Chapter 1 Vectors

*** Scalars & Vectors: -**

(1) Scalars : Those physical quantities, which have magnitude only but no particular direction in space, are called scalars. Examples of scalar quantities are mass, volume, density, temperature, electric potential, charge etc.

(2) Vectors : Those physical quantities, which have directions as well as magnitude, are called vectors. Moreover, a vector quantity must satisfy the parallelogram law of addition. Familiar examples of vectors are displacement, velocity, acceleration, force, momentum, magnetic induction, magnetic and electric intensities etc.

*** Law of Scalar Addition and Multiplication: -**

(1) Law of Commutation : $A+B=B+A$

$A\times B=B\times A$

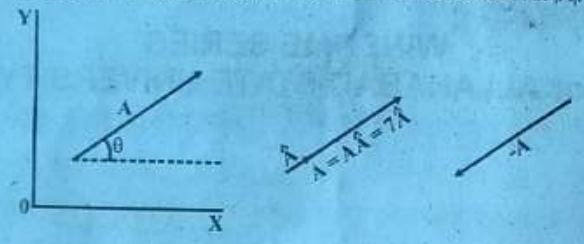
(2) Law of Association : $A+(B+C)=(A+B)+C$

$A\times(B\times C)=(A\times B)\times C$

(3) Law of Distribution : $A\times(B+C)=A\times B+A\times C$

$(A+B)\times C=A\times C+B\times C$

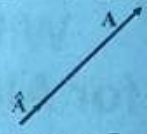
*** Representation of Vector :** A vector is represented by a letter and putting an arrow over it. For Ex. $\vec{A}$. The negative of vector A, is the vector – A, having the same magnitude as A but being oppositely directed to it. It is, therefore, represented by an arrow equal in length to that representing A but antiparallel to it, as shown in fig. (c). The magnitude of a vector is also called as mod A or |A|.



(a) (b) (c)

Unit Vector : $\vec{A}$ vector, whose magnitude is unity, is called a unit vector. A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$ and is given by Figure.

$\hat{A}=\frac{\vec{A}}{A}$ i.e., $A=A\hat{A}$.



*** Product & Division of Vector by Scalar :** The product of a vector $\vec{A}$ and a

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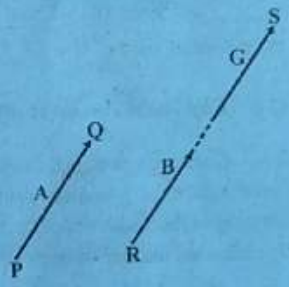
scalar m is a vector m $\vec{A}$ whose magnitude is m times the magnitude of $\vec{A}$ and which is similarly or oppositely directed to $\vec{A}$ according as the scalar m is positive or negative. Thus,

$|m\ \vec{A}|=mA$.

The division of vector $\vec{A}$ by a non-zero scalar m is defined as the multiplication of vector $\vec{A}$ by 1/m.

*** Equality of Vectors :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, if they have equal magnitudes and are parallel, pointing in the same direction figure. Thus, if $\vec{A}$ and $\vec{B}$ are equal vectors, then

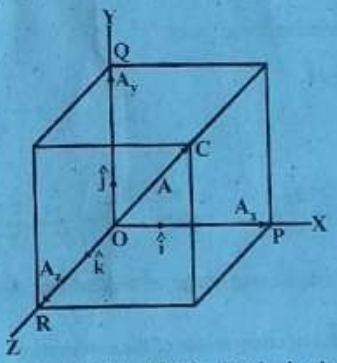
$\vec{A}=\vec{B}$



*** Parallel OR Collinear Vector :** Those vectors, which have the same line of action or lines of action parallel to the same line are called collinear or parallel vectors; they need not have the same direction.

*** Components of Vector :** The components of a vector $\vec{A}$ are any number of vectors whose sum is $\vec{A}$. If $\vec{A}_x, \vec{A}_y$ and $\vec{A}_z$ are the vector intercepts along X, Y and Z axes respectively, then

$\vec{A}=\vec{A}_x+\vec{A}_y+\vec{A}_z$ .....(i)



Let $\hat{i},\hat{j},\hat{k}$ represent vectors of unit magnitude along the X, Y and Z axes respectively; these unit vectors along the axes will be frequently used in the later work. Now, if $|A_x|$, $|A_y|$ and $|A_z|$ are the magnitudes of the vectors $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ respectively, then

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$\vec{A}_x = |A_x|\hat{i}, \vec{A}_y = |A_y|\hat{j}, \vec{A}_z = |A_z|\hat{k}$

Therefore, equation (i) can be written as

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad \text{.....(ii)}$$

magnitude of vector $|A| = \sqrt{|A_x|^2 + |A_y|^2 + |A_z|^2}$

Q.1 Explain the concept of differentiation of a vector with respect to time. (2006)

Ans. Consider a particle P moving along a curve such that its position vector at any instant t is $\vec{r}$ and as t changes to t + δt, changes $\vec{r} + \delta\vec{r}$ to. . Then is a function of the scalar vector t

By definition the derivative of $\vec{r}$ with respect to t is

$$\frac{d\vec{r}}{dt} = \underset{\delta t \to 0}{\text{Lim}} \frac{\delta\vec{r}}{\delta t}$$

$\frac{d\vec{r}}{dt}$ is itself a vector and in this case represents the velocity $\vec{v}$ of the particle similarly the derivative with respect to t is will represent the acceleration of the particle is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

If $\vec{r}$ be given in Cartesian coordinates

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

then $\vec{v} = \frac{d\vec{r}}{dt}$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \qquad \text{and} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

here $\frac{dx}{dt}$ — etc represent the magnitudes of the component of the velocity and $\frac{d^2x}{dt^2}$ — etc magnitudes of components of the acceleration.

Q.2 Obtain the differentiation of the sum and product of two vectors. If the surface to which Stoke's theorem is applied is a closed surface, the length of the boundary edges is zero and then $\int \text{curl}\vec{A}.d\vec{s} = 0$ (2006)

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Ans. The rules for differentiation of sum and product of scalar apply to vector differentiation i.e.

(i) $\frac{d}{dt}\left(\vec{A}+\vec{B}\right) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$ (ii) $\frac{d}{dt}\left(\vec{A}.\vec{B}\right) = \vec{A}\frac{d\vec{B}}{dt} + \vec{B}\frac{d\vec{A}}{dt}$

(iii) $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$

Proof - (i) If $\vec{V} = \vec{A} + \vec{B}$ As t change to t + δt

We have

$$\vec{V} + \delta\vec{V} = \left(\vec{A} + \delta\vec{A}\right) + \left(\vec{B} + \delta\vec{B}\right)$$

$$\delta\vec{V} = \delta\vec{A} + \delta\vec{B} \qquad \left[\vec{V} = \vec{A} + \vec{B}\right]$$

Dividing by δt and proceeding to the limit,

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\vec{A} + \vec{B}\right) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

(ii) As t change to t + δt

$$\vec{A}.\vec{B} + \delta\left(\vec{A}.\vec{B}\right) = \vec{A}.\vec{B} + \vec{A}.\delta\vec{B} + \vec{B}.\delta\vec{A} + \delta\vec{A}.\delta\vec{B}$$

As $\delta\vec{A}$, $\delta\vec{B}$ approach zero when $(\delta t \to 0)$ product $\delta\vec{A}.\delta\vec{B}$ may be neglects in the limit

Hence

$$\frac{\delta\left(\vec{A}.\vec{B}\right)}{\delta t} = \vec{A}.\frac{\delta\vec{B}}{\delta t} + \vec{B}.\frac{\delta\vec{A}}{\delta t}$$

as $\delta t \to 0$

$$\left[\frac{d}{dt}\left(\vec{A}.\vec{B}\right) = \vec{A}.\frac{d\vec{B}}{dt} + \vec{B}\frac{d\vec{A}}{dt}\right]$$

(iii) $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$

Proof Let

$$\vec{V} = \vec{A}\times\vec{B}$$

When V change to V + δV, then $\vec{A}$ changes to $\vec{A}+\delta\vec{A}$ and B changes to $\vec{B}+\delta\vec{B}$

$$\vec{V} + \delta\vec{V} = \left(\vec{A} + \delta\vec{A}\right)\times\left(\vec{B} + \delta\vec{B}\right)$$

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$$= \vec{A}\times\vec{B} + \vec{A}\times\delta\vec{B} + \delta\vec{A}\times\vec{B} + \delta\vec{A}\times\delta\vec{B}$$

$$= \vec{V} + \vec{A}\times\delta\vec{B} + \delta\vec{A}\times\vec{B} + \delta\vec{A}\times\delta\vec{B}$$

$$\delta\vec{V} = \vec{A}\times\delta\vec{B} + \delta\vec{A}\times B + \delta\vec{A}\times\delta\vec{B}$$

Dividing by δt and proceeding to the limit

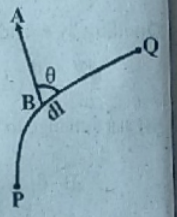
$$\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

*** Vector Integration: -**

(1) Line Integral : Let PQ be any curve, drawn in a vector field and dl an element of length along this curve at any point B. Let $\vec{A}$ represent the vector at B in a direction making an angle θ with dl. In general if $\vec{A}$ varies in magnitude and direction from point to point along the curve, the integral

$$\int_P^Q \vec{A}.d\vec{l} = \int_P^Q A\cos\theta\, dl \qquad .....(i)$$

is defined as the line integral of $\vec{A}$ along the curve PQ.



In terms of cartesian components,

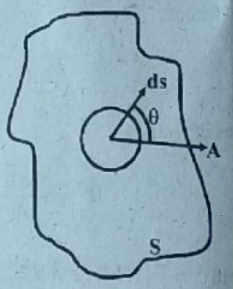
$$\int_P^Q \vec{A}.d\vec{l} = \int_P^Q \left(A_x dx + A_y dy + A_z dz\right) \qquad .....(ii)$$

If a force $\vec{F}$ is acting on a moving particle then the line integral $\int \vec{F}\,.\,d\vec{l}$ over the path described by the particle is the work done by the force.

(2) Surface Integral : Consider any element of infinitesimal area ds upon a surface S in a vector field. If $\vec{A}$ be the value of the vector at the middle point of the element, then the integral over the surface

$$\iint_S \vec{A}.d\vec{s} = \iint_S A\cos\theta\, ds \qquad .....(iii)$$

is defined as the surface integral or total flux of vector $\vec{A}$ through the whole surface S.



In cartesian components,

$$\iint_S \vec{A}.d\vec{s} = \iint_S \left(A_x ds_x + A_y ds_y + A_z ds_z\right) \qquad .....(iv)$$

If V denotes the vector velocity of a moving fluid, in which a fixed surface S is drawn, the $\iint_S V.ds$ will represent the rate of flow a liquid through the surface S.

(3) Volume Integral : If we consider a closed surface in space enclosing a

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volume V, then

$$\iiint_V \vec{A}dV = \int_x\int_y\int_z \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)dx\, dy\, dz \qquad .....(v)$$

is defined as the volume integral.

*** Partial Derivative—Gradient: -**

Partial Derivatives : The differentiation of a function such as $\phi(x_1, x_2, x_3 ....)$ of two or more independent variables, x_1, x_2, x_3 etc., with respect to one of them, keeping the others constant, is called partial differentiation. The derivatives thus obtained are referred to as partial derivatives and are denoted by the symbol ∂ instead of d.

Thus, if we have a continuously differentiable scalar point function φ(x, y, z), i.e., a function of the coordinates x, y, z, its partial derivatives along the three coordinate axes are

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\phi}{\partial z}.$$

Gradient : The vector function $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar point function φ and is denoted by grad φ. Thus,

$$\text{grad}\ \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$\therefore \text{differential } d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right) = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{i}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

Or. $d\phi = dr\ \text{grad}\ \phi$.

It can easily be shown that grad φ at any point is quite independent of the choice of the coordinate axes. It follows, therefore, that the gradient of a scalar point function is a vector point function.

If φ be a constant, obviously the partial derivatives ∂φ/∂x, ∂φ/∂y and ∂φ/∂z will all be zero and hence grad φ = 0. And, conversely, if grad φ = 0, the partial derivatives are all zero and hence function φ is a constant. Thus, grad φ = 0 only if φ be constant.

The Operator ∇: - **(2019)**

We have in vector algebra a differential operator, denoted by ∇ (i.e., inverted Δ). It is called **'del'** or **'nabla'**. It operates distributively and is formally assumed to have the character of a vector. And since it is assumed to have the character of a vector, its product ∇φ with a scalar φ is a vector.

If we use Cartesion orthogonal coordinates, we may write

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

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So that, $\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \nabla\phi$, a vector.

*** Divergence And Curl: -**

From a vector point function $\vec{f}$ we obtain two more point functions : a scalar one, called divergence and a vector one, called curl (or rotation).

Thus, if $\vec{f}$ represents a vector field, i.e., if $\vec{f}$ be a continuously differentiable vector point function, the function

$$\hat{i}\cdot\frac{\partial\vec{f}}{\partial x} + \hat{j}\cdot\frac{\partial\vec{f}}{\partial y} + \hat{k}\cdot\frac{\partial\vec{f}}{\partial z},$$

which is a scalar, is called divergence of $\vec{f}$ and is written as div $\vec{f}$. And the function

$$\hat{i}\times\frac{\partial\vec{f}}{\partial x} + \hat{j}\times\frac{\partial\vec{f}}{\partial y} + \hat{k}\times\frac{\partial\vec{f}}{\partial z},$$

which is a vector, is called curl of $\vec{f}$ and is written as curl $\vec{f}$.

It can easily be shown that both div f and curl $\vec{f}$ are quite independent of the orthogonal unit triad (i.e., unit vectors) that may be chosen. They are thus essentially point functions.

Now, clearly, $\text{div}\,\vec{f} = \hat{i}\cdot\frac{\partial\vec{f}}{\partial x} + \hat{j}\cdot\frac{\partial\vec{f}}{\partial y} + \hat{k}\cdot\frac{\partial\vec{f}}{\partial z} = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\vec{f} = \nabla\vec{f}$

So that, divergence of a vector point function $\vec{f}$ is the scalar product of the Del operator with $\vec{f}$. And,

$$\text{curl}\,\vec{f} = \hat{i}\times\frac{\partial\vec{f}}{\partial x} + \hat{j}\times\frac{\partial\vec{f}}{\partial y} + \hat{k}\times\frac{\partial\vec{f}}{\partial z} = \left(\hat{i}\times\frac{\partial}{\partial x} + \hat{j}\times\frac{\partial}{\partial y} + \hat{k}\times\frac{\partial}{\partial z}\right)\times\vec{f} = \nabla\times\vec{f}$$

i.e., curl of a vector point function $\vec{f}$ is the vector product of the Del operator with $\vec{f}$. The term $\nabla\times$, [read as Del cross (or Nabla cross)], is therefore also used in place of the term 'curl'.

Both divergence and curl of $\vec{f}$ may be expressed in terms of the components of vector $\vec{f}$, as shown below :

If $\vec{f} = \hat{i}f_x + \hat{j}f_y + \hat{k}f_z$, we have

$$\text{div}\,\vec{f} = \sum\hat{i}\cdot\frac{\partial\vec{f}}{\partial x} = \sum\hat{i}\cdot\left[\hat{i}\frac{\partial f_x}{\partial x} + \hat{j}\frac{\partial f_y}{\partial x} + \hat{k}\frac{\partial f_z}{\partial x}\right] = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$$

$$\text{And curl}\,\vec{f} = \sum\hat{i}\times\frac{\partial f}{\partial x} = \sum\hat{i}\times\left[\hat{i}\frac{\partial f_x}{\partial x} + \hat{j}\frac{\partial f_y}{\partial x} + \hat{k}\frac{\partial f_z}{\partial x}\right]$$

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$$= \hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{j}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

*** Physical Significance: -**

(i) If div $\vec{V} = 0$ every where in some region R of space, then $\vec{V}$ is called solenoidal vector point function.

(ii) If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is said to be an irrotational-vector.

(iii) grad ϕ is a vector normal to surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

*** Gauss Theorem: -** **(2007-08)**

According to this theorem, the volume integral of the divergence of a vector field $\vec{A}$ taken over any volume V is equal to the surface integral of $\vec{A}$ taken over the closed surface surrounding the volume. That is

$$\iiint_v (\nabla\cdot\vec{A})\,dV = \iint_s \vec{A}\cdot ds \qquad \text{.....(i)}$$

Proof. In cartesian coordinates,

$$\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

$dV = dx\,dy\,dz$,

and $\vec{A}\cdot d\vec{s} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\cdot\left(ds_x\hat{i} + ds_y\hat{j} + ds_z\hat{k}\right)$

$= A_x\,ds_x + A_y\,ds_y + A_z\,ds_z = A_x\,dy\,dz + A_y\,dx\,dz + A_z\,dx\,dy$

Hence, Gauss's theorem in cartesian coordinates is

$$\iiint_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)dx\,dy\,dz$$

$$= \iint_S \left(A_x ds_x + A_y ds_y + A_z ds_z\right) \qquad \text{.....(ii)}$$

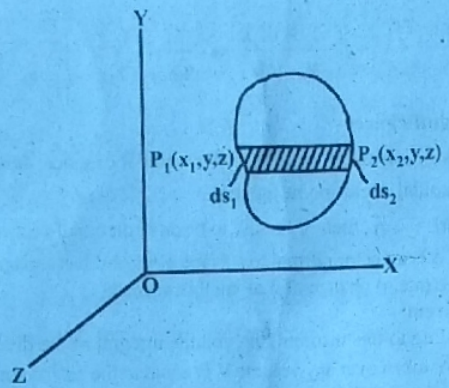
Let us consider the first part of the left-hand side integral. As shown in fig., integrate it with respect to x along a stripe of cross-section dy dz extending from $P_1(x_1, y, z)$ to $P_2(x_2, y, z)$, i.e.,

$$\iiint_v \frac{\partial A_x}{\partial x}dx\,dy\,dz = \iint_S \left[A_x\right]_{P_1}^{P_2} dy\,dz = \iint S\left[A_x(P_2) - A_x(P_1)\right]dy\,dz$$

$$= \iint_s \left[A_x(x_2, y, z) - A_x(x_1, y, z)\right]dy\,dz$$

Evidently, at P_1, $A_x\,dy\,dz = -A_x\,ds_x$ and at P_2, $A_x\,dy\,dz = A_x\,ds_x$ (since at P_1 and P_2, the X-components of area have opposite directions).

