6) ***(2014)***

Ans. $\int_C F.\, dr = \int_{AB} F.\, dr + \int_{BC} F.\, dr + \int_{CA} F.\, dr$

Now, equations of the line AB, in parametric form, are

$$\frac{x-2}{0-2} = \frac{y-0}{3-0} = \frac{z-0}{0-0}, \quad \text{or } x = 2 - 2t, y = 3t, z = 0$$

and the points A and B correspond to $t = 0$ and $t = 1$

$$\therefore \int_{AB} F.dr = \int_0^1 \{(2 - 2t + 3t)(-2dt) + (4 - 4t - 0)(3dt) + (3t + 0)0\}$$



$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \{(2t-0)(2dt) + (4t^2-6+6t)(0+0+6-6t)(-6dt)\}$$

$$= \int_0^1 (44t^2 - 36)\,dt = (22t^3 - 36t) = -14$$

For C_2: $\frac{x-0}{2-0} = \frac{y-0}{0-3} = \frac{z-6}{0-6}$, or $x = 2t,\ y = 0,\ z = 6 - 6t$

For C_2, $t = 0$ and for A, $t = 1$

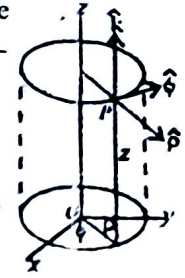
$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \{(2t-0)(2dt) + (4t^2-6+6t)(0+0+6-6t)(-6dt)\}$$

$$= \int_0^1 (44t^2 - 36)\,dt = (22t^3 - 36t) = -14$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 1 - 36 - 16 = -21$$

Q. 6. Define cylindrical polar coordinates (ρ, ϕ, z) of a point P. (2015)

Ans. The transformation equations connecting the cartesian coordinates (x, y, z) and the cylindrical polar coordinates (ρ, θ, ϕ) of a point P are given by



$$x = \rho\cos\phi,\ y = \rho\sin\theta\sin\phi,\ z = z$$

$$\rho > 0,\ 0 \le \phi \le 2\pi,\ -\infty < z < \infty$$

Consequently, the position vector r of the point P, is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\rho\cos\phi)\mathbf{i} + (\rho\sin\phi)\mathbf{j} + z\mathbf{k}$$

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3$$

$$\frac{\partial \mathbf{r}}{\partial \rho}d\rho + \frac{\partial \mathbf{r}}{\partial \phi}d\phi + \frac{\partial \mathbf{r}}{\partial z}dz$$

$$(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j})d\rho + (-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j})d\phi + \mathbf{k}\,dz$$

$$(\cos\phi\,d\rho - \rho\sin\phi\,d\phi)\mathbf{i} + (\sin\phi\,d\rho + \rho\cos\phi\,d\phi)\mathbf{j} + \mathbf{k}\,dz$$

$$ds^2 = dr\cdot dr = (\cos\phi\,d\rho - \rho\sin\phi\,d\phi)^2 + (\sin\phi\,d\rho + \rho\cos\phi\,d\phi)^2 + (dz)^2$$

$$= (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2 \quad \text{...(2)}$$

Hence, for cylindrical coordinates

$u_1 = \rho, u_2 = \phi, u_3 = z$. $\hat{e}_1 = \hat{\rho}$. $\hat{e}_2 = \hat{\phi}$. $\hat{e}_3 = \hat{k}$

$h_1 = 1, h_2 = \rho, h_3 = 1$ ...(3)

Substituting these values, we get for the scalar function $f(\rho, \phi, z)$ and vector function $\mathbf{a} = \mathbf{a}(\rho, \phi, z) = a_\rho\,\hat{\rho} + a_\phi\,\hat{\phi} + a_z\,\hat{k}$ that

$$\text{grad } f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{k}\frac{\partial a_z}{\partial z} \quad \text{...(4)}$$

$$\text{div } \mathbf{a} = \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial a_\phi}{\partial \phi} + \frac{\partial a_z}{\partial z} \quad \text{...(5)}$$

$$\text{Curl } \mathbf{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix} \quad \text{...(6)}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{...(7)}$$

Q. 7. Find the work done by the force $\vec{F}(x, y, z) = 4xy + 8y + 2$ in displacing a unit mass, along a circle once, $x^2 + y^2 = 4$, $z = 0$. (2015)

Ans. Work $= \int_C \vec{F}.d\vec{r}$

$$= \int_C \left(4xy\,\hat{i} + 8y\,\hat{j} + 2\hat{k}\right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_C 4xy.dx + \int_C 8\,y\,dy + \int 2dz = \int_C 4x\sqrt{4 - x^2}\,dx - \int_C 8\,y\,dy - 0$$

Let $4 - x^2 = t^2$ then, $-2x\,dx = 2t\,dt$

$\therefore x\,dx = -t\,dt$.

$$\therefore \int_C 4(-t)(t\,dt) + \int_C 8y\,dy = \left[-\frac{4t^3}{3} + \frac{8y^2}{2}\right]_C$$

$$= \left[\frac{-4(4 - x^2)^{3/2}}{3} + \frac{8y^2}{2}\right]_C$$

Q. 8. State & proof Greens theorem. **(2015, 2016).**

Ans. Green's theorem: Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot n\, dS.$$

Proof: By divergence theorem,

we have $\iiint_V F\, dV = \iint_S F \cdot n\, dS$.