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$$\therefore \quad \iiint_v \frac{\partial A_x}{\partial x} dx\, dy\, dz = \iint_s A_x\, ds_x \qquad \text{.....(iii)}$$

Similarly, $\iiint_v \frac{\partial A_y}{\partial y} dx\, dy\, dz = \iint_s A_y\, ds_y$ .....(iv)

and $\iiint_v \frac{\partial A_y}{\partial z} dx\, dy\, dz = \iint_s A_z\, ds_z$ .....(v)

Hence, adding (iii), (iv) and (v), we have

$$\iiint_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\, dy\, dz = \iint_s \left(A_x ds_x + A_y ds_y + A_z ds_z\right)$$

$$\iiint_v \left(\nabla \cdot \vec{A}\right) dV = \iint_s \vec{A} \cdot d\vec{s} \text{ or } \iiint_v \left(\text{div } \vec{A}\right) dV = \iint_S A \cdot ds$$

This proves the Gauss's theorem of divergence.

*** Stokes Theorem: -** **(2009, 16)**

This theorem states that the line integral of vector $\vec{A}$ around any closed curve C is equal to the surface integral of curl $\vec{A}$ taken over any surface S of which the curve is boundary edge, i.e.

$$\oint_C \vec{A}.d\vec{l} = \iint_S \text{curl } \vec{A}.d\vec{s} = \iint_S \left(\vec{\nabla} \times \vec{A}\right).ds$$

Proof: -

If $\vec{A}$ a vector be a function of position, then its line integral along a closed curve C is given by

$$L = \oint_C \vec{A}.d\vec{l}$$

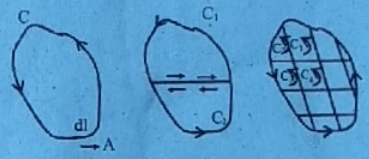
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Now, divide the enclosed area into two parts by a line pq so as to form two closed curve C_1 and C_2. If L_1 and L_2 are the line integrals for the two paths C_1 and C_2, then their sum $(L_1 + L_2)$ will be equal to L for the curve C. the reason lies in the fact that the part pq is traversed from p to q for the curv C_1 and from q to p for the path C_2 so that its contribution is cancelled. Thus if we divide the area enclosed by the curve C in to number of smaller areas δS_1, δS_2.....bounded by the curves $C_1, C_2, C_3, \ldots, C_n$..... then

$$\oint_C \vec{A}.d\vec{l} = \sum \oint_{C_n} \vec{A}.d\vec{l}$$

The curve C may be divided in such away those small areas ∂S_1, ∂S_2 may lie in different planes. But from the definition of curl $\vec{A}$ at a point inside ∂S_n, we have

$$\text{curl } \vec{A}.\delta s_n = \oint_{C_n} \vec{A}.d\vec{l}$$



Thus $\oint_C \vec{A}.d\vec{l} = \sum \oint_{C_n} \vec{A}.d\vec{l} = \sum \text{curl } \vec{A}.\delta\vec{s}_n$

i.e. $\oint_C \vec{A}.d\vec{l} = \iint_S \text{curl } \vec{A}.d\vec{s}$

This is stoke's theorem. It should be noted that the boundary curve or the enclosed surface need not to be plane one.

Q.3 Prove that Curl grad $\phi = 0$

ϕ is a scalar field, $\vec{A}$ is a vector field. **(2007)**

Ans. (i) Curt grad $\phi = \vec{\nabla} \times \left(\vec{\nabla}\phi\right)$

$$= \vec{\nabla} \times \left(\frac{\partial \phi}{\partial x} x + \frac{\partial \phi}{\partial y}\hat{J} + \frac{\partial \phi}{\partial z}\hat{K}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

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$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{K}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = 0$$

$\therefore \phi$ is perfect differential

$$\Rightarrow \quad \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}, \quad \text{and} \quad \frac{\partial^2\phi}{\partial x\partial z} = \frac{\partial^2\phi}{\partial z\partial x}$$

and $$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$$

Hence Curl grad $\phi = 0$.

Q.4 Prove that Div curl $\vec{A} = 0$

ϕ is a scalar field, $\vec{A}$ is a vector field. **(2007)**

Ans. We have

$$\text{Div curl } \vec{A} = \vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)$$

$$= \vec{\nabla}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{\nabla}\left[\hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[\hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial y\partial x} + \frac{\partial^2 A_x}{\partial y\partial z} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial z\partial y}$$

$$= \left(\frac{\partial^2 A_z}{\partial y\partial z} - \frac{\partial^2 A_y}{\partial z\partial y}\right) + \left(\frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_z}{\partial x\partial z}\right) + \left(\frac{\partial^2 A_x}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial y\partial x}\right) = 0$$

Since $\vec{A}$ is perfect deferential

$$\Rightarrow \quad \frac{\partial^2 A_x}{\partial y\partial z} = \frac{\partial^2 A_z}{\partial z\partial y} \quad \text{and} \quad \frac{\partial^2 Ay}{\partial z\partial x} = \frac{\partial^2 A_y}{\partial x\partial z}$$

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and $$\frac{\partial^2 Az}{\partial x\partial y} = \frac{\partial^2 Az}{\partial y\partial x}$$

Hence div curl $\vec{A} = 0$.

Q.5 Prove that

$$\vec{\nabla}\times\left(\phi\vec{A}\right) = \left(\vec{\nabla}\phi\right)\times\vec{A} + \phi\left(\vec{\nabla}\times\vec{A}\right) \qquad \textbf{(2008)}$$

Ans. We have

$$\text{Curl}\left(\phi.\vec{A}\right) = \vec{\nabla}\times\left(\phi.\vec{A}\right)$$

$$= \vec{\nabla}\times\left(\phi A_x\hat{i} + \phi A_y\hat{j} + \phi A_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\phi A_z - \frac{\partial}{\partial z}\phi A_y\right) - \hat{j}\left(\frac{\partial}{\partial x}\phi A_z - \frac{\partial}{\partial z}\phi A_x\right) + \hat{k}\left(\frac{\partial}{\partial x}\phi A_y - \frac{\partial}{\partial y}\phi A_x\right)$$

$$- \hat{i}\left[\phi\frac{\partial A_z}{\partial y} + A_z\frac{\partial\phi}{\partial y} - \phi\frac{\partial A_y}{\partial z} - A_y\frac{\partial\phi}{\partial z}\right] - \hat{j}\left[\phi\frac{\partial A_z}{\partial x} + A_z\frac{\partial\phi}{\partial x} - \phi\frac{\partial A_x}{\partial z}\right.$$

$$\left. - A_x\frac{\partial\phi}{\partial z}\right] + \hat{K}\left[\phi\frac{\partial A_y}{\partial x} + A_y\frac{\partial\phi}{\partial x} - \phi\frac{\partial A_x}{\partial y} - A_x\frac{\partial\phi}{\partial y}\right]$$

$$= \phi\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$

$$+ \left[\hat{i}\left(A_z\frac{\partial\phi}{\partial y} - A_y\frac{\partial\phi}{\partial z}\right) - \hat{j}\left(A_y\frac{\partial\phi}{\partial x} - A_x\frac{\partial\phi}{\partial z}\right) + \hat{k}\left(A_y\frac{\partial\phi}{\partial x} - A_x\frac{\partial\phi}{\partial y}\right)\right]$$

$$= \phi\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax & Ay & Az \end{vmatrix} + \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ Ax & Ay & Az \end{vmatrix} = \phi.\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{\nabla}\phi\right)\times\vec{A}$$

$$= \phi.\text{curl}\,\vec{A} + (\text{grad}\phi)\times\vec{A}$$

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Q.6 Prove that

$$\text{Div}\left(\vec{A}\times\vec{B}\right)=\vec{B}\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\left(\vec{\nabla}\times\vec{B}\right)$$ (2008

Ans. Let $\vec{V}=\vec{A}\times\vec{B}$

$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

So that

$$V_x\hat{i}+V_y\hat{j}+V_z\hat{K}=\hat{i}\left(A_yB_z-A_zB_y\right)-\hat{J}\left(A_xB_z-A_zB_x\right)+\hat{K}\left(A_xB_y-A_yB_x\right)$$

Now

$$\text{div}\ \vec{V}=\text{div}\ \vec{A}\times\vec{B}$$

$$\frac{\partial}{\partial x}V_x+\frac{\partial}{\partial y}V_y+\frac{\partial}{\partial z}V_z=\frac{\partial}{\partial x}\left(A_yB_z-A_zB_y\right)+\frac{\partial}{\partial y}\left(A_zB_x-A_xB_z\right)+\frac{\partial}{\partial z}\left(A_xB_y-A_yB_x\right)$$

$$=B_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+B_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+B_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$-A_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)-A_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-A_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)$$

$$=\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\A_x&A_y&A_z\end{vmatrix}-\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right).$$

$$=\begin{vmatrix}\hat{i}&\hat{J}&\hat{K}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\Bx&By&Bz\end{vmatrix}$$

$$=\vec{B}\text{Curl}\vec{A}-\vec{A}\text{Curl}\vec{B}=\vec{B}\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\left(\vec{\nabla}\times\vec{B}\right)$$

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Q.7 If $\vec{V}=\vec{\omega}\times\vec{r}$; show that $\vec{\omega}=\frac{1}{2}\text{Curl}\vec{V}$. where $\vec{\omega}$ is a constant. (2007, 16)

Sol. We have the relative $\vec{v}=\vec{\omega}\times\vec{r}$, where $\vec{v}$ is the velocity of any particle of the body with r, the position vector relative to a fixed point and $\vec{\omega}$, the angular velocity of the body.

Obviously, $\vec{v}$ is a vector point function and vector $\vec{\omega}$ is the same for all the particles.

Now, $\text{Curl}\vec{V}=\text{Curl}\left(\vec{\omega}\times\vec{r}\right)$

$$=\sum\hat{i}\frac{\partial}{\partial x}\left(\vec{\omega}\times\vec{r}\right)=\sum\hat{i}\left(\vec{\omega}\times\frac{\partial\vec{r}}{\partial x}\right)$$

$$=\sum\hat{i}\left(\vec{\omega}\times\hat{i}\right)=\sum\left(\hat{i}.\hat{i}\right)\vec{\omega}-\sum\left(\hat{i}.\vec{\omega}\right)\hat{i}$$

$$=3\vec{\omega}-\vec{\omega}$$

$$\text{Curl}V=2\vec{\omega}\qquad\left\{\because\sum\hat{i}.\hat{i}=\hat{i}.\hat{i}+\hat{i}.\hat{i}+\hat{i}.\hat{i}=1+1+1=3\right\}$$

$$\Rightarrow\quad\vec{\omega}=\frac{1}{2}\text{Curl}\vec{V}$$

Q.8. An electron (rest mass = 9.1 × 10^{-31} kg) is moving with a speed of 0.8C (C = speed of light = 3 × 10^8 m/sec). What is its total energy? (2014, 17)

Other Related Questions -

Q. Find the increase in the mass of a particle of rest mass 1 gm when it is moving with speed 0.8 C. Hence find its Kinetic energy. (2018)

Sol. $$m=\frac{m_0}{\sqrt{1-v^2/c^2}}=\frac{10^{-3}}{\sqrt{1-(0.8)^2}}=\frac{10^{-3}}{0.6}=1.667\times10^{-3}\ \text{kg}$$

Increase in mass $= 1.667\times10^{-3}-10^{-3}=0.667\times10^{-3}\,\text{kg}$

$$\therefore\quad\text{K.E.}=(m-m_0)c^2=0.667\times10^{-3}\times\left(3\times10^8\right)^2=6\times10^{13}\,\text{J}$$

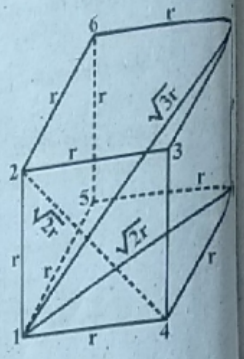
Q.9. Calculate the gravitational potential energy if a mass of 10 Kg is placed at each corner of a cube having each edge equal to 0.25 metre. (2011, 18)

Ans. Figure shows a cube of each edge, 0.25 metre in length, with a mass of 10 kg placed at each corner. The gravitational potential energy or self energy of the system is thus obviously given by

$$U_s=-G\sum_{i>j}^{8}\sum_{j=1}^{8}\frac{m_im_j}{r_{ij}}\quad\text{where } m_i=m_j=10\,\text{kg}.$$

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Now, each of the twelve distances, $r_{21}, r_{41}, r_{51}, r_{32}, r_{62}, r_{42}, r_{23}, r_{54}, r_{65}, r_{65}, r_{36}$ and r_{87} (all edges of the cube) is equal to r, each of the twelve distances, $r_{31}, r_{61}, r_{81}, r_{42}, r_{52}, r_{32}, r_{63}, r_{83}, r_{54}, r_{74}, r_{75}$ and r_{86} (all face-diagonals of the cube) is equal to $\sqrt{r^2+r^2}=\sqrt{2}r$, and each of the four distances, r_{71}, r_{82}, r_{53} and r_{64} (all long diagonals, or distances between opposite corners, of the cube) is equal to $\sqrt{2r^2+r^2}=\sqrt{3r^2}=\sqrt{3}r$.



$\therefore \quad U_s = -Gm^2(12/r + 12/\sqrt{2}r + 4/\sqrt{3}r)$

$$= -\frac{4Gm^2}{r}\left(3+3/\sqrt{2}+1/\sqrt{3}\right) = -\frac{4Gm^2}{r}$$

Substituting the given values of G, m and r, therefore, we have gravitational potential energy of the system,

$$U_s = -\frac{4\times 6.67\times 10^{-11}\times(10)^2}{0.25} = -1.067\times 10^{-7} \text{ joule.}$$

Q.10. If $\vec{r}$ is the position vector of a point. Then prove that $\text{grad}(\log r)=\frac{\hat{n}}{r}$ where $\hat{n}$ is the unit vector along $\vec{r}$. **(2015)**

Ans. Now $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow r^2 = |\vec{r}|^2 = x^2+y^2+z^2$ .....(i)

From eq.(i) $\quad \frac{\partial r}{\partial x}=\frac{x}{r}, \frac{\partial r}{\partial y}=\frac{y}{r}, \frac{\partial r}{\partial z}=\frac{z}{r}$ .....(ii)

From eq.(ii), $\quad \frac{\partial}{\partial x}\log r = \frac{1}{r}\frac{\partial r}{\partial x}=\frac{x}{r^2}, \frac{\partial}{\partial y}\log r=\frac{y}{r^2}, \frac{\partial}{\partial z}\log r=\frac{z}{r^2}$

Hence, $\quad \text{grad}(\log r) = \hat{i}\frac{\partial}{\partial x}\log r+\hat{j}\frac{\partial}{\partial y}\log r+\hat{k}\frac{\partial}{\partial z}\log r = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^2}$

$$= \frac{\vec{r}}{r}\cdot\frac{1}{r} = \frac{\hat{n}}{r}$$

Q.11. A particle of rest mass m_0 trevelling with a speed of 0.6C collides with an identical particle at rest and sticks to it. Find the velocity of resulting particle. **(2015)**

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Ans. Since the moving mass collides with a identical particle at rest and sticks to it then from the law of conservation of momentum we have

$$m_1u_1 = (m_1+m_2)v$$

$$\Rightarrow \quad v = \frac{m_1u_1}{m_1+m_2} = \frac{m_0\times 0.6C}{m_0+m_0} = 0.3C$$

Q.12. A rigid body is spinning with an angular velocity of 4 radians per second about an axis (3j - k) passing through the point A (1, 3, - 1) m. Find the velocity of the particle at the point P (4, - 2, 1). **(2016)**

Sol. Let $\hat{n}$ be the unit vector in the direction of 3j - k. Then.

$$\hat{n} = \frac{3j-k}{\sqrt{0+9+1}} = \frac{1}{\sqrt{10}}(3j-k),$$

angular velocity of the particle P, say, $\omega = \omega\hat{n} = \frac{4}{\sqrt{10}}(3j-k)$.

The position vector of P with reference to point i + 3j - k, i.e.,

$$r = (4i-2j+k)-(i+3j-k) = 3i-5j+2k$$

Hence velocity (linear) of particle $P = r\times\omega = (3i-5j+2k)\times\frac{4}{\sqrt{10}}(3j-k)$

$$= \frac{4}{\sqrt{10}}(i+3j+9k)$$

and its magnitude $= \frac{4}{\sqrt{10}}\sqrt{1+9+81} = 4\sqrt{\frac{91}{10}} = 12$ units very nearly.

Q.13. Show that the component of a vector $\vec{V}$ along $\vec{U}$ may be expressed as a vector $\frac{(\vec{V}.\vec{U})\vec{U}}{U^2}$. **(2018)**

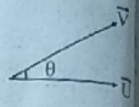
Ans. We know that dot product of two vectors is given by,

$$\vec{V}.\vec{U} = |\vec{V}||\vec{U}|\cos\theta$$

$$\Rightarrow \quad \frac{\vec{V}.\vec{U}}{|\vec{V}||\vec{U}|} = \cos\theta$$

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$$\Rightarrow \quad \frac{\vec{V}.\vec{U}}{|\vec{V}||\vec{U}|}|\vec{V}| = |\vec{V}|\cos\theta$$



But $|\vec{V}|\cos\theta$ is component of $\vec{V}$ along $\vec{U}$.

So, projection of $\vec{V}$ along $\vec{U}$ (which is a vector quantity).

$$= \frac{\vec{V}.\vec{U}}{|\vec{U}|}.\hat{U} = \frac{\vec{V}.\vec{U}}{|\vec{U}|}\frac{\vec{U}}{|\vec{U}|} = \frac{\vec{V}.\vec{U}}{U^2}.\vec{U}$$

$$\Rightarrow \quad \frac{(\vec{V}.\vec{U})\vec{U}}{U^2}$$

Q.14. Show that $\vec{\nabla}r = \hat{n}$ where $\hat{n}$ is the unit vector along $\vec{r}$, the position vector. **(2018)**

Ans. Position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

magnitude of position vector $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

del operator, $\vec{\nabla} = \hat{x}\frac{d}{dx} + \hat{y}\frac{d}{dy} + \hat{z}\frac{d}{dz}$

therefore $\vec{\nabla}r = \hat{x}\frac{d}{dx}r + \hat{y}\frac{d}{dy}r + \hat{z}\frac{d}{dz}r$

$$= \hat{x}\frac{d}{dx}\left(x^2+y^2+z^2\right)^{1/2} + \hat{y}\frac{d}{dy}\left(x^2+y^2+z^2\right)^{1/2} + \hat{z}\frac{d}{dz}\left(x^2+y^2+z^2\right)^{1/2}$$

$$= \frac{\hat{x}}{2\left(x^2+y^2+z^2\right)^{1/2}}2x + \frac{\hat{y}}{2\left(x^2+y^2+z^2\right)^{1/2}}2y + \frac{\hat{z}}{2\left(x^2+y^2+z^2\right)^{1/2}}2z$$

$$= \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\left(x^2+y^2+z^2\right)^{1/2}} = \frac{\vec{r}}{r} = \hat{n}$$

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Chapter 2 Frame Of Reference

*** Reference Frame: -**

A system of coordinate axes which defines the position of a particle or an event in two or three dimensional space is called a frame of reference. The essential requirement of a frame of reference is that it should be rigid.

The simplest frame of reference is, of course, the familiar Cartesian system of coordinates, in which the position of the particle is specified by its three coordinates x, y, z along the three perpendicular axes, such that if r be the position-vector of the particle with respect to the origin, we have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the three axes.

So that, velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

and its acceleration by $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$.

The origin of the frame of reference may not always be coincident with the position of the observer. The frame of reference must, however, be at rest with respect to him.

Obviously, for complete identification of an event in a reference frame, i.e., for determination of its exact location as well as the exact time of its occurence, we must, in addition to the usual three spatial coordinates x, y and y, have yet another coordinate, viz., that of time t, perpendicular to the former three.

A reference frame with such four coordinates, x, y, z and t, is referred to as a **space-time frame**,– the four axes defining a four dimensional continuum, called **space-time**.

*** Newton's Law of Motion: -**

First Law : A body must continue in its state of rest or of uniform motion along a straight line unless acted upon by an external force.

Second Law : The rate of change of momentum is proportional to the impressed force and takes place in the direction of the force.

Third Law : To very action, there is an equal and opposite reaction. Thus, if $\vec{F}_{12}$ and $\vec{F}_{21}$ be the forces exerted on each other by two interacting bodies respectively, we have $\vec{F}_{12} = \vec{F}_{21}$.

*** Inertial Frame of Reference: -** **(2016)**

Those unaccelerated frames of reference, in which Newton's first and second laws hold, are called inertial frames. In such frames, if a body is not

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acted by external force, it continues in its state of rest or uniform translatory motion (law of inertia) and hence they are called inertial frames. Thus, in an inertial frame, if a body is not experiencing any external force, its acceleration $\bar{a}$ is given by

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = 0 \qquad (\because \bar{F} = m\bar{a} = 0) \quad \text{.....(i)}$$

In component form, this equation can be written as

$$\frac{d^2x}{dt^2} = 0, \frac{d^2y}{dt^2} = 0, \frac{d^2z}{dt^2} = 0.$$

Let us consider an inertial frame S and another frame S′, which is moving with constant velocity $\bar{v}$ relative to S. Initially (at t = 0) if the positions of the origins of the two frames coincide, then in the two frames the position vectors of any particle P at any instant t can be related by the following expressions :

$\bar{r} = OO' + \bar{r}$ or $\bar{r} = \bar{v}t + \bar{r}'$

or $\bar{r}' = \bar{r} - \bar{v}t$

As v is constant, $\therefore \frac{d^2\bar{r}'}{dt^2} = \frac{d^2\bar{r}'}{dt^2}$ or $\bar{a}' = \bar{a}$

That is, a particle experiences the same acceleration in two frames, which are moving with constant relative velocity. Now, if the acceleration of the particle in frame S is zero, its acceleration in S′ is also zero. But S is an inertial frame, S′ should also be inertial. Thus, we conclude that if a frame is an inertial frame, then all those frames, which are moving with constant velocity relative to the first frame, are also inertial.

Necessarily, the inertial frames are unaccelerated frames because if the frame is accelerated, a particle, whose velocity is constant, will appear accelerated in this frame.

Note : The law of inertia is valid in all those reference systems, which are moving with constant velocity relative to the absolute system.

*** Galilean Transformation: -** **(2005, 10, 13, 17)**

A point, or a particle at any instant, in space has different coordinates in different reference systems. The equations which provide the relationship between the coordinates of two reference systems are called transformation equations.

We have seen above that a frame S′, which is moving with constant velocity $\bar{v}$ relative to an inertial frame S, is itself inertial.

In view of convenience, if we assume (i) that the origins of the two frames coincide at t = 0, (ii) that the coordinate axes of the second frame are parallel to that of the first and (iii) that the velocity of the second frame relative

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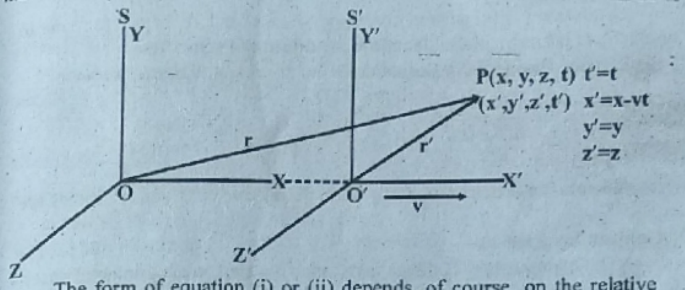
to the first is $\bar{v}$ along X-axis, then the position vectors of a particle at any instant t are related by the equation

$$\bar{r}' = \bar{r} - \bar{v}t \quad \text{.....(i)}$$

In the component form, the coordinates are related by the equations

$$x' = x - vt;\ y' = y;\ z' = z \quad \text{.....(ii)}$$

Equation (i) or (ii) expresses the transformation of coordinates from one inertial frame to another and hence they are referred as Galilean transformations.



The form of equation (i) or (ii) depends, of course, on the relative motion of two frames of reference, but it also depends upon certain assumptions regarding the nature of time and space. It is assumed that the time t is independent of any particular frame of reference, i.e., if t and t′ be the times recorded by the observers O and O′ of an event occurring at P, then t′ = t. This assumption has been expressed in the above transformation equation by the absence of the transformation equation t′ = t. If we add it, then the Galilean transformation equations are

$$x' = x - vt;\ y' = y;\ z' = z;\ t' = t \quad \text{.....(iii)}$$

The other assumption, regarding the nature of the space, is that the distance between two points (or two particles) is independent of any particular frame of reference. Evidently, this assumption is expressed by the form of the transformation equation (i) or (iii). If a rod has length L in the frame S with the end coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

If at the same time the end coordinates of the rod in frame S′, are (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2), then

$$L' = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}$$

But for any time t, from equation (iii), we have

$x_2' - x_1' = x_2 - x_1, y_2' - y_1' = y_2 - y_1, z_2' - z_1' = z_2 - z_1$

$\therefore$ $L = L'$ .....(iv)

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Thus, the length or distance between two points is invariant under Galilean transformation.

Differentiating equation (i) with respect to time, we get

$$\frac{d\vec{r}}{dt} = \vec{v} + \frac{d\vec{r}'}{dt} = \vec{v} + \frac{d\vec{r}'}{dt'} \qquad [\because t = t']$$

or $\vec{V} = \vec{v} + \vec{V}'$ .....(v)

where $\vec{V}$ and $\vec{V}'$ are the observed velocities in S and S' frames respectively.

Equation (v) transforms the velocity of a particle from one frame to another and is known as **Galilean law of addition of velocities**.

Again differentiating equation (v) with respect to time t, we have

$$\frac{d\vec{V}}{dt} = 0 + \frac{d\vec{V}'}{dt'} = \frac{d\vec{V}'}{dt'} \qquad (\because t = t')$$

or $\vec{a} = \vec{a}'$ .....(vi)

Hence the accelerations of a particle relative to S and S' frames are equal.

*** Galilean Invariance: -** **(2013, 17)**

The hypothesis of Galilean invariance is based on experimental observations and can be stated as follows :

The basic laws of Physics are identical in all reference systems which move with uniform velocity with respect to one another.

According to this hypothesis, if a windowless spaceship is moving with uniform speed relative to the fixed stars, then all experiments performed in the spaceship and all the phenomena in the spaceship will appear the same to an observer confined to it as if the spaceship were stationary. Only by looking through a window, the observer can tell that he is in uniform motion relative to the stars. Thus, the absolute velocity has no physical meaning; only the velocity relative to other objects is meaningful.

The hypothesis of Galilean invariance is a fundamental postulate and is entirely consistent with the theory of special relativity.

Since any two inertial systems, which are moving with constant relative velocity, can be connected by Galilean transformations, we can modify the hypothesis of Galilean invariance by giving the following statement :

The basic laws of physics are invariant in form in two reference systems connected by Galilean transformations.

This statement is somewhat special than the hypothesis of Galilean invariance in the sense that it means the assumptions that the time and the space interval are independent of the frame of reference. Theory of Special Relativity that the Galilean transformations are not correct, but the appropriate exact transformation equations are the Lorentz transformation equations for connecting any two frames in uniform relative motion. Thus, the hypothesis of invariance may be stated as :

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"The basic laws of Physics are invariant in form in two reference frames connected by Lorentz transformations."

Transformation of force from one inertial system to another - Suppose that the force $\vec{F}$ on a particle of mass m in the frame S is represented by the law

$$\vec{F} = m\vec{a}$$

But according to the postulate that the laws of Physics are the same in the frame S and in another frame S', which is in uniform motion relative to S, we have

$$\vec{F}' = m\vec{a}'$$

in the frame S', provided that mass m is independent of velocity.

The acceleration of the particle is the same in the two frames, connected by Galilean transformation , i.e.,

$$\vec{a}' = \vec{a}$$

Therefore, $\vec{F} = m\vec{a} = m\vec{a} = \vec{F}'$ .....(i)

This means that if the relation $\vec{F} = m\vec{a}$ (Newton's second law) is used to define the force, then in all inertial frames the force $\vec{F}$ will have the same magnitude and direction, independent of the relative velocities of the reference frames and the form of the Newton's equation (second law) is the same. Thus, Newton's second law and hence Newton's first law ($\dot{\vec{F}} = 0$) and third law (involving forces), i.e., the laws of Mechanics are identical in all the reference systems, having uniform relative motion.

*** Non Inertial Frame And Pseuds Force: -** **(2005, 14, 15, 16, 19)**

Those frames of reference in which Newton's law of inertia does not hold, are called **non-inertial frames**. All the accelerated and rotating frames are the non-inertial frames of reference. According to Newton's second law, the applied force $\vec{F}$ on a body of mass m is given by

$$\vec{F}_i = m\vec{a}_i \qquad \text{.....(i)}$$

where $\vec{a}_i$ is the acceleration, observed in an inertial frame. This equation, as it stands, is not valid for accelerated frames because due to its own acceleration, the accelerated observer will observe an acceleration $\vec{a}_N$ and hence

$$\vec{F}_i \neq m\vec{a}_N$$

If no external force is acting on a particle, even then in the accelerated frame it will appear that a force is acting on it. Suppose that S is an inertial frame and another frame S' is moving with an acceleration $\vec{a}_0$ relative to S. The acceleration of a particle P, on which no external force is acting, will be zero in the frame S; but in frame S' the observer will find that an acceleration $-\vec{a}_0$ is acting on it. Thus, in frame S', the observed force on the particle is $-m\vec{a}_0$, where m is the mass of the particle.

Such a force, which does not really act on the particle but appears due to the acceleration of the frame, is called a **fictitious or pseudo force**. Hence

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fictitious force on the particle P is

$$\vec{F}_0 = -m\vec{a}_0 \qquad \text{.....(ii)}$$

Here, accelerated frame S′ is non-inertial, because in this frame Newton's law of inertia is not valid.

*** Centrifugal Force: -**

Let a mass m be at rest in a non-inertial frame. So that in this frame the value of the observed acceleration is $\vec{a}_N = 0$. The non-inertial frame is rotating uniformly with angular velocity ω about an axis fixed with respect to an inertial frame. Hence the expression for the centripetal acceleration of mass m with respect to inertial frame is

$$\vec{a}_0 = -\vec{\omega}^2\vec{r}$$

where $\vec{r}$ is directed outward to the particle from the axis.

The fictitious force on the mass m is

$$\vec{F}_0 = -m\vec{a}_0 = m\vec{\omega}^2\vec{r}$$

This fictitious force F_0 is called **centrifugal force**, and is obviously directed away from the centre. The centrifugal force is balanced by some actual force (e.g., inward tension in a rotating string with a stone) in order to produce zero acceleration in the rotating non-inertial frame.

*** Rotating Frame of Reference: -**

Q.1. Show that Corialis force acting on a body of mass m in a rotating frame is $-2m\vec{\omega}\times\vec{V}_r$, where $\vec{\omega}$ is the angular velocity of rotating frame and $\vec{V}_r$ is the velocity of the body in rotating frame. (2006)

Other Related Questions -

Q. Write short note on coriolis force. (2015)

Ans. The earth itself rotates about its axis in 24 hours. Therefore, the frame fixed to the earth will also rotate with it and so it will be a non-inertial frame.

Suppose that a frame S′(X_r, Y_r, Z_r) is rotating with an angular velocity $\vec{\omega}$ relative to an Inertial frame S(X_i, Y_i, Z_i). We assume that the both of the frames have common origin O and common Z – axis.

The position of a vector particle P in both frames will be the same i.e. $R_i = R_r = R$, because the origins are coincident. Now, if the particle P is stationary in the frame S, the observer in the rotating frame S′ will see that the particle is moving oppositely with linear velocity $\vec{\omega}\times\vec{R}$. Thus, if the velocity of the particle in the frame S is $(dR/dt)_i$, then its velocity $(dR/dt)_r$ in the rotating frame will given by

$$\left(\frac{d\vec{R}}{dt}\right)_r = \left(\frac{d\vec{R}}{dt}\right)_i - \vec{\omega}\times\vec{R}$$

or

$$\left(\frac{d\vec{R}}{dt}\right)_i = \left(\frac{d\vec{R}}{dt}\right)_r + \vec{\omega}\times\vec{R} \qquad \text{.....(i)}$$

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This equation holds for all vector and relates the time derivatives of a vector in the frame S and S′. Therefore, equation (i) may be written in the form of operator equation.

$$\left(\frac{d}{dt}\right)_i = \left(\frac{d}{dt}\right)_r + \vec{\omega} \qquad \text{.....(ii)}$$

Put $\frac{d\vec{R}}{dt} = \vec{v}$ for the velocity of the particle, we get

$$\vec{v}_i = \vec{v}_r + \vec{\omega}\times\vec{R} \qquad \text{.....(iii)}$$

If we operate equation (ii) on velocity vector $\vec{v}$, we have.

$$\left(\frac{d\vec{v}_i}{dt}\right)_i = \left(\frac{d\vec{v}_i}{dt}\right)_r + \vec{\omega}\times\vec{v}_i$$

Putting the value of $\vec{v}_i$ in the right hand side of above relation, we get

$$\left(\frac{d\vec{v}_i}{dt}\right)_i = \frac{d}{dt}\left(\vec{v}_r + \vec{\omega}\times\vec{R}\right)_r + \vec{\omega}\times\left(\vec{v}_r + \vec{\omega}\times\vec{R}\right)$$

$$= \left(\frac{d\vec{v}_r}{dt}\right)_i + \frac{d\vec{\omega}}{dt}\times\vec{R} + \vec{\omega}\times\left(\frac{d\vec{R}}{dt}\right)_r + \vec{\omega}\times\vec{v}_r + \vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right)$$

Put $\frac{d\vec{v}}{dt} = \vec{a}$ and $\left(\frac{d\vec{R}}{dt}\right)_r = \vec{v}_r$

We get

$$\vec{a}_i = \vec{a}_r + 2\vec{\omega}\times\vec{v}_r + \vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) + \frac{d\vec{\omega}}{dt}\times\vec{R}$$

In case of earth $\vec{\omega}$ is constant, so $\frac{d\vec{\omega}}{dt} = 0$,

Then $\vec{a}_i = \vec{a}_r + 2\vec{\omega}\times\vec{v}_r + \vec{\omega}\left(\vec{\omega}\times\vec{R}\right)$ .....(iv)

If m is the mass particle, then the force in the rotating frame is

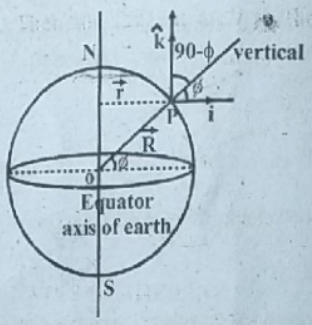
$$m\vec{a}_r = m\vec{a}_i - 2m\vec{\omega}\times\vec{v}_r - \vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right)$$

$$m\vec{a}_r = \vec{F}_i + \vec{F}_0$$

Fictitious force $\vec{F}_0 = -2m\vec{\omega}\times\vec{V}_r - m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right)$ .....(v)

where - $2m\vec{\omega}\times\vec{v}_r$ is coriolis force and $-m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right)$, the centrifugal force.

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All are the familiar with the centrifugal force. It is the only fictitious force, acting on a particle which is at rest $(\vec{v}_r = 0)$ in the rotating frame.

We may write the centrifugal force as

$$-m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) = m\vec{\omega}^2 r$$

where r is the vector from the axis of earth to the particle and normal to it, because

$$\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) = \left(\vec{\omega}.\vec{R}\right)\vec{\omega} - \left(\vec{\omega}.\vec{\omega}\right)\vec{R}$$

$$= \omega^2 R\sin\phi\hat{k} - \omega^2 R\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right)$$

$$\left[\because \vec{\omega} = \omega\hat{k} \text{ and } \vec{R} = R\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right)\right]$$

$$= -\omega^2 R\cos\phi\hat{i} = -\omega^2 r$$

Coriolis force $\left(-2m\vec{\omega}\times\vec{v}_r\right)$ is a fictitious force which acts on a particle only if it is in motion with respect to the rotating frame. Hence, in the rotating frame if a particle moves with velocity $\vec{v}_r$, then it always experiences force $\left(-2m\vec{\omega}\times\vec{v}_r\right)$ perpendicular to its path opposite to the direction of vector product $\vec{\omega}\times\vec{v}_r$.

Q.2. Discuss the origin and importance of Coriolis force. (2018)

Ans. Coriolis force, in classical mechanics, an inertial force described by the 19th-century French engineer-mathematician Gustave-Gaspard Coriolis in 1835. Coriolis showed that, if the ordinary Newtonian laws of motion of bodies are to be used in a rotating frame of reference, an inertial force - acting to the right of the direction of body motion for counterclockwise rotation of the

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reference frame or to the left for clockwise rotation - must be included in the equations of motion.