Putting $F = \phi \nabla \psi$, we get $\nabla \cdot F = \nabla \cdot (\phi \nabla \psi)$

$= \phi(\nabla \cdot \nabla \psi) + (\nabla \phi) \cdot (\nabla \psi) = \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)$.

Also. $F \cdot n = (\phi \nabla \psi) \cdot n$

$\therefore$ divergence theorem gives

$$\iiint_V [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)]\, dV = \iint_S (\phi \nabla \psi) \cdot n\, dS \quad ...(1)$$

This is callled Green's first identity or theorem.

Interchanging ϕ and ψ in (1), we get

Subtracting (2) from (1), we get

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot n dS \quad ...(3)$$

This is called Green's second identity or Green's theorem in symmetrical form.

Since $\nabla \psi = \frac{\partial \Psi}{\partial n} n$ and $\nabla \phi = \frac{\partial \phi}{\partial n} n$, therefore

$$(\phi \nabla \psi - \psi \nabla \phi) \cdot n = \left(\phi \frac{\partial \Psi}{\partial n} n - \Psi \frac{\partial \phi}{\partial n} n \right) \cdot n$$

Hence (3) can also be written as

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S \left(\phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \phi}{\partial n} \right) dS.$$

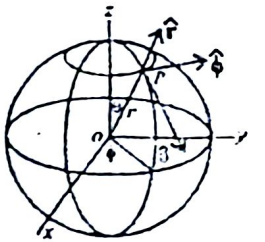
Q. 9. Define the spherical polar co-ordinates (r, q , f) and express the unit base vector $\hat{r}, \hat{\phi}, \hat{\theta}$ along the co-ordinate curve in terms of $\hat{i}, \hat{j}, \hat{k}$, the unit base vectors along the x, y, z axes respectively. **(2016).**

Ans. The transformation equations connecting the cartesian coordinates (x, y, z) and the spherical polar coordinates (r, θ, ϕ) of a point P are given by

$x = r \sin q \cos f$, $y = r \sin q \sin f$, $z = r \cos q$

$r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\theta$

Hence, if r is the position vector of the point P,



then

$$r = x\hat{i} + y\hat{j} + z\hat{k} = (r \sin\theta \cos\phi)\,\hat{i} + (r \sin\theta \sin\phi)\,\hat{j} + (r \cos\theta)\,\hat{k}$$

$$\therefore dr = \frac{\partial r}{\partial r} dr + \frac{\partial r}{\partial \theta} d\theta + \frac{\partial r}{\partial \phi}$$

$= (\hat{i} \sin\theta \cos\phi + \hat{j} \sin\theta \sin\phi + k \cos\theta)\, dr + (\hat{i}\, r \cos\theta \cos\phi + \hat{j}\, r \cos\theta \sin\phi - r \sin\theta)\, d\theta + (-r \sin\theta \sin\phi + r \sin\theta \cos\phi)\, d\phi$

$= (\sin\theta \cos\phi\, dr + r \cos\theta \cos\phi\, d\theta - r \sin\theta \sin\phi\, d\phi) + (\sin\theta \sin\phi\, dr + r \cos\theta \sin\phi\, d\theta + r \sin\theta \cos\phi\, d\phi) + (\cos\theta\, dr - r \sin\theta\, d\theta)$

$\therefore ds^2 = dr \cdot dr = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2$ ...(2)

Hence, for spherical coordinates,

$u_1 = r, u_2 = \theta, u_3 = \phi$; $\hat{e}_1 = \hat{r}, \hat{e}_2 = \hat{\theta}, \hat{e}_3 = \hat{\phi}$,

$h_1 = 1, h_2 = r, h_3 = r \sin\theta$ ...(3)

for the scalar function $f(r, \theta, \phi)$ and the vector function $a(r, \theta, \phi) = a_r \hat{r} + a_\theta \hat{\theta} + a_\phi \hat{\phi}$, reduce to yield

$$\text{grad} f = \hat{r} \frac{\partial f}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial f}{\partial \hat{\phi}} \quad ...(4)$$

$$\text{div}\, a = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 a_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta a_\theta) + \frac{1}{r \sin\theta} \frac{\partial a_\phi}{\partial \phi} \quad ...(5)$$

$$\text{Curl}\, a = \frac{1}{r \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin\theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ a_r & r a_\theta & r \sin\theta a_\phi \end{vmatrix} \quad ...(6)$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} \quad ...(7)$$

Q. 10. Verify the stokes theorem for the function $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$, where S is the circular disc $x^2 + y^2 \leq 1, z = 0$. **(2016)**

Ans. The equation of curve, C are $x^2 + y^2 = 1, z = 0$

Suppose $x = \cos t, y = \sin t, z = 0, 0 \leq t 2\pi$ are parametric equation of

C. Then,

$\oint_c F.dr = \oint_c (-y\hat{i} + x\hat{j} + z\hat{k}).(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \oint (-ydx + xdy + z^2dz) = \oint -ydx + xdy + 0.$

$= \oint -\sin t.(-\sin t)\,dt + \cos t(\cos t)\,dt = \oint_0^{2\pi} \sin^2 t + \cos^{2t})\,dt$

$= \oint_0^{2\pi} dt = 2\pi.$

Now, ...(1)

let us evaluate $\iint$ Curl F.n dS.