The effect of the Coriolis force is an apparent deflection of the path of an object that moves within a rotating coordinate system. The object does not actually deviate from its path, but it appears to do so because of the motion of the coordinate system.

The Coriolis effect is most apparent in the path of an object moving longitudinally. On the Earth an object that moves along a north-south path, or longitudinal line, will undergo apparent deflection to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. There are two reasons for this phenomenon: first, the Earth rotates eastward; and second, the tangential velocity of a point on the Earth is a function of latitude (the velocity is essentially zero at the poles and it attains a maximum value at the Equator). Thus, if a cannon were fired northward from a point on the Equator, the projectile would land to the east of its due north path. This variation would occur because the projectile was moving eastward faster at the Equator than was its target farther north. Similarly, if the weapon were fired toward the Equator from the North Pole, the projectile would again land to the right of its true path. In this case, the target area would have moved eastward before the shell reached it because of its greater eastward velocity. An exactly similar displacement occurs if the projectile is fired in any direction.

The Coriolis deflection is therefore related to the motion of the object, the motion of the Earth, and the latitude. For this reason, the magnitude of the effect is given by $2v\omega\sin\phi$, in which v is the velocity of the object, ω is the angular velocity of the Earth, and ϕ is the latitude.

The Coriolis effect has great significance in astrophysics and stellar dynamics, in which it is a controlling factor in the directions of rotation of sunspots. It is also significant in the earth sciences, especially meteorology, physical geology, and oceanography, in that the Earth is a rotating frame of reference, and motions over the surface of the Earth are subject to acceleration from the force indicated. Thus, the Coriolis force figures prominently in studies of the dynamics of the atmosphere, in which it affects prevailing winds and the rotation of storms, and in the hydrosphere, in which it affects the rotation of the oceanic currents. It is also an important consideration in ballistics, particularly in the launching and orbiting of space vehicles. In modern physics, application of a quantity analogous to Coriolis force appears in electrodynamics wherever instantaneous voltages generated in rotating electrical machinery must be calculated relative to the moving reference frame: this compensation is called the Christoffel voltage.

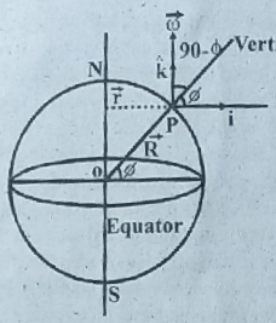
Q.3. A particle of mass m is Moving in the horizontal plane with velocity V in the Earth's rotating frame. Calculate the horizontal component of the Coriolis force at (i) a point on the equator and (ii) the north pole. (2007)

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Ans. We know that fictitious force

$$\vec{F}_0 = -2m\vec{\omega}\times\vec{v}_r - m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) \quad \text{.....(i)}$$

where $-2m\vec{\omega}\times\vec{v}_r$ is the coriolis force and $-m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right)$, the centrifuga forces.



All are familiar with the centrifugal force. It is the only fictitious force acting on a particle which is at rest $(\vec{v}_r = 0)$ in the rotating frame. Centrifugal force as

$$-m\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) = m\omega^2\vec{r} \quad \text{.....(ii)}$$

where $\vec{r}$ is the vector from the axis of earth to the particle and normal to it because.

$$\vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) = \left(\vec{\omega}.\vec{R}\right)\vec{\omega} - \left(\vec{\omega}.\vec{\omega}\right)\vec{R}$$

$$= \omega^2\vec{R}\sin\phi\hat{k} - \omega^2\vec{R}\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right)$$

$\because$ $\vec{\omega} = \omega\hat{k}$ and $\vec{R} = R\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right) = -\omega^2 R\cos\phi\hat{i}$

$$= -\omega^2\vec{r}$$

The coriolis force is a fictitious force which acts on a particle only if it is in motion with respect to the rotating frame. Hence in the rotating frame if a particle moves with velocity $\vec{v}_r$, then it always experiences a force $\left(-2m\vec{\omega}\times\vec{v}_r\right)$ perpendicular to its path opposite to direction of vector product $\vec{\omega}\times\vec{v}_r$. The coriolis force causes a moving particle in the northern hemisphere to deflect towards the right of its path, in the southern hemisphere the deflection is towards the left of the path.

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When the velocity $\vec{v}_r$ of the moving particle in the rotating frame is horizontal, the horizontal component of coriolis force will be only due to vertical component $\omega\sin\phi$ of angular velocity where ϕ is the latitude of the place. Hence the magnitude of horizontal coriolis force is $2m\omega v_r\sin\phi$ and is zero for $\phi = 0$ i.e. at the equator. Coriolis force is that the freely falling bodies deviate from their true vertical path. This deviation is always towards east direction in either of hemisphere of the earth. This deviation is due to the fact that the vertical fall means a velocity along vertical direction. Coriolis force starts to act on it in the horizontal direction and deviates the freely falling particle from the true vertical direction.

Q.4. Show that time intervals are invariant in Galilean transformations. (2005)

Ans. Invariance of Time Interval : -

Let two events occurring in space at the instants t_1 and t_2 respectively relative to frame S.

Time interval between the two events as measured in frame S is given by

$$\Delta t = t_2 - t_1$$

Let the same events occur at the instants t'_1 and t'_2 respectively relative to frame S'. The time interval between the same two events as measured in frame S' is given by

$$\Delta t' = t'_2 - t'_1$$

According to Galilean transformation $t' = t$

Or $t'_1 = t_1$

and $t'_2 = t_2$

$\therefore$ $\Delta t' = t'_2 - t'_1 = t_2 - t_1 = \Delta t$

Hence the time interval between two events is invariant under Galilean transformation.

*** Intertial Mass: - (2014)**

We are aware of the property of inertia possessed by a material body, i.e., its property of resisting any change in its position of rest or uniform motion along a straight line. Thus, we know that some effort or force has to be applied to put into motion a block, of whatever material, lying at rest on a smooth, frictionless horizontal surface, or to stop or even to show it down or decrease its acceleration if it be in motion, and also that the bulkier the block, i.e., the greater its mass, the greater the effort required for the purpose. Obviously, gravity does not come into the picture in this case (the motion being perpendicular to the direction in which the gravity acts). In fact, the effort or the force required to set the block in motion or to stop it would still be the same if the experiment were performed in gravity-free space.

This inertness or reluctance of the block, or a material body, in general, to change its state of rest or uniform motion along a straight line is thus an

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inherent property of the body in virtue of its mass which, since it represents the inertia of the body, is called its inertial mass. It can obviously be measured in terms of the ratio between the force (F) applied and the acceleration (a) produced in the body. For, in accordance with Newton's second law of motion we have

$F = ma$ and, therefore, $m = F/a$.

Here, m represents the inertial mass of the body and may, therefore, more appropriately, be denoted by the symbol m_i.

(2014)

*** Gravitational Mass: -**

We can also measure the mass of a body in terms of the gravitational force of attraction exerted on it by another body, such as the earth. For, as we know, we have to exert an upward force to prevent a body from falling under the gravitational force of the earth and again, the bulkier the body, the greater the force required to hold it. Here, clearly, the property of inertia plays no part but only the action of gravity. This mass of a body, measured in terms of the gravitational force of attraction on it due to another body, usually the earth, is called its gravitional mass. Thus, if F and F' be the forces of attraction exerted by a given body A, of mass M, on two bodies B and C, of masses m and m' respectively, at the same distance R from it, we have

$F = mMG/R^2$ and $F' = m'MG/R^2$, whence, $F/F' = m/m'$.

Here, m and m' are the gravitational masses of the two bodies and may be denoted by m_g and m'_g respectively to distinguish them from their inertial masses m_i and m'_i. So that, $F/F' = m_g/m'_g$.

Thus, we see that the gravitational masses of the two bodies B and C, are directly proportional to the forces of attraction exerted on them by the body A.

If the body A happens to be the earth, clearly $F = w$ and $F' = w'$, where w and w' are the weights of bodies B and C respectively at the given place. We, therefore, have

$$w = \frac{m_g M}{R^2} G \text{ and } w' = \frac{m'_g M}{R^2} G, \text{ whence, } \frac{w}{w'} = \frac{m_g}{m'_g}$$

Or, the ratio between the gravitational masses of two bodies is equal to the ratio between their weights at a given place.

It follows, therefore, that the mass of a body, as measured by a spring balance or by balancing it in one pan against a standard mass in the other, of a physical balance, is its gravitational mass.

*** Principle of Equivalence: -**

We know that how the mass of a body, when subjected to a gravitational attraction but no acceleration, (i.e., its gravitational mass) comes out to be the same as when it is subjected to an acceleration but no gravitational attraction (i.e., its inertial mass). This gave Einstein the idea that a gravitational field can be imitated by a field of acceleration and this, ultimately, led to the formulation of his general theory of relativity, wherein he showed that a non-accelerating or

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inertial frame of reference in which there is a gravitational field is physically equivalent to a reference frame accelerating uniformly with reference to the inertial frame but in which there is no gravitational field. This means, in other words, that experiments carried out in the two frames, under the same conditions, will yield identical results. This is called the principle of equivalence.

Thus, if we have two reference frames S and S', the former an inertial or non-accelerating frame on the surface of the earth, so as to have a uniform gravitational field in it, and the latter, a frame of reference moving away from the earth with an acceleration $-g$ with respect to S into far space where there is no gravitational field in it, all physical phenomena occurring in the two frames of reference will be identical. For instance, just as an object dropped in the inertial frame S will fall freely with an acceleration g with respect to it, so also will an object dropped in frame S' fall freely with an acceleration g with respect to it; or, just as a person sitting in frame S will experience an upward force equal to his weight mg by way of reaction of the floor, so will a person sitting in frame S' experience an identical reaction equal to his weight. So that, if the frame S be a closed ship, a person inside it would not be able to say from any observations he makes inside it whether he is in a uniformly accelerating frame without a gravitational field or in a non-accelerating frame with a uniform gravitational field.

Einstein thus showed that just as, in accordance with his special theory of relativity, we can only talk of relative, and not absolute, velocity of a frame of reference, so also in accordance with the principle of equivalence, we can only talk of relative and not absolute acceleration of a frame of reference.

The principle of equivalence thus demands the equality of inertial and gravitational mass. Indeed, this equality of inertial and gravitational mass is itself sometimes referred to as the principle of equl-valence.

It thus follows from this principle that in a reference frame accelerating towards the inertial frame of the earth, with an acceleration g (e.g., a falling lift), all particles, originally subject to its gravitational field, will become free because of the gravitational effect getting cancelled by the effect due to acceleration. This explains the weightlessness of a person inside an earth satellite.

Centre of Mass: - **(2006)**

When we consider the motion of a system consisting of a large number of particles, say, N, there is one point in it which behaves as though the entire mass of the system (i.e., the sum of the masses of all the N individual particles) were concentrated there and its motion is the same as would ensue if the external forces acting on the system were applied directly to it. This point is called the centre of mass of the system.

*** Motion of Centre of Mass: -**

Q.5. Show that in absence of external force, the velocity of center of mass remains constant. **(2006)**

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Let us consider the motion of a system of N particles of individual masses $m_1, m_2, ...m_N$ and total mass M, it being assumed that no mass enters or leaves the system during its motion, so that M remains constant, i.e., leaving out, for the moment, cases like that of a rocket where the mass goes on decreasing (due to expulsion of the hot gases) throughout its motion. Then, as we have seen, we have the relation

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + ...m_N\vec{r}_N,$$

where $\vec{R}$ is the position vector of the centre of mass and $\vec{r}_1, \vec{r}_2, ...\vec{r}_N$, the position vectors of the individual particles.

Differentiating this expression with respect to t, we have

$$Md\vec{R}/dt = m_1 d\vec{r}_1/dt + m_2 d\vec{r}_2/dt + ...m_N d\vec{r}_N/dt \quad .....(i)$$

Clearly $d\vec{R}/dt = \vec{V}$, the velocity of the centre of mass and $d\vec{r}_1/dt$, $d\vec{r}_2/dt$ ... $d\vec{r}_N/dt$, the velocities $\vec{v}_1, \vec{v}_2, ...\vec{v}_N$ of the individual particles. So that,

$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + ...m_N\vec{v}_N = \sum m_N\vec{v}_N = \vec{P} \quad .....(ii)$$

indicating that the total linear momentum of the system is equal to the product of the total mass of the particles and the velocity of the centre of mass of the system.

It follows, therefore, that the velocity of the centre of mass, i.e.

$$\vec{V} = \sum m_N\vec{v}_N/\vec{M} \quad .....(iii)$$

Since, in the absence of any external force, the momentum remains conserved, we have MV = a constant.

Or, V = a constant, [M being constant.

i.e., the velocity of the centre of mass of a system remains constant if no external force be applied to it.

*** Centre of Mass of Reference Frame: -** **(2019)**

A frame of reference carried by the centre of mass of an isolated system of particles (i.e., a system not subjected to any external forces) is called the centre-of-mass or the C-frame of reference. In this frame, obviously, $\vec{R}$ (the position vector of the centre of mass) is equal to zero and hence the velocity of the centre of mass, i.e., $\vec{V} = d\vec{R}/dt$ also equal to zero. In consequence, the linear momentum $\vec{P}$ of the system too, given by $\vec{P} = \overrightarrow{MV} = \sum \overrightarrow{m_k v_k}$, is zero. The C-frame of reference is, therefore, also called the zero-momentum frame. And; since, in the absence of external forces, the centre of mass, and hence the C-frame, moves with constant velocity, it is also an inertial frame.

***Newton's Law of Gravitation: -**

This law was first announced by Newton in the year 1687 in his monumental work Principia, hailed as the greatest production of the human mind.

The law states that every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product

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of their masses and inversely proportional to the square of the distance between them.

Thus, if the masses of two particles, distant r from each other be m and m', the force of gravitational attraction between them, say; $F \propto mm'/r^2$.

Or, $$F = \frac{mm'}{r^2}G,$$

where G is a universal constant, called the universal gravitational constant, or usually simply the Gravitational constant.

*** Gravitational Potential: -** **(2005, 12)**

The gravitational potential V at a point distant r from a body of mass m is equal to the amount of work done in moving a unit mass from infinity (where the gravitational force and potential are zero) to that point. Thus,

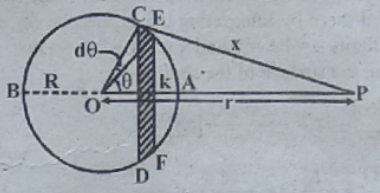
$$V = -\int_{\infty}^{r} Edr = -\int_{\infty}^{r} \frac{m}{r^2}G = -\frac{m}{r}G.$$

Q.6. Derive the expressions for gravitational potentials and fields at a point (a) Outside and (b) On the surface of thin uniform spherical shell. (2019)

*** GRAVITATIONAL POTENTIAL AND FIELD DUE TO A SPHERICAL SHELL (2012, 14)**

1. Gravitational Potential: -

(a) At a Point Outside the Shell : Let P be a point, distant r from the centre O of a spherical shell of radius R figure and surface density (i.e., mass per unit area of surface) σ.



(i)

Join OP and cut out a slice CEFD in the form of a ring by two planes CD and EF close to each other and perpendicular to the radius OA of the shell, and let angle EOP be θ and the small angle COE = dθ.

Clearly, radius of the ring, EK = OE sin θ = R sin θ, so that its circumference = 2πR sin θ and its width = CE = R dθ.

Therefore, surface area of the ring = circumference × width = 2πR sin θ. R dθ and hence its mass = 2π R sin θ. Rdθ σ = $2\pi R^2 \sin\theta\, d\theta\, \sigma$.

If EP = x, every point of the slice or the ring is at a distance x from P and, therefore, potential at P due to the ring, say,

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$$dV = -\frac{\text{mass of slice}}{x}G = \frac{2\pi R^2 \sin\theta d\theta\sigma}{x}G. \qquad \text{.....(i)}$$

Now, in ΔOEP, $EP^2 = OE^2 + OP^2 - 2OE.OP\cos\theta$. Or, $x^2 = R^2 + r^2 - 2Rr\cos\theta$, which, on differentiation, gives $2xdx = 0 + 0 + 2Rr\sin\theta d\theta$, whence, $x = R\,r\sin\theta d\theta\,dx$. [R and r being constants

Substituting this value of x in expression (i) above, we have

$$dV = -\frac{2\pi R^2 \sin\theta d\theta\sigma}{Rr\sin\theta\, d\theta}Gdx = -\frac{2\pi R\sigma G}{r}dx.$$

The integral of this between the limits $x = AP = (r - R)$ and $x = BP = (r + R)$ then clearly gives the potential V at P due to the whole shell. Thus,

$$V = \int_{(r-R)}^{(r+R)} -\frac{2\pi R\sigma G}{r}dx = -\frac{2\pi R\sigma G}{r}\int_{(r-R)}^{(r+R)}dx$$

$$= -\frac{2\pi R\sigma G}{r}\left[x\right]_{(r-R)}^{(r+R)} = -\frac{2\pi R\sigma G}{r}2R = -\frac{4\pi R^2\sigma G}{r}.$$

Clearly, $4\pi R^2$ is the surface area of the shell and, therefore, $4\pi R^2\sigma$, its mass M. We, therefore, have gravitational potential at P due to the whole shell, $V = -\frac{M}{r}G$, i.e., the same as due to a mass M at O.

In other words, the mass of the shell behaves as though it were concentrated at its centre.

(b) At a Point on the Surface of the Shell : In figure above, if we imagine the point P to be at A, i.e., on the surface of the shell itself, we obtain the gravitational potential there by integrating the expression for dV (obtained above) between the limits $x = PA = 0$ and $x = PB = 2R$. So that, gravitational potential at a point on the surface of the shell, i.e.,

$$V = \int_0^{2R} -\frac{2\pi R\sigma G}{r}dx = -\frac{2\pi R\sigma G}{r}\left[x\right]_0^{2R}$$

$$= -\frac{4\pi R^2\sigma G}{r} = -\frac{M}{r}G = -\frac{M}{R}G. \qquad [\because \text{here } r = R]$$

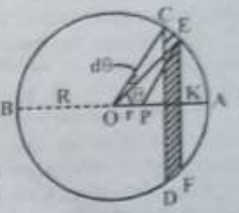
Again, therefore, the mass of the shell behaves as though it were concentrated at its centre.

Q. Obtain an expression for the gravitational field due to a uniform spherical shell of radius R at a point having distance $r < R$ from the centre.

(c) At a Point Inside the Shell : Let the point P now lie anywhere inside the spherical shell (figure), such that $OP = r$. Then, proceeding as in case (a) above, we have potential at P due to the ring CEFD, i.e.,

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$$dV = -2\pi\frac{R\sigma G}{r}dx.$$



Differentiating this expression between the limits $x = PA = (R - r)$ and $x = PB = (R + r)$, we therefore have potential at P due to the whole shell, i.e.,

$$V = \int_{(R-r)}^{(R+r)} -\frac{2\pi R\sigma G}{r}dx = -\frac{2\pi R\sigma G}{r}\left[x\right]_{(R-r)}^{(R+r)}$$

$$= -\frac{2\pi R\sigma G}{r}\cdot 2r = -4\pi R\sigma G$$

Or, multiplying and dividing by R, we have

$$V = -\frac{4\pi R^2\sigma G}{R} = -\frac{M}{R}G, \qquad [\because 4\pi R^2\sigma = M.$$

the same as at a point on the surface of the shell.

Since the point P has been taken anywhere inside the shell, it follows that the gravitational potential at all points inside a spherical shell is the same and is numerically equal to the value of the gravitational potential on the surface of the shell itself, which is also its maximum (negative) value.

2. Gravitational Field: - **(2005)**

(a) At a Point Outside the Shell : Since the potential at a point outside the shell, distant r from its centre figure(i), i.e., when $r > R$, is given by $V = -MG/r$, we have intensity of the gravitational field at the point,

$$E = -dV/dr = -\frac{d}{dr}\left(-\frac{M}{r}G\right) = -\frac{M}{r^2}Gr$$

Or, $E = -\frac{M}{R^2}G$,

i.e., the same as though the whole mass of the shell were concentrated at its centre.

Force on a point mass m. If instead of a unit mass at P figure (i), we have a point-mass m there, its potential energy will be given by $U = mV = -mMG/r$ and the force acting on it by $F = -dU/dr = -(mM/r^2)G$, the same as due to a mass M at O.

Thus, for all points lying outside it (i.e., for all values of $r > R$), a spherical shell behaves as though its whole mass were concentrated at its centre.

(b) At a Point on the Outer Surface of the Shell : As we know, the gravitational potential at a point on the outer surface of the shell is given by $V = -MG/R$.

Therefore, intensity of the gravitational field at the point, i.e.,

$$E = -\frac{d}{dr}\left(-\frac{MG}{R}\right) = -\frac{M}{R^2}G,$$

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again, as though the mass of the shell were concentrated at its centre O.

(c) At a Point Inside the Shell : We have seen that the gravitational potential at all points inside a spherical shell is the same.

Now, the gravitational field at a point is given by the space rate of change of potential (i.e., the potential gradient) there. Therefore, field at the point is given by $E = -dV/dr$.

Since V is constant for all points inside the shell, $dV/dr = 0$, i.e., the field in the interior of the shell is zero at all points. In other words, there is no gravitational field inside a spherical shell.

Force acting on a mass m at a point inside the shell. Since the gravitational field inside a spherical shell is zero, the force acting on a unit mass or any mass m at any point inside the shell is zero. This may also be seen from the following:

Considering a mass m at P, we have its P.E. $= U = mV = -mMG/R$, where M is the mass of the shell and R, its radius. And, therefore, force acting on mass m, i.e., $F = -dU/dR = 0$.

The gravitational field at a point on the inner surface of the shell is zero, that at a point on its outer surface is—MG/R^2, i.e., there is a sudden change in the value of the field from—MG/R^2 on its outer surface to zero on its inner surface. There is thus a definite, sharp discontinuity in the gravitational intensity at the surface of a spherical shell.

*GRAVITATIONAL POTENTIAL AND FIELD DUE TO A SOLID SPHERE

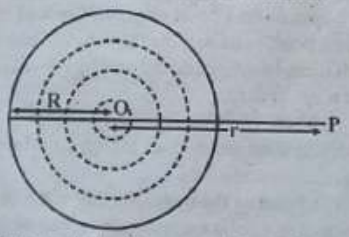
Related Questions -

Q.7. Obtain the expression for the intensity of gravitational field due to a solid sphere of mass M and radius R at a point at the surface and centre of the sphere. (2015)

Q. Find the gravitational potential and field due to a solid sphere at a point outside the sphere. (2013)

I. Gravitational Potential: -

(a) At a Point Outside the Solid Sphere : Let P be a point distant r from the centre O of a solid sphere, of mass M and radius R, outside the sphere, i.e., with $r > R$, figure, where the gravitational potential due to the sphere is to be determined.



Imagine the sphere to consist of a number of spherical shells (shown dotted), one inside the other, concentric with the sphere, and of masses m_1, m_2, m_3 etc. The, gravitational potential at P due to each spherical shell = - (mass of spherical shell) × G/R. So that potentials at P due to the different shells are $-m_1G/R$, $-m_2G/R$, $-m_3G/R$ etc. And, therefore, potential at P due to all the shells constituting the sphere, i.e., due to the whole solid sphere is

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given by $V = -(m_1 + m_2 + m_3 + ...)\,G/R$, because potential is a scalar quantity.

Clearly, $(m_1 + m_2 + m_3 + ...) = M$, the mass of the solid sphere. So that, gravitational potential at P due to the solid sphere, i.e., $V = -\frac{M}{r}G$.

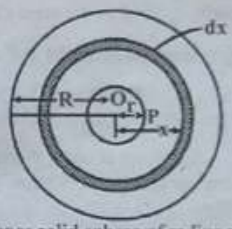
Again, therefore, the sphere behaves as though its whole mass is concentrated at its centre.

(b) At a Point on the Surface of the Solid Sphere : Clearly, if the point P lies on the surface of the solid sphere, we have $r = R$, the radius of the sphere.

So that, gravitational potential at a point on the surface of a solid sphere $= -\frac{M}{R}G$.

(c) At a Point Inside the Solid Sphere : (2007)

Let the point P now lie inside the solid sphere at a distance r from the centre O of the sphere, figure, i.e., now $r < R$.



The solid sphere may be imagined to be made up of an inner solid sphere of radius r surrounded by a number of spherical shells, concentric with it and with their radii ranging from r to R. The potential at P due to the whole solid sphere is then clearly equal to the sum of the potentials at P due to the inner solid sphere and all the spherical shells outside it.

Clearly, point P lies on the surface of the inner solid sphere of radius r and inside all the spherical shells of radii greater than r. So that, potential at P due to the inner solid sphere of radius r

$$= -\frac{\text{mass of the sphere}}{r}G = -\frac{4}{3}\pi r^3\rho G/r = -\frac{4}{3}\pi r^2\rho G,$$

because mass of the inner solid sphere $= \frac{4}{3}\pi r^3\rho$, where ρ is the volume density of the sphere.

To determine the potential at P due to all the outer shells, let us consider one such shell of radius x and thickness dx, i.e., of volume = area × thickness $= 4\pi x^2 dx$ and hence of mass $= 4\pi x^2 dx\rho$.

Since potential at a point inside a shell is the same as that at a point on its surface, we have potential at P due to this shell

$$= -\frac{4\pi x^2 dx\rho}{x}G = -4\pi x dx\rho G.$$

∴ potential at P due to all the shells

$$= \int_r^R -4\pi\rho G x dx = -4\pi\rho G\int_r^R x dx = -4\pi\rho G\left[\frac{x^2}{2}\right]_r^R$$

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$$= -4\pi\rho G\left(\frac{R^2 - r^2}{2}\right) = -\frac{4}{3}\pi\rho G.\frac{3\left(R^2 - r^2\right)}{2} = -\frac{4}{3}\pi\rho G\frac{\left(3R^2 - 3r^2\right)}{2}$$

∴ potential at P due to the whole solid sphere = potential at P due to inner solid sphere + potential at P due all the outer spherical shells

$$= -\frac{4}{3}\pi r^2\rho G - \frac{4}{3}\pi\rho G\left(\frac{3R^2 - 3r^2}{2}\right) = -\frac{4}{3}\pi\rho G\left(r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\left(\frac{3R^2 - r^2}{2}\right) = -\frac{4}{3}\pi R^3\rho G\left(\frac{3R^2 - r^2}{2R^3}\right).$$

[Multiplying and dividing by R^3.

Clearly, $\frac{4}{3}\pi R^3\rho$ is the mass of the whole solid sphere, i.e., M.

∴ gravitational potential at P due to the solid sphere, i.e.,

$$V = -\frac{M\left(3R^2 - r^2\right)}{2R^3}G.$$

It follows at once, therefore, that if the point P lies at the centre of the sphere, we have r = 0. So that, gravitational potential at the centre of the solid sphere

$$= -M\left(\frac{3R^2}{2R^3}\right)G = -\frac{3}{2}.\frac{M}{R}G.$$

But $-\frac{M}{R}G$, as we know, is the gravitational potential on the surface of the sphere.

We thus have gravitational potential at the centre of solid sphere $= \frac{3}{2}$ time the gravitational potential on its surface. Or, gravitational potential at the centre of the solid sphere : gravitational potential on the surface of the sphere : : 3 : 2.

This means, in other words, that the gravitational potential due to a solid sphere has its maximum (negative) value at its centre.

2. Gravitational Field: -

(a) At a Point Outside the Solid Sphere : We know that the gravitational potential at a point P outside a solid sphere distant r from its centre (i.e., with r > R) is given by V = - MG/r. And, since intensity of the gravitational field at a point is equal to the potential gradient there, we have gravitational field due to a solid sphere at a point P distant r from its centre (r > R), i.e.,

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$$E = -dV/dr = -\frac{d}{dr}\left[-\frac{MG}{r}\right] = -\frac{MG}{r^2},$$

the same as though the whole mass (M) of the sphere were concentrated at its centre.

Alternatively, we may imagine the sphere to consist of a number of concentric spherical shells one inside the other, with their radii ranging from 0 to R. Then, considering one such spherical shell of radius x and thickness dx and hence of mass $4\pi x^2 dx\rho$, we have gravitational field at a point P distant r from the centre of the shell = $-4\pi x^2 dx\rho G/r^2$

And, therefore gravitational field at P due to the whole solid sphere

$$= -\frac{4\pi\rho G}{r^2}\int_0^R x^2 dx = -\frac{4}{3}\pi R^3\rho G/r^2.$$

Clearly, $\frac{4}{3}\pi R^3\rho = M$, the mass of the sphere. So that, gravitational field at P due to the solid sphere, i.e., $E = -MG/r^2$.

Force on a point-mass m. If we replace the unit mass at P by a point-mass m, we have P.E. of the mass, U = mV = -mMG/R.

And, therefore force acting on the point-mass m, i.e.,

$F = -dU/dr = -mMG/r^2$.

(b) At a Point on the Surface of the Solid Sphere : For a point on the surface of the solid sphere, obviously, r = R, the radius of the sphere. We, therefore, have gravitational field at a point on the surface of the solid sphere, i.e.,

$E = -MG/R^2$.

(c) At a Point Inside the Solid Sphere :

Other Related Questions -

Q.8. Obtain an expression for the Gravitational field due to a uniform soild sphere at a point inside the sphere. (2016, 18)

As we know, the gravitational potential at a point inside a solid sphere distant r from its centre (i.e., with r < R) is given by $V = -MG(3R^2 - r^2)/2R^3$. Since the gravitational field at a point is given by the potential gradient there, we have gravitational field due to a solid sphere at a point P inside it, distant r from its centre, i.e.,

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left[-MG\frac{\left(3R^2 - r^2\right)}{2R^3}\right]$$

$$= -MG\frac{2r}{2R^3} = -\frac{MG}{R^3}r, \text{ showing that } E \propto r.$$

Thus, the intensity of the gravitational field at a point inside a solid sphere is directly proportional to the distance of the point from the centre of

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the sphere.

Obviously, therefore, at the centre of the sphere, since $r = 0$, the intensity of the gravitational field is zero, whereas the gravitational potential there, as we have seen under above, has its maximum (negative) value.

If we put $r = R$, the radius of the sphere, we have $E = MG/R^2$ as the field on the surface of the sphere in either case, i.e., whether the point P lies on the outer or the inner side of the surface of the sphere.

This shows clearly that there is a continuity in the intensity of the gravitational field at the surface of a solid sphere, unlike the case ofa spherical shell, where there is a sharp discontinuity in the intensity of the gravitational field at the surface of the shell.

Force on a point-mass m. As usual, the potential energy of a point-mass m at a point P inside the solid sphere at a distance r from its centre is given by $U = mV = -mMG\left(\frac{3R^2 - r^2}{2R^3}\right)$ and, therefore, force acting on the mass, i.e., $F = -\frac{dU}{dr} = -\frac{mMG}{R^3}r$, i.e., the force acting on a mass m placed at a point inside a solid sphere is also directly proportional to its distance from the centre of the sphere and is, therefore, zero at the centre of the sphere.

Q.9. Obtain the expression for the gravitational self energy of a solid sphere of mass M and radius R. **(2008)**

Ans. Gravitational self potential energy of body is defined as the amount of work done in assembling the body from infinitesimal elements, which are initially infinite distance apart from each other.

Let us consider the body is spherical. Its mass M and Radius R the body can be imagined to grow from a single point elements as a sphere of increasing radius at any stage of process, there is an inner care – sphere of radius r, over which an infinitesimal shell of thickness grows. If δM be the mass of shell

$$\delta m = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\sigma\right).dv = 4\pi r^2\sigma.dr$$

$\therefore$ The gravitational potential energy of the shell due to gravitational field of the inner-sphere is given by.

$$U_{shell} = \frac{-GM\delta m}{r}$$

$$= -G\left(\frac{4}{3}\pi r^3\sigma\right)\left(\frac{4\pi r^2\sigma.dr}{r}\right) = -\frac{1}{3}G(4\pi\sigma)^2.r^4dr$$

As the body builds up by successive addition of shell, r changes, from 0 to R. Hence total energy of all the shells will be

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$$U_{star} = \int_0^R U_{shell}dr = \int_0^R -\frac{1}{3}G(4\pi\sigma)^2.r^4dr$$

$$= -\frac{1}{3}G(4\pi\sigma)^2\int_0^R r^4dr = -\frac{1}{3}G(4\pi\sigma)^2.\frac{1}{5}R^5$$

Putting $\frac{4}{3}\pi R^3\sigma = M(\text{mass of star})$

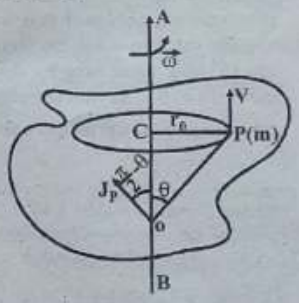
$$U_{star} = -\frac{3}{5}\frac{GM^2}{R}$$

Q.10. Show that $\vec{J}$ and $\vec{\omega}$ are not necessarily in the same direction for a rigid body. **(2008)**

Other Related Questions -

Q. For a body of arbitrary shape and mass distribution show that the angular momentum is given by $\vec{J} = \sum_n M_n\left[\vec{\omega}r_n^2 - \vec{r}_n(\vec{r}_n.\vec{\omega})\right]$. **(2010, 13, 14, 15, 17)**

Ans. Let us consider a rigid body rotating with an angular velocity $\vec{\omega}$ about an axis AB, passing through a fixed point O. The direction of the angular velocity $\vec{\omega}$ will be along the axis of rotation AB. All the particles of the body will move in circular paths about the axis AB.



Let at any instant $\vec{r}$ be the position vector of a particle P of the body. Speed the particle $P = \omega r_0$

$= \omega r \sin POC$

velocity of the particle P, $v = \vec{\omega}\times\vec{r}$ .....(i)

The direction of this velocity $\vec{v}$ at any instant will be perpendicular to the position vector and tangential to the circular path.

The angular momentum of the particle P about the point O is given by

$\vec{J}_p = \vec{r}\times m\vec{v}$

whose direction is perpendicular to $\vec{r}$ and $\vec{v}$ evidently this direction will not coincide in general to the angular velocity vector $\vec{\omega}$.

Hence the angular momentum $\vec{J}$ of the entire body about the point O will be obtained by summing over $\vec{J}_p$ for all the particles of the body.

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$$\bar{J} = \sum m\bar{r} \times \bar{v} = \sum m\bar{r} \times (\bar{\omega} \times \bar{r})$$

whose direction, in general, will not be along $\bar{\omega}$.

$$\bar{J} = \sum m.\{(\bar{r}.\bar{r})\bar{\omega} - (\bar{r}.\bar{\omega})\bar{r}\} \quad \ldots\ldots\ldots(ii)$$

or $$\bar{J} = \sum m\{r^2\bar{\omega} - (\omega r\cos\theta)\bar{r}\} \quad \ldots\ldots\ldots(iii)$$

From equation (ii) it is clear that $\bar{J}$ and $\bar{\omega}$ in general are not in the same direction.

Q.11. What are the principle moments of inertia and principal axes of a rigid body? Explain these in terms of inertia tensor. (2008)

Ans. Angular momentum $\bar{J}$ of the entire body about the point O

$$\bar{J} = \sum m\bar{r} \times \bar{v}$$

$$\bar{J} = \sum m\bar{r} \times (\bar{\omega} \times \bar{r}) \quad \ldots\ldots\ldots(i)$$

From above it is clear that in general $\bar{J}$ & $\bar{\omega}$ are not in the same direction. It can be proved that what ever the shape of body may be, there are three mutually perpendicular axes, such that the angular momentum of the body is along the angular velocity, when the body is rotated about that axis. The moments of Inertia about these axes are denoted by I_1, I_2 and I_3 and are called principal moments of Inertia with corresponding axes as principal axes. Thus if a body is rotating about a principal axis, then, $\bar{J}$ and $\bar{\omega}$ are in the same direction, i.e.

$$\bar{J} = I\bar{\omega} \quad \ldots\ldots\ldots(ii)$$

where $\bar{J} = J_x\hat{i} + J_y\hat{j} + J_z\hat{k}$

$$\bar{\omega} = \omega_x\hat{i} + \omega y\hat{j} + \omega_z\hat{k}$$

$$\Rightarrow \quad J_x\hat{i} + J_y\hat{j} + J_z\hat{k} = I(\omega_x\hat{i} + \omega y\hat{j} + \omega_z\hat{k})$$

$$\Rightarrow \left.\begin{aligned} J_x &= I\omega_x \\ J_y &= I\omega_y \\ J_z &= I\omega_z \end{aligned}\right\} \quad \ldots\ldots\ldots(iii)$$

We know that in inertia tensor

$$\left.\begin{aligned} J_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ J_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ J_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}\right\} \quad \ldots\ldots\ldots(iv)$$

Comparing equation (iii) and (iv), we get

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$$\left.\begin{aligned} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z &= I\omega_x \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z &= I\omega_y \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z &= I\omega_z \end{aligned}\right.$$

Or $$\left.\begin{aligned} (I_{xx} - 1)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z &= 0 \\ I_{xy}\omega_x + (I_{yy} - 1)\omega_y + I_{yz}\omega_z &= 0 \\ I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - 1)\omega_z &= 0 \end{aligned}\right\}$$

Or $$\begin{bmatrix} I_{xx} - 1 & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - 1 & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0 \quad \ldots\ldots\ldots(vi)$$

For these equations to have nontrivial solutions, the determinant of the coefficient must varnish.

$$\begin{bmatrix} I_{xx} - 1 & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - 1 & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - 1 \end{bmatrix} = 0$$

This is known as characteristic equation and its solution are know as eigen values, the three solution $I = I_1, I_2, I_3$ are three principal moment of inertia.