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z^2 \end{vmatrix}$$

Curl $F = O\,\hat{i} + O\,\hat{j}\,(1+1)\,\hat{k} = 2\,\hat{k}$.

$\iint_S$ Curl F. n dS $= \iint_S (2\,\hat{k}).k\,ds.$

$\Rightarrow \iint_S (2)\,dX.$

$\Rightarrow 2\iint ds$ $\{\because S = \text{Areas of circle} = \pi(1)^2 = \pi\}$

$\Rightarrow 2\pi.$

Hence from (1) & (2), the theorem is verified. ...(2)

Q. 11. ***Evaluate $\iint_S \bar{F}.\hat{n}\,dS$ where $\bar{F} = 4x\,\hat{i} - 3y^2\,\hat{j} + z^2\,\hat{k}$, and S is the entire surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ & $z = 3$.*** ***(2016, 2019).***

Ans. From Gausss Divergence theorem.

$\iint_S \bar{F}.\hat{n}\,dS = \iiint_V \nabla . \bar{F}\,dv.$

$\bar{F} = 4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k}.$

$$\nabla . F = \text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\nabla . F = 4 - 4 + 2z = 2z.$

$$\iiint_V (\text{div } F)\,dV = \int_{z=0}^{3}\int_{y=-2}^{+2}\int_{x=-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} 2z\,dx\,dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=-2}^{2}\left[2zx\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy\,dz = \int_{z=0}^{3}\int_{y=-2}^{2} 4z\sqrt{4-y^2}\,dy\,dz$$

$$= \int_{y=-2}^{2}\left[4.\frac{z^2}{2}\sqrt{4-y^2}\right]_{z=0}^{3} dy = 18\int_{y=-2}^{2}\left(\sqrt{4-y^2}\right)dy$$

$$= 18\int_0^2 \sqrt{4-y^2}\,dy = 18\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$= 18\,[\sin^{-1} 1] = 18 \times 2 \times \frac{\pi}{2} = 18\,\pi.$

Q. 12. ***Establish the Green's theorem is a particular case of stokes theorem. If R is plane region enclosed by the Curve C, then using Green's theorem for plane regions, establish that the area A(R) of the region R can be expressed as a line integral. Hence obtain P(x, y) and Q(x, y) such that,***

$$A(R) = \int_c (P\,dx + Q\,dy)$$ ***(2017)***

Ans. Let $F(x, y)\,\hat{i} = F_1(x, y)\,\hat{j} + F_2(x, y)\,\hat{k}$ be a vector function which is continuously differentiable in a simply connected bounded closed simple closed region R whose boundary is a piecewise smooth closed curve C. Then dS = dx $\hat{i}$ dy, dr = dx dy $\hat{j}$ and

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\therefore\ \hat{n}.\text{Curl } F = \hat{k}\,\text{curl } F = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ ($\hat{k} \perp$ xy - plane)

$F.dr = (F_1\,\hat{i} + F_2\,\hat{j}).(dx\,\hat{i} + dy\,\hat{j}) = F_1\,dx + F_2\,dy$

Consequently, the Stoke's theorem takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

which is the theorem. Hence, the Green's theorem in the plane is a special case of Stoke's theorem.

Q. 13. *Verify the stokes theorem for following details-*

$\vec{F} = -y^3 \hat{i} + x^2 \hat{j}$

S: circular disc $x^2 + y^2 \leq 1$, $z = 0$

(2017)

Ans. The boundary C of S is a circle in xy-plane of radius 1 & centre of origin.

Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leq t < 2\alpha$ are parametric equations of C.

Then, $\oint_C F.\, dr = (-y^3 \hat{i} + x^3 \hat{j})(dx \hat{i} + dy \hat{j} + dz \hat{k})$.

$$= \oint_C (-y^3 \hat{i} + x^3 \hat{j}) = \int_{t=0}^{2\pi} \left\{ y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt} \right\} dt$$

$$= \int_0^{2\pi} [-\sin^3 t(-\sin t) + \cos^3 t (\cos t)\, dt = \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\left\{ \frac{3.1}{4.2} \cdot \frac{\pi}{2} + \frac{3.1}{4.2} \cdot \frac{\pi}{2} \right\} = \frac{3\pi}{2}$$

$$\text{Also } \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

Here n = k, because surface is in xy-plane

$\therefore (\nabla \times F).\, n = (3x^2 + 3y^2)\, k.k = 3(x^2 + y^2)$

$\therefore \iint_S (\nabla \times F).\, n\, dS = 3 \iint_S (x^2 + y^2)\, ds$

$$= 3\int_{0=0}^{2\pi} \int_{r=0}^{1} r^2\, r\, d\theta\, dr = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3\pi}{2}$$

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