Q.12. Show that the angular momentum $\bar{J}$ of a system of particles can be expressed in the form

$$\bar{J} = \bar{J}_{cm} + \bar{R} \times \bar{P}$$

where $\bar{J}_{cm}$ = angular momentum about the center of mass $\bar{R}$ = Position vector of the center of mass $\bar{P}$ = total linear momentum. (2009, 12, 15, 18)

Ans. The total angular momentum $\bar{J}$ of a system of particles can be expressed in a convenient and important form by using the velocity of the center of mass and velocity of particle relative to the center of mass.

If $\bar{R}$ and $\bar{v}$ are the position vector and velocity of the center of mass, then the position vector and velocity of particle with respect to the reference plate are given by

$$\bar{r}_n = \bar{R} + \bar{r}_{nc}$$

and $$\bar{v}_n = \bar{V} + \bar{v}_{nc}$$

Hence the total angular momentum for a system of N particles is –

$$\bar{J} = \sum_{n=1}^{N} m_n(\bar{R} + \bar{r}_{nc}) \times (\bar{V} + \bar{v}_{nc})$$

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$$= \sum m_\alpha (\vec{R} \times \vec{V}) + \sum m_\alpha (R \times V_{\alpha c}) + \sum m_\alpha (\vec{r}_{\alpha c} \times \vec{V}) + \sum m_\alpha (\vec{r}_{\alpha c} \times \vec{v}_{\alpha c}) \quad \text{.....(i)}$$

But $\vec{r}_{\alpha c} = \vec{r}_\alpha - \vec{R}$

Or $m_{\alpha c} \vec{r}_{\alpha c} = m_\alpha \vec{r}_\alpha - m_\alpha \vec{R}$

$$\sum m_{\alpha c} \vec{r}_{\alpha c} = \sum m_\alpha \vec{r}_\alpha - \sum m_\alpha \vec{R} = \sum m_\alpha \vec{r}_\alpha - M\vec{R}$$

Where $M = \Sigma m$ = mass of all the particles of the system.

$$M\vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots\ldots\ldots + \sum m_n \vec{r}_n = \sum m_n \vec{r}_n$$

$$\therefore \quad \sum m_{\alpha c} \vec{r}_{\alpha c} = 0$$

Similarly $\sum m_{\alpha c} v_{\alpha c} = 0$

∴ Angular momentum

$$\vec{J} = \vec{R} \times M\vec{V} + \sum (\vec{r}_{\alpha c} \times m_{\alpha c} v_{\alpha c}) \quad \text{.....(ii)}$$

From above equation $\sum (\vec{r}_{\alpha c} \times m_{\alpha c} v_{\alpha c})$ represents the angular momentum of system about the center of mass (J_{cm}) and $M\vec{V} = \vec{P}$ the total linear momentum.

Hence $\vec{J} = \vec{J}_{cm} + \vec{R} \times \vec{P}$

Q.13. Find the center of mass of a homogeneous semicircular plate of radius r. **(2009, 18)**

Ans. AOB is the diameter of semicircular lamina OC is the radius of the lamina. Center of mass of the lamina lies on it.

Area of the lamina $= \frac{\pi R^2}{2}$ if M be the mass of lamina, its mass per unit are $= \frac{2M}{\pi R^2}$ let us consider a strip of thinness dx at a distance x from the origin, then length of strip $= 2\sqrt{R^2 - X^2}$

Area of strip $= 2\sqrt{R^2 - x^2}.dx$

Mass of strip $= 2\frac{\sqrt{R^2 - x^2}}{\pi R^2}.dx.2M$

$$X = \frac{\int x dm}{M} = \frac{\int_0^r x.2\sqrt{R^2 - x^2}.dx.\frac{2M}{\pi R^2}}{M}$$

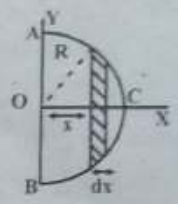
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$$X = \frac{4}{\pi R^2}\int_0^R x\sqrt{R^2 - x^2}\,dx \quad \text{.....(i)}$$

Put $R^2 - x^2 = t^2$

$0 - 2x\,dx = 2t\,dt$

$x\,dx = -t\,dt$



Now $\int_0^R x\sqrt{R^2 - x^2}\,dx$

$$= -\int_R^0 t^2 dt = \int_0^R t^2 dt = \left[\frac{t^3}{3}\right]_0^R$$

$$= \frac{1}{3}(R^3 - 0) = \frac{R^3}{3}$$

From eq.(i), we get

$$X = \frac{4}{\pi R^2}.\frac{R^3}{3} = \frac{4R}{3\pi}$$

Hence, the center of mass of the semicircular lamina will lie on the line of symmetry at a distance $\frac{4R}{3\pi}$ from the center O of the lamina.

Q.14 Write notes moment of inertia tensor. **(2007, 18)**

Ans. Moment of Intertia Tensor: -

Angular momentum,

$$\vec{J} = \sum m\{(\vec{r}.\vec{r})\vec{\omega} - (\vec{r}.\vec{\omega})\vec{r}\} \quad \text{.....(i)}$$

From equation (i), $\vec{J}$ and $\vec{\omega}$ in general are not in the same direction.

Now $\vec{J} = \sum m\left[\left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) r^2 - \left\{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right)\right\}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$ .....(ii)

where $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $r^2 = x^2 + y^2 + z^2$

If J_x, J_y, J_z are the component of $\vec{J}$ along X, Y, Z axes respectively, then

$$J_x = \sum m\left[r^2\omega_x - \left(x\omega_x + y\omega_y + z\omega_z\right)x\right] \quad \text{.....(iii)}$$

or $J_x = \omega_x \sum m\left(r^2 - x^2\right) - \omega_y \sum mxy - \omega_z \sum mxz$

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Similarly $I_{yx} = -\omega_x \sum mxy + \omega_y \sum m\left(r^2 - y^2\right) - \omega_z \sum myz$ .....(iv)

and $I_{zx} = -\omega_x \sum mzx - \omega_y \sum myz + \omega_z \sum m\left(r^2 - z^2\right)$ .....(v)

Equation (iii), (iv) and (v) can be written as

$I_x = I_{xx}\omega_x + I_{xy}\omega_y - I_{xz}\omega_z$ .....(vi)

$I_y = I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$ .....(vii)

$I_z = I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$ .....(viii)

where $I_{xx} = \sum m\left(r^2 - x^2\right) = \sum m\left(y^2 + z^2\right)$

$I_{yy} = \sum m\left(r^2 - y^2\right) = \sum m\left(x^2 + z^2\right)$

$I_{zz} = \sum m\left(r^2 - z^2\right) = \sum m\left(x^2 + y^2\right)$

$I_{xy} = -\sum mxy = I_{yx}, I_{yz} = -\sum myz = I_{zy}, I_{zx} = -\sum mzx = I_{xz}$ .....(ix)

The equation I_{xx}, I_{yy} and I_{zz} are called moment of inertia of body about X, Y and Z axis respectively. The quantities I_{xy}, I_{yz}, I_{zx}.....etc. are known as products of coordinates, are called products of inertia.

Equation (vi), (vii) and (viii) can be written as

$$\begin{bmatrix} I_x \\ I_y \\ I_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{.....(x)}$$

$\Rightarrow \quad \vec{L} = \vec{I}\vec{\omega}$

The nine elements I_{xx}, I_{xy}.....I_{zz} of the 3×3 matrix may be regarded as components of a single entity $\vec{I}$. This entity $\vec{I}$ is called inertia tensor. Since $I_{xy} = I_{yx}$, $\vec{I}$ is symmetric tensor.

Q.15 Two particles of mass m_1 and m_2 are at positions $\vec{r}_1$ and $\vec{r}_2$ are moving with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. When is the velocity of the centre of mass constant? (2010)

Ans. Let us consider the motion of a system of N particles of individual masses m_1, m_2.....m_N and total mass M, it being assumed that no mass enters or leaves the system during its motion, so that M remains constant, i.e., leaving out, for the moment, cases like that of a rocket where the mass goes on decreasing (due to expulsion of the hot gases) throughout its motion. Then as we have seen, we have the relation

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots m_N\vec{r}_N$$

where R is the position vector of the centre of mass and r_1, r_2.....r_N the position

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vectors of the individual particles.

Differentiating this expression with respect to t, we have

$M\vec{dR}/dt = m_1 d\vec{r}_1/dt + m_2 d\vec{r}_2/dt + \ldots\ldots m_N d\vec{r}_N/dt$ .....(1)

Clearly $d\vec{R}/dt = \vec{V}$, the velocity of the centre of mass and $d\vec{r}_1/dt, d\vec{r}_2/dt$ ...$d\vec{r}_N/dt$, the velocities $\vec{v}_1, \vec{v}_2$...$\vec{v}_N$ of the individual particles. So that

$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots m_N\vec{v}_N = \Sigma m_i\vec{v}_i = \vec{P}$ .....(2)

indicating that the total linear momentum of the system is equal to the product of the total mass of the particles and the velocity of the centre of mass of the system.

It follows, therefore, that the velocity of the centre of mass, i.e.

$\vec{V} = \Sigma m_i\vec{v}_i/M$ .....(3)

Since, in the absence of any external force, the momentum remains conserved, we have $M\vec{V}$ = a constant.

Or $\vec{V}$ = a constant [M being constant]

i.e., the velocity of the centre of mass of a system remains constant if no external force be applied to it.

Q.16 Show that the force $\vec{F} = \left(x^2 + y^2\right)\hat{i} + 3xy\hat{j}$ is non-conservative. (2010)

Ans. For a conservative force

$\text{Curl } \vec{F} = 0$

Here, $\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & 3xy & 0 \end{vmatrix}$

$\Rightarrow \quad \hat{i}\left[0 - \frac{\partial}{\partial z}3xy\right] - \hat{j}\left[0 - \frac{\partial}{\partial x}\left(x^2 + y^2\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}3xy - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right]$

$\Rightarrow \quad 0 - 0 + \hat{k}\left[3y - (2y)\right]$

$\Rightarrow \quad \hat{k} \neq 0$

That means force is non-conservative.

Q.17 For a conservative force, show that $\vec{F} = -\vec{\nabla}U$. (2011)

Ans. As we know,

$U(r) = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r}$

Expressing $\vec{F}$ and $d\vec{r}$ in rectangular components, we have

$U(\vec{r}) = -\int\left(\hat{i}, \hat{i} + F_y\hat{j} + F_z\hat{k}\right) \cdot \left(dx\,\hat{i} - dy\,\hat{j} + dz\,\hat{k}\right)$

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or $U(\vec{r}) = -\int F_x dx - \int F_y dy - \int F_y dz$ .....(i)

Differentiating this equation partially with respect to x, y, z, we get

$$F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}, F_z = -\frac{\partial U}{\partial z}$$

Hence the force $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z = -\hat{i}\frac{\partial U}{\partial x} - \hat{j}\frac{\partial U}{\partial y} - \hat{k}\frac{\partial U}{\partial z}$

or $\vec{F} = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right) = -\text{grad } U \text{ or } -\nabla U$ .....(ii)

Thus, a conservative force is the negative gradient of potential energy U.

Q.18. Show that the force $\vec{F} = (2xy + z^2)\hat{i} + x^2\hat{j} + 2xz\hat{k}$ is conservative. Determine its potential function. **(2015, 16)**

Ans. For a conservative force

Curl F = 0

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^2) & x^2 & 2zx \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}2xz - \frac{\partial}{\partial z}x^2\right] - \hat{j}\left[\frac{\partial}{\partial x}2zx - \frac{\partial}{\partial z}(2xy + z^2)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy + z^2)\right]$$

$$= \hat{i}[0] - \hat{j}[2z - 2z] + \hat{k}[2x - 2x] = \hat{i}[0] - \hat{j}[0] + \hat{k}[0] = 0$$

This means force is conservative.

The potential energy function of $\vec{F}$ is give by

$$U = -\int \vec{F}.d\vec{r} = -\int (F_x dx + F_y dy + F_z dz)$$

$$= -\int (2xy + z^2)dx + x^2 dy + 2xz dz$$

$$= -\int (2xy dx + x^2 dy) + (z^2 dx + 2zx dz)$$

$$= -\int \{d(x^2y) + d(z^2x)\} = -\int d(x^2y + z^2x) = -(x^2y + z^2x)$$

Q.19. Prove that the force, $\vec{F} = -m\omega^2\vec{r}$ is conservative. **(2013, 18)**

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Sol. Given $\vec{F} = -m\omega^2\vec{r}$

Since curl $\vec{r} = 0$, we have

curl $\vec{F} = 0$

Hence the force $\vec{F}$ is conservative.

Q.20. A particle moves in the potential energy field $U(x) = Ax - Bx + Cx^2$. Find the expression for force as a function of position. For what value of x does the force vanish.

Sol. $U(x) = Ax - Bx - Cx^2$

Now $F_x = -\frac{\partial U}{\partial x} = -[A - B - 2Cx]$

Now $F_x = 0 \Rightarrow A - B - 2Cx = 0 \Rightarrow x = \frac{A - B}{2C}$.

Q.21. If a particle of mass m is moving on a circular path in a plane with a uniform speed, show that the force acting on it is conservative. **(2011, 17)**

Ans. In figure, a central force has been shown to act away from (or toward) the centre. In the central force field any two arbitrary paths APB and AQB have been drawn between two points A and B. Taking centre O, two dashed sectors of circles have been drawn with radii r and r + dr respectively. Now take two points P and Q on APB and AQB paths respectively, lying between the two sectors figure. Let $\vec{F}_1$ and $\vec{F}_2$ be the central forces at P and Q and let $d\vec{r}_1$ and $d\vec{r}_2$ be the displacements of the particle between two sectors along two paths respectively. If θ_1 and θ_2 are the angles between $\vec{F}_1.d\vec{r}_1$ and $\vec{F}_2.d\vec{r}_2$ respectively, then

$\vec{F}_1.d\vec{r}_1 = F_1 dr_1 \cos\theta_1$ and $\vec{F}_2.d\vec{r}_2 = F_2 dr_2 \cos\theta_2$

As $\vec{F}_1$ and $\vec{F}_2$ lie at equal distances from the centre O, their magnitudes F_1 and F_2 are equal. Also, the separation of the circles measured along the direction of $\vec{F}_1$ is equal to the separation measured along $\vec{F}_2$, hence the projections of $d\vec{r}_1$ and $d\vec{r}_2$ on $\vec{F}_1$ and $\vec{F}_2$ respectively are equal i.e., $dr_1 \cos\theta_1 = dr_2 \cos\theta_2$. Thus

$F_1 dr_1 \cos\theta_1 = F_2 dr_2 \cos\theta_2$ or $\vec{F}_1.d\vec{r}_1 = \vec{F}_2.d\vec{r}_2$

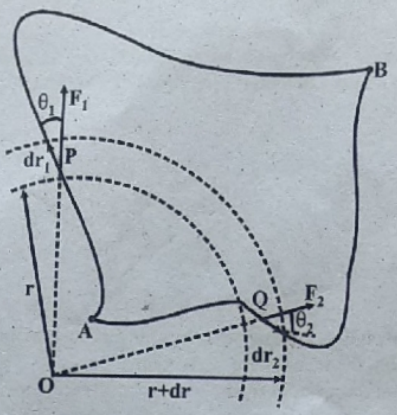
Similarly, the same argument can be given for every pair of segments of the paths APB and AQB so that

$$\int_A^B F \cdot dr = \int_A^B F \cdot dr$$

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(Path APB) (Path AQB)

This means that the work done $W = \int_A^B \vec{F} \cdot d\vec{r}$ by a central force is independent of the path. Hence the central force is conservative.



Q.22. Write short notes on work energy theorem. **(2010, 13, 16, 17)**

Other Related Questions -

Q. Prove $W(A \rightarrow B) = K_B - K_A$ **(2011)**

Ans. Work-Energy Theorem: -

The energy of a particle is defined as the capacity of doing work and it is measured by the amount of work which it can do in the position in which it is. The energy due to the speed of a particle is called its kinetic energy. It is generally represented by the letter K.

According to Newton's second law of motion,

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

where $d\vec{v}/dt$ is the acceleration, produced in a particle of mass m by the force $\vec{F}$.

The amount of work done on the particle, when it is taken from the point A to the point B, is given by

$$W = \int_A^B \vec{F} \cdot d\vec{r} = m\int_A^B \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

Now, $d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}dt$

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and so $\frac{d\vec{v}}{dt}.d\vec{r} = \left(\frac{d\vec{v}}{dt}.\vec{v}\right)dt = \frac{d}{dt}\left(\frac{1}{2}\vec{v}.\vec{v}\right)dt = d\left(\frac{v^2}{2}\right)$

$$\therefore \quad W = \frac{m}{2}\int_A^B d\left(v^2\right) = \frac{m}{2}\left[v^2\right]_A^B = \frac{m}{2}\left(v_B^2 - v_A^2\right)$$

where v_A and v_B are the speeds of the particle at the point A and B respectively.

Thus $W = \int_A^B \vec{F}.d\vec{r} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$.

The energy $\frac{1}{2}mv^2$ is called kinetic energy of the particle. Thus

W = Kinetic energy (K_B) at B - Kinetic energy (K_A) at A
= Change in kinetic energy

Thus, the work done by a force on a particle is equal to the change in its kinetic energy. This is known as work energy theorem.

Q.23. Differentiate between inertial mass and gravitational mass. **(2016)**

Ans. Although inertial mass, passive gravitational mass and active gravitational mass are conceptually distinct, no experiment has ever unambiguously demonstrated any difference between them. In classical mechanics, Newton's third law implies that active and passive gravitational mass must always be identical (or at least proportional), but the classical theory offers no compelling reason why the gravitational mass has to equal the inertial mass. That it does is merely an empirical fact.

Albert Einstein developed his general theory of relativity starting from the assumption that this correspondence between inertial and (passive) gravitational mass is not accidental: that no experiment will ever detect a difference between them (the weak version of the equivalence principle). However, in the resulting theory, gravitation is not a force and thus not subject to Newton's third law, so "the equality of inertial and active gravitational mass remains as puzzling as ever".

The equivalence of inertial and gravitational masses is sometimes referred to as the "Galilean equivalence principle" or the "weak equivalence principle". The most important consequence of this equivalence principle applies to freely falling objects. Suppose we have an object with inertial and gravitational masses m and M, respectively. If the only force acting on the object comes from a gravitational field g, combining Newton's second law and the gravitational law yields the acceleration

$$a = \frac{M}{m}g$$

This says that the ratio of gravitational to inertial mass of any object

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is equal to some constant K if and only if all objects fall at the same rate in a given gravitational field. This phenomenon is referred to as the "universality of free-fall". (In addition, the constant K can be taken to be 1 by defining our units appropriately.)

The first experiments demonstrating the universality of free-fall were conducted by Galileo. It is commonly stated that Galileo obtained his results by dropping objects from the Leaning Tower of Pisa, but this is most likely apocryphal; actually, he performed his experiments with balls rolling down nearly frictionless inclined planes to slow the motion and increase the timing accuracy. As of 2008, no deviation from universality, and thus from Galilean equivalence, has ever been found, at least to the precision 10-12. More precise experimental efforts are still being carried out.

The universality of free-fall only applies to systems in which gravity is the only acting force. All other forces, especially friction and air resistance, must be absent or at least negligible. For example, if a hammer and a feather are dropped from the same height through the air on Earth, the feather will take much longer to reach the ground; the feather is not really in free-fall because the force of air resistance upwards against the feather is comparable to the downward force of gravity. On the other hand, if the experiment is performed in a vacuum, in which there is no air resistance, the hammer and the feather should hit the ground at exactly the same time (assuming the acceleration of both objects towards each other, and of the ground towards both objects, for its own part, is negligible).

A stronger version of the equivalence principle, known as the Einstein equivalence principle or the strong equivalence principle, lies at the heart of the general theory of relativity. Einstein's equivalence principle states that within sufficiently small regions of space-time, it is impossible to distinguish between a uniform acceleration and a uniform gravitational field. Thus, the theory postulates that the force acting on a massive object caused by a gravitational field is a result of the object's tendency to move in a straight line (in other words its inertia) and should therefore be a function of its inertial mass and the strength of the gravitational field.

Q.24. The gravitational potentials of two thin spherical shells of the same surface density at their respective centres are in the ratio of 3:4. If the two shells coalsce into a single one and surface density remains unchanged, obtain the gravitational potential at a point inside the new shell. (2017)

Ans. As given -

$$\frac{(-GM_1/R_1)}{(-GM_2/R_2)} = \frac{3}{4}$$

$$\therefore \quad \frac{M_1}{M_2}\cdot\frac{R_2}{R_1} = \frac{3}{4}$$

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$$\text{or} \quad \frac{\rho(4\pi R_1^2)}{\rho(4\pi R_2^2)}\cdot\frac{R_2}{R_1} = \frac{3}{4}$$

$$\text{or} \quad \frac{R_1}{R_2} = \frac{3}{4}$$

So, let $R_1 = 3R$ and $R_2 = 4R$

Now let R_3 is the radius of bigger shell then,

$$4\pi R_1^2 + 4\pi R_2^2 = 4\pi R_3^2$$

$$\text{or} \quad R_3 = \sqrt{R_1^2 + R_2^2} = 5R$$

$$\Rightarrow \quad \frac{V_3}{V_1} = \frac{-GM_3/R_3}{-GM_1/R_1} = \frac{M_3}{M_1}\cdot\frac{R_1}{R_3}$$

$$= \frac{M_3}{M_1}\cdot\frac{R_1}{R_3} = \frac{(\rho 4\pi R_3^2)R_1}{(\rho 4\pi R_1^2)R_3}$$

$$= \frac{R_3}{R_1} = \frac{5R}{3R} = \frac{5}{3}$$

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Chapter 3 Motion Under Central Forces

Central Forces—Inverse Square Law Forces: -

Related Question -

Q. What are central forces. Prove that central force is conservative. (2012)

Q. A particle moves in a force field given by $\vec{F} = f(r)\vec{r}$. Prove that the angular momentum of the particle is conserved. (2015)

Q. A particle moves in a force field given by $\vec{F} = f(r)\hat{r}$. Prove that force is conservative and the angular momentum of the particle is conserved. (2019)

A central force between two particles is one which is directed along the line joining the two particles and whose magnitude is a function of the distance r between them.

Thus, if r be the distance between the two particles, the central force operating between them is given by $\vec{F} = F_{(r)}\vec{r} = F_{(r)}\vec{r}/r$, where $F_{(r)}$ is a function of distance r and $\vec{r}$ or $\vec{r}/r$, the unit vector along r.

This central force is -ve or +ve, i.e., of attraction or repulsion, according as $F_{(r)} < 0$ or > 0.

Now, if one of the particles be fixed in its position, the central force acting on the other is $\vec{F} = F_{(r)}\vec{r}/r$. If, therefore, J be the angular momentum of the second particle about the fixed particle or the origin, we have

torque acting on it, $\tau = d\vec{j}/dt = \vec{r} \times \vec{F} = r \times F_{(r)}\vec{r}/r = 0$.

Since the torque $\tau = 0$, it is clear that $\vec{j}$ = constant.

In other words, if a particle be moving under the influence of a central force, its angular momentum ($\vec{j}$) remains conserved, (meaning, obviously that there is no change in its magnitude and direction).

Further, the constancy of $\vec{j}$ also implies that the motion of the particle remains confined to a plane. For if p be the linear momentum of the moving particle, we have $\vec{J} = \vec{r} \times \vec{p}$. So that, $\vec{r}\vec{J} = \vec{r}.(\vec{r} \times \vec{p})$.

Or, since in a triple vector product, we may interchange the dots and crosses, we have $\vec{r}.\vec{J} = (\vec{r} \times \vec{r}).\vec{p} = 0$

[$\because$ the cross product $\vec{r} \times \vec{r} = 0$.]

This shows clearly that vectors $\vec{r}$ and $\vec{j}$ are perpendicular to each other or that the motion of the particle is confined, to a plane.

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The general form of a central force is represented by what may be called the **inverse nth power law**, viz., $\vec{F} = \frac{C}{r^n}\vec{r}$, where C is a constant.

The force is -ve or attractive or +ve or repulsive according as $C < 0$ or > 0.

Since a central force is a conservative force, we have

$F_{(r)} = -\text{grad } U = -\frac{dU}{dr} = \frac{C}{r^n}$ where U is the P.E. of the particle.

So that, $dU = -\frac{C}{r^n}dr$, whence, on integration, we have

$U_{(r)} = \frac{C}{(n-1)r^{(n-1)}} + C_1$, where C_1 is a constant of integration.

Since at $r = \infty$, P.E. = 0, we have $C_1 = 0$. And, therefore,

$U_{(r)} = \frac{C}{(n-1)r^{(n-1)}}$.

Now, if n = -1 and C negative, we have $F_{(r)} = -Cr$ and $U_{(r)} = \frac{C}{2}r^2$, which as we know, is the case of a simple harmonic oscillator, the force being always directed towards the fixed point or the origin.

And, if n = 2, we have $F_{(r)} = C/r^2$ and $U = C/r$, where C may be positive or negative if the force be an electrostatic one, depending upon the same or opposite signs of the two point-charges, and always negative if the force be a gravitational one.

Clearly, in this case, the force between the two particles (or point-charges) is inversely proportional to the square of the distance between them. This force is, therefore, known as the central inverse square law force, or more commonly, simply as **the Inverse square law force** and the law pertaining to it, the Inverse square law, viz., that the force between two particles (or two point-charges) is inversely proportional to the square of the distance between them.

This is the one law that occurs most frequently in the entire field of physics. One notable example of it is the force of gravitational attraction between two masses m_1 and m_2, viz., $F = -\frac{m_1 m_2}{r^2}G\,\vec{r}$, which is the familiar Newton's law of gravitation, and where the constant $C = -m_1 m_2 G$, the force being always one of attraction.

Another equally familiar example is Coulomb's law of electrostatic attraction or repulsion (in C.G.S. Gaussian units) between two point-charges q_1 and q_2 (in air or vacuum), viz., $F = \frac{q_1 q_2}{r^2}\vec{r}$, where the constant C is obviously equal to $q_1 q_2$, its sign being +ve or -ve according as the force is one of

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repulsion or attraction, i.e., according as the two charges are of the same or opposite signs.

The inverse square law of force may also be expressed as the inverse first power law of potential energy. For, the inverse square law force (F) being a central, and hence a conservative, force, we have $F_{(r)} = -dU/dr = C/r^2$, whence, $dU = -(C/r^2)dr$, which, on integration, gives $U = \frac{C}{r} + C_1$, where, C_1 is a constant of integration.

Since we assume P.E. to be zero for an infinite distance between the two particles, the constant $C_1 = 0$ and we, therefore, have $U_{(r)} = C/r$, i.e., the potential energy of system of two particles at a separation of r is C/r.

Thus, if $C = -m_1 m_2 G$, we have gravitational potential energy of a system of two particles of masses m_1 and m_2 given by $U = -m_1 m_2 G/r$ And, if $C = q_1 q_2$, we have electrostatic potential energy of a system of two charges q_1 and q_2 given by $U = q_1 q_2/r$ in C.G.S. Gaussian units.

Q.1 If an earth satellite moves to lower orbit and there is some dissipation of energy, yet the satellite speed increases explain. (2006)

Ans. For a satellite in orbit or radius r,

Potential energy $V = -\frac{Gm_1m_2}{r}$

Kinetic energy $K = \frac{1}{2}\cdot\frac{Gm_1m_2}{r}$

Total energy = Potential energy + kinetic energy.

$E = U + K$

$$E = -\frac{Gm_1m_2}{r} + \frac{1}{2}\frac{Gm_1m_2}{r} \quad \Rightarrow \quad E = \frac{-1}{2}\cdot\frac{Gm_1m_2}{r}$$

So if due to reduction of r, energy reduces by an amount ΔE, the change in kinetic energy ΔK will be given by $(K + \Delta K) = -[E - \Delta E]$ or $\Delta K = \Delta E$.

Therefore if there is a reduction in total energy, the kinetic energy increases by an equal amount.

It also follows that potential energy decreases by an amount, twice the decreases in total energy.

Q.2 Show that the areal velocity of particle moving under influence of a central force remains constant. (2009)

Ans. A central force, acting on a particle, depends only upon the magnitude of its distance from a fixed center. If $\vec{r}$ is the instantaneous position vector of the particle relative to the fixed center, then the central force is represented by the relation.

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$\vec{F} = f(r)\hat{r}$, where f(r) is a scalar function of distance r and $\hat{r} = \frac{\vec{r}}{r}$. The torque acting on the particle is

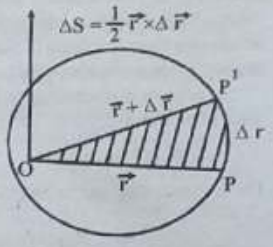
$$\vec{\Gamma} = \frac{d\vec{J}}{dt} = \vec{r}\times\vec{F} = \vec{r}\times f(r)\hat{r}$$

$$= f(r)\left(\vec{r}\times\frac{\vec{r}}{r}\right) = 0$$

So that $\vec{J} = \vec{r}\times m\vec{v}$

$\vec{J}$ = Constant .....(i)

Thus, the angular momentum of a particle moving under central force is conserved.



If $\vec{J}$ is constant, it should always be perpendicular to the plane containing $\vec{r}$ and $\vec{v}$. From above it is clear that the path of a particle in the influence of a central force lies in a plane.

Let us consider O be the center of force, when the vectors, $\vec{r}$ changes to $\vec{r}+\Delta\vec{r}$ the vector area $\Delta\vec{s}$ swept by the radius vector in this time is

$$\Delta\vec{S} = \frac{1}{2}\vec{r}\times\Delta\vec{r}$$

This area is swept in Δt time, therefore dividing both the sides of this equation by Δt and taking limits as $\Delta t \to 0$ then

$$\frac{d\vec{s}}{dt} = \frac{1}{2}\vec{r}\times\vec{v} = \frac{\vec{J}}{2m}$$

This expression gives the areal velocity $\left(\frac{d\vec{s}}{dt}\right)$ of the particle. But $\vec{J}$ is constant for a particle, moving under central force.

Here equation (ii) indicates that the areal velocity remains constant, when the particle moves in the influence of central force.

Q.3. A particle is subjected to a central force. Prove that
(a) Its angular momentum is a constant of motion.
(b) The particle moves in a fixed plane, and
(c) The radius vector sweeps out equal areas in equal time. (2016)

Ans. (a) Angular Momentum under a central force is always conserved:

In the absence of external torque, the angular momentum of a particle (or of a system) remains constant. This the principle of conservation of angular momentum.

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Now, a central force is always directed towards or away from a fixed point, (i.e. its line of action always passes through a fixed point). As such, it cannot have a moment about this point. Thus any particle moving under the action of a central force will not have any torque acting upon it and hence its angular momentum about the centre of force will remains constant.

To prove it mathematically, let us consider a particle subjected to a central force of the form

$$\vec{F} = f(r)\hat{r}$$

The torque acting on the particle is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = \vec{r} \times f(r)\frac{\vec{r}}{r} = \frac{f(r)}{r}\vec{r} \times \vec{r} = 0.$$

because the vector product of two parallel vectors $(\vec{r} \times \vec{r})$ is zero.

Thus, if $\vec{J}$ be the angular momentum of the particle, then

$$\frac{d\vec{J}}{dt} = \vec{\tau} = 0$$

or $\vec{J}$ = constant.

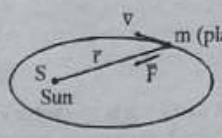
Thus when a particle moves under the action of a central force, the angular momentum relative to the centre of force is a constant of motion.

Many forces in nature are central. For example, the earth moves around the sun under a central force directed towards the sun. The earth's angular momentum relative to the sun is therefore constant. Similarly, in hydrogen atom the angular momentum of the electron relative to the nucleus is constant.

(b) Motion under a central force takes place in a fixed plane: To prove this, suppose at some instant the position and velocity vectors, $\vec{r}$ and $\vec{v}$ of the particle lies in the x-y plane. The angular momentum is given by

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

will then be directed along the z- axis (by definition of cross- product). Since $\vec{J}$ is constant both in magnitude and direction, it will always remain directed along the z - axis. For this, $\vec{r}$ and $\vec{v}$, that is, the orbit of the particle must always lie in the x-y plane.



As an example, we can consider the motion of a planet, say earth, around the sun. The planet moves under the (central) gravitational force $\vec{F}$ exerted by the sun. Consequently, the angular momentum $\vec{J}(= \vec{r} \times m\vec{v})$ of the planet with respect sun S is constant in magnitude and direction.

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Since both $\vec{r}$ and $\vec{v}$ are always perpendicular to $\vec{J}$, the orbit of the planet always lies in a plane perpendicular to $\vec{J}$.

(c) The areal velocity of a particle moving under a central force remains constant :-

Again, we consider the planetary motion. Let S be the centre of the sun and P that of the planet (mass m) in its orbit (Fig.). Let $\vec{r}$ be the radius vector of the planet with respect to S. Suppose in a small time - interval Δt, the planet moves from P to Q where the radius vector is $\vec{r} + \Delta\vec{r}$. The vector area $\Delta\vec{A}$ swept by the radius vector in time - interval Δt is $\Delta\vec{A}$ = area of ΔSPQ =

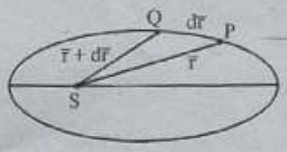
$$= \frac{1}{2}\vec{r} \times \Delta\vec{r}.$$

Dividing both sides by Δt, we get $\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$.

Proceeding to the limit $\Delta t \to 0$

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt}$$

$$= \frac{1}{2}\vec{r} \times \vec{v} = \frac{1}{2m}\vec{r} \times m\vec{v}$$



$\vec{r} \times m\vec{v}$ is the angular momentum $\vec{J}$ of the planet with respect to the sun.

i.e. $\vec{J} = \vec{r} \times m\vec{v}$

Therefore $\frac{d\vec{A}}{dt} = \frac{1}{2m}\vec{J}$

Now $\vec{J}$ is constant for a central force. Therefore

$$\frac{d\vec{A}}{dt} = \text{constant}$$

Thus, in planetary motion the areal velocity $(d\vec{A}/dt)$ is constant. In other word, the radius vector of the planet sweeps out equal areas in equal times. This is known as Kepler's second law of planetary motion.

Q.4. Reduce the two body problem to a single body problem and obtain the expression of reduced mass. **(2008, 16)**

Other Related Questions -

Q. Write short note on reduction of two body problem to a single body problem. **(2010, 11, 13, 14, 15, 17)**

Ans. Let us consider two particles of masses m_1 and m_2, whose instantaneous position vectors is an inertial frame with origin O are r_1 and r_2.

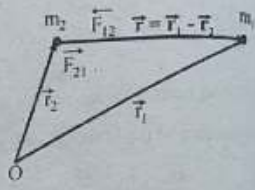
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Thus vector distance of m_1 and m_2 is $\vec{r} = \vec{r}_1 - \vec{r}_2$. If both particle interact via a force (central force) collinear with vector $\vec{r}$, then forces acting on the two bodies (particle) are given by

$$\vec{F}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2}, \quad \vec{F}_{21} = m_2 \frac{d^2\vec{r}_2}{dt^2} \qquad \text{.....(i)}$$

Where $\vec{F}_{12}$ & $\vec{F}_{21}$ are the forces acting on particle m_1 by m_2 & m_2 by m_1 respectively.

Now according to Newton's third law of motion $\vec{F}_{12} = -\vec{F}_{21} = \vec{F}$



Therefore, $\frac{d^2\vec{r}_1}{dt^2} = \frac{\vec{F}}{m_1}$ .....(ii)

& $\frac{d^2\vec{r}_2}{dt^2} = -\frac{\vec{F}}{m_2}$ .....(iii)

Subtracting relation (iii) from relation (ii), we get

$$\Rightarrow \quad \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = \vec{F}\left\{\frac{1}{m_1} + \frac{1}{m_2}\right\}$$

If $\hat{r}$ is unit vector along $\vec{r}$ then $\vec{F} = F\hat{r}$

Where F is negative for attractive force & position for repulsive force. For central force, $\vec{F}$ is any function of distance r.

Now $\frac{d^2\vec{r}}{dt^2} = \left\{\frac{1}{m_1} + \frac{1}{m_2}\right\} F\hat{r}$

If $\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \to \frac{1}{\mu}$ then above equation will becomes –

$$\frac{d^2\vec{r}}{dt^2} = \frac{F\hat{r}}{\mu} = \frac{\vec{F}}{\mu} \quad \Rightarrow \quad \boxed{\vec{F} = \mu\frac{d^2\vec{r}}{dt^2}} \qquad \text{.....(iv)}$$

Where μ is called as reduced mass. Equation (iv) represents a one-body problem since it is similar to the equation of motion of a single particle of mass μ at vector distance $\vec{r}$ from one of the two particles, considered as a fixed center of force, and μ is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

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Q.5 Write short note on reduced mass. **(2006, 12)**

Ans. Reduced Mass: -

We know that, forces acting on the two particles are given by,

$$\vec{F} = \mu\frac{d^2\vec{r}}{dt^2}$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

This equation represents a one body problem, because it is similar to the equation of motion of a single particles of mass μ at a vector distance $\vec{r}$ from one of two particles. Hence the original two body problem, involving the calculation of $\vec{r}_1$ and $\vec{r}_2$ has been reduced to the calculation of a single vector $\vec{r}$ as the function of time. The quantity $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is therefore called the **reduced mass** of the two particles, having masses m_1 and m_2. The reduced mass μ and a value less than m_1 or m_2.

In hydrogen atom, an electron of mass m revolves round a proton of much heavier mass M. The reduced mass of the hydrogen atom will be given by.

$$\mu = \frac{mM}{m+M}$$

$$\mu = Mm\,(m+M)^{-1} \quad \Rightarrow \quad \mu = m\left(1 + \frac{m}{M}\right)^{-1}$$

$$\mu = m\left(1 - \frac{m}{M}\right)$$

But $\frac{\text{mass of the electron}}{\text{Mass of the proton}} = \frac{1}{1836}$

$$\Rightarrow \quad \mu = m\left(1 - \frac{1}{1836}\right)$$

Here m should be replaced by the reduced mass μ is the expression of energy, period etc. For hydrogen atom, error will be very small because,

$$m\left(1 - \frac{1}{1836}\right) \cong m \text{ approximately.}$$

Q.6 Positron and Electron particles of the same mass but opposite charge make an atom named positronium. Calculate the reduced mass of positronium. **(2007)**

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Ans. Positronium is a combination of positron and an electron similar to a hydrogen atom. A positron is a particle which has mass equal to the electron mass but has positive charge. The reduced mass of the positronium is given by

$$\mu' = \frac{m_1 m_2}{m_1 + m_2}$$

$m_1 = m_2 = m$ = mass of electron

$$\mu' = \frac{m.m}{m+m} = \frac{m}{2} = \text{mass of positron}$$

Thus the reduced mass of the positronium particle is one-half the mass of electron.

Q.7 State the Keplor's laws of planetary motion. **(2008)**

Ans. Kepler's Law: -

Kepler stated three laws of planetary motion as follows.

(1) Law of elliptical orbits : Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.

(2) Law of areas : The radius vector drawn from the sun to a planet, sweeps out equal areas in equal times i.e. the areal velocity of the radius vector constant. i.e. v_a = constant

(3) The harmonic law : The Square of the period of revolution of the planet around the sun is proportional to the cube of the semi major axis of the ellipse.

i.e. $T^2 \propto a^3$

Q.8 Show that the Newton's law of universal gravitation can be deduced from the Kepler's law. **(2007)**

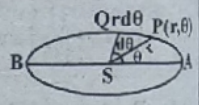
Ans. Newton's deduction from kepler's law: -

Newton deduced his famous law of universal gravitation from kepler's law with the help of his laws of motion.

Let planet P moving in on elliptical orbit around the sun, the sun being at one of the foci S of the ellipse. Let P be the position of the planet at any time t and Q at the time (t + dt). Then the area swept out by the radius vector SP in time dt is equal to the area ΔSPQ i.e.

area swept out $= \frac{1}{2} SP \times PQ$ [here SP = r, and PQ = r.dθ]

hence area swept out $= \frac{1}{2} \times r \times r d\theta = \frac{1}{2} r^2 d\theta$



the radial velocity of the radius vector $= \frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right)$

According to kepler's second law, this must be a constant. Let it be equal to h/2 then

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$$r^2 \frac{d\theta}{dt} = h \qquad \ldots\ldots(1)$$

The planet is moving on the curved path, hence it is obvious from Newton's first formula that a force is acting upon it along the radius. Consequently acceleration is acting on the planet in the direction of the force. Resolving this acceleration into its two components, along and perpendicular to the radius vector, we have

(i) Component a_r, along the radius vector i.e. the radial acceleration of the planet is given by

$$a_r = \left(\frac{d^2 r}{dt^2}\right) - r\left(\frac{d\theta}{dt}\right)^2 \qquad \ldots\ldots(2)$$

(ii) Component a_t, perpendicular to the radius vector, i.e. the transverse acceleration of the planet is given by

$$a_t = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) \qquad \ldots\ldots(3)$$

from eq(i) $r^2 \frac{d\theta}{dt} = \text{a constant} \Rightarrow a_t = 0$

Thus the planet has no transverse acceleration and only the redial acceleration is acting on it i.e. the force on the planet is directed towards the sun from eq. (1) we have

$$\left(\frac{d\theta}{dt}\right) = \frac{h}{r^2} = hu^2 \qquad \ldots\ldots(4)$$

where u = 1/r

Now

$$\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}.hu^2 = -h\frac{du}{d\theta}$$

$$\left(\frac{d^2 r}{dt^2}\right) = -h\left(\frac{d^2 u}{d\theta^2}\right)\left(\frac{d\theta}{dt}\right) = -h^2 u^2 \left(\frac{d^2 u}{d\theta^2}\right) \qquad \ldots\ldots(5)$$

Substituting the value of $d\theta/dt$ and d^2r/dt^2 from eq.(4) and eq.(5) in eq.(2) we get

$$a_r = -h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2}\right) \qquad \ldots\ldots(6)$$

Let the equation of the elliptical orbit be

$$\frac{\ell}{r} = 1 + e\cos\theta \quad \text{or} \quad \ell u = 1 + e\cos\theta \qquad \ldots\ldots(7)$$

Where ℓ is the latus rectum and e the eccentricity of the ellipse. Differentiating eq.(7) w.r.t. θ, we have

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$$\ell\left(\frac{d^2u}{d\theta^2}\right) = -e\cos\theta \qquad \ldots\ldots(8)$$

adding eq.(7) and eq.(8) we get

$$\ell\left(u+\frac{d^2u}{d\theta^2}\right) = 1 \quad \Rightarrow \quad u+\frac{d^2u}{d\theta^2} = \frac{1}{\ell}$$

substituting this value in eq.(6) we get

$$a_r = -\frac{h^2u^2}{\ell} = -\frac{h^2}{\ell}\cdot\frac{1}{r^2} \qquad \text{let } \frac{h^2}{\ell} = k\text{(a constant)}$$

$$a_r = \frac{-k}{r^2} \quad \Rightarrow \quad a_r \propto -\frac{1}{r^2}$$

Thus, the acceleration and hence force acting on the planet is inversely proportional to the square of its distance from the sun. Negative sign indicates that force is one of attraction.

Let a and b be semi-major and semi-minor axes of the elliptical orbit of the planet. The periodic time t of the planet, i.e. the time for one revolution round the sun is given by,

$$T = \frac{\text{area of the ellipse}}{\text{areal velocity}} = \frac{\pi ab}{\frac{1}{2}r^2\frac{d\theta}{dt}}$$

$$= \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{h} \qquad \left(\because r^2\frac{d\theta}{dt} = h = \text{constant}\right)$$

$$T^2 = \frac{4\pi^2a^2b^2}{h^2}$$

but $b^2/a = \ell$, the latus rectum of the ellipse

$$T^2 = \frac{4\pi^2\ell}{h^2}a^3 = \frac{4\pi^2}{k}a^3 \qquad \left(\because k = \frac{h^2}{\ell}\right)$$

Now in accordance with the kepler's third law $T^2 \propto a^3$ for every planet. Thus $4\pi^2/k$ is constant and therefore k is constant for every planet. In other words, k is independent of the nature of the planet.

Q.9. For central inverse square law forces, derive a relation between the total energy E and the eccentricity e. Show that E < 0 for closed orbits.
(2013, 17

Other Related Questions -

Q. Obtain expression for energy of the particle in terms of its angular momentum and eccentricity of the orbit: Show that E < 0 for bound orbits.
(201

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Q. For motion under the central inverse square law force obtain an expression for the total energy E and the eccentricity of the orbit. Show that E < 0 for closed orbits. (2019)

Sol. If the force acting on the moving particle towards the fixed particle or the origin O be an inverse square law force, we have force per unit mass towards the centre of force, i.e., $P = K/r^2$, where K is a constant.

Our differential equation for the orbit now becomes $\left(d^2u/d\theta^2\right)+u = K/h^2$, the general solution of which is

$$u = K/h^2 + A\cos(\theta-\theta_0), \qquad \ldots\ldots(i)$$

Where A and θ_0 are constants of integration.

This, in polar coordinates, is the equation of the most general orbit described under a central inverse square law force.

Now, the potential energy per unit mass,

$U = \int P\,dr = \int\left(K/r^2\right)dr = -K/r = -Ku$, the constant of integration having been so chosen that $U = 0$ at ∞.

From relations $\left(\frac{du}{d\theta}\right)^2+u^2 = \frac{2(E-U)}{h^2}$ and eq.(i), we get

$$A^2+\frac{K^2}{h^4}+2A\frac{K}{h^2}\cos(\theta-\theta_0) = \frac{2}{h^2}\left[E+\frac{K^2}{h^2}+KA\cos(\theta-\theta_0)\right]$$

or, $$A^2+\frac{K^2}{h^4}+\frac{2AK\cos(\theta-\theta_0)}{h^2} = \frac{2E}{h^2}+\frac{2K^2}{h^2}+\frac{2KA\cos(\theta-\theta_0)}{h^2}$$

or, $$A^2 = \frac{K^2}{h^4}+\frac{2E}{h^2},$$

Thus obtaining the value of constant A in terms of the total energy and angular momentum per unit mass.

If we rotate the base line $\theta = 0$ to make $\theta_0 = 0$ and $A > 0$, relation above will assume the form

$$u = \frac{K}{h^2}\left(1+\sqrt{1+\frac{2Eh^2}{K^2}}\cos\theta\right) \qquad \ldots\ldots(ii)$$

From the focus-directrix property of a conic, we know that its equation in polar coordinates may be written as

$$u = \frac{1}{\ell}(1+e\cos\theta), \text{ or, } \frac{\ell}{r} = (1+e\cos\theta), \qquad \ldots\ldots(iii) \quad [\because u = 1/r],$$

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Where ℓ is the semi latus rectum (i.e., half the focal chord parallel to the directrix) and e, the eccentricity, with θ measured from the perpendicular dropped on the directrix from the focus.

The conic may be of any one of the following types:

(i) ellipse, if $e < 1$, (ii) parabola, if $e = 1$, (iii) hyperbola, if $e > 1$, (in this case, relation (iii) gives only the branch adjacent to the focus) and (iv) circle, if $e = 0$.

If may be noted that relations and become identical if we choose

$$\ell = h^2/K \text{ and } e = \sqrt{1+\frac{2Eh^2}{K^2}}. \qquad \text{.....(iv)}$$

Thus, we arrive at the result that the orbit described by a particle, attracted by a central inverse square law force towards a fixed point or origin, is a conic having the fixed point or centre of force as focus, (its semi latus rectum and eccentricity being given by relation in terms of E and h).

The orbit may take the following forms:

(i) If $E < 0$ or - ve, it is an ellipse (a closed orbit), (ii) if $E = 0$, it is a parabola and (iii) if $E > 0$, it is a hyperbola.

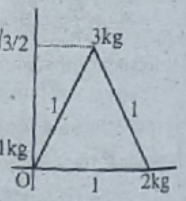
Q.10. Locate the center of mass of a system of 3 particles of masses 1 Kg, 2 Kg and 3 Kg, placed at the corners of an equilateral triangle of 1 meter side. (2013)

Sol. Imagine the triangle to lie in xy plane, with one side along the axis of x, as shown in fig then we have.

$$x = \frac{\sum m_k x_k}{\sum m_k} = \frac{1(0)+2(1)+3\left(\frac{1}{2}\right)}{1+2+3} = \frac{3.5}{6} = 0.58\text{m}.$$

and $$y = \frac{\sum m_k y_k}{\sum m_k} = \frac{1(0)+2(0)+3\left(\sqrt{3}/2\right)}{1+2+3}$$

$$= \frac{3\sqrt{3}}{12} = 0.43\text{m}.$$



Hence the center of mass is located at (0.58, 0.43).

Q.11. Starting from the equation

$$\mu\ddot{\vec{r}} = -\frac{GM_1M_2}{r^2}\hat{r}$$

Obtain equation for the radial and transverse motion. (2014, 15, 17, 19)

Ans. In the case of two bodies of masses m_1 and m_2, interacting through gravitational force as the central force along the line joining them, we may determine the motion of m_1 relative to m_2 by taking m_2 to lie at the fixed origin of

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an internal frame and replacing m_1 by their reduced mass μ, i.e., we can take

$$\mu\frac{d^2\vec{r}}{dt^2} = -G\frac{m_1m_2}{r^2}\hat{r} \qquad \text{.....(i)}$$

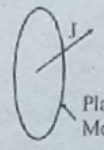


Fig. (a)

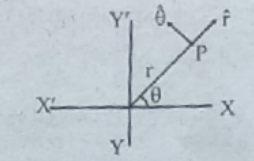


Fig. (b)

Where r is the unit vector along r, thus reducing the two-body problem effective to a one body-problem.

Now, as we know, for a particle moving about a fixed centre of force, the angular momentum remains conserved. So that, the angular momentum in our effective or equivalent one-body problem, viz.

$$J = r \times \mu\frac{dr}{dt} \qquad \text{.....(ii)}$$

must also remain conserved in both magnitude and direction.

Of course, the value of the angular momentum is its usual value for the system of the two bodies whose position vectors with respect to their centre of mass (R_{cm}) are r_1 and r_2, i.e.,

$$J = r_1 \times m_1\frac{dr_1}{dt} + r_2 \times m_2\frac{dr_2}{dt} \qquad \text{.....(iii)}$$

Since the reduced mass $\mu = m_1m_2/(m_1+m_2)$, we may writte relation (ii) in the form

$$J = \frac{m_1m_2}{m_1+m_2}(r_1-r_2)\left(\frac{dr_1}{dt}-\frac{dr_2}{dt}\right)$$

In a centre-of-mass frame, the centre of mass being at rest, $dR_{cm}/dt = 0$ and, as we have seen before,

$$\frac{dr_1}{dt} = -\frac{m_2}{m_1}\frac{dr_2}{dt} \text{ and } \frac{dr_2}{dt} = -\frac{m_1}{m_2}\frac{dr_1}{dt}.$$

So that, in that case, $$J = \frac{m_1m_2}{m_1+m_2}(r_1-r_2)\left(\frac{m_1}{m_2}\frac{dr_1}{dt}-\frac{m_2}{m_1}\frac{dr_2}{dt}\right)$$

Now angular momentum is said to be conserved when both its magnitude and direction remain constant. Thus, in order that J in expression (ii) above may remain constant (or conserved), $r \times dr/dt$ must necessarily remain constant in direction. In other words, the motion must be in a plane, as indicated in figure (a).

It is found to be easier and more useful to describe the motion of a particle in a plane in terms of the polar coordinates r and θ, where r is the unit vector from the fixed origin directed towards the particle (P) and θ, the unit vector in the plane, perpendicular to r, as shown in figure (b).

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The velocity of the particle P will thus have a component along r as also a component along $\hat{\theta}$, so that,

$$\frac{dr}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta}$$

We may, therefore, write expresion (Ii) for J above as

$$J = r \times \mu\left[\frac{dr}{dt}\hat{r} + r\omega\hat{\theta}\right]$$

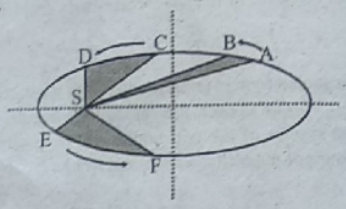
or since $r \times \hat{r} = 0$ and $\hat{r} \times \hat{\theta} = \hat{z}$, the unit vector normal to the plane of motion of the particle, we have

$$J = \mu r^2\omega\hat{z} \text{ or } J = \mu r^2\omega.$$

Q.12. Use the transverse equation of motion to prove that the radial vector sweeps out equal area in equal time. **(2015)**

Ans. Go through Q.3 page no. 54.

Let us consider a planet of mass m moving in the gravitational field of sun. According to Newton's law of gravitation the force acting on the planet due to the sun is



$$F = -\frac{GMm}{r^2}\hat{r}$$

Where M is the mass of the sun and r is the distance of the planet from the sun.

The gravitational force is a central force, hence the angular momentum J must be conserved in magnitude ($J = mvr = m\omega r^2$) and direction. In consequence, the motion of planet must be in a plane and the areal velocity $dS/dt = J/2m$ should be constant this is Kepler's second law. This expression gives the areal velocity (dS/dt) of a particle. But J is constant for a particle, moving under central force, so from relation we see that the areal velocity remains constant, when the particle moves in the influence of a central force.

The velocity of planet revolving around the sun is not uniform but changes such that the line joining the planet to sun sweeps equal areas in equal times.

For example if the planet moves from A to B, C to D, E to F in equal time interval Δt, then the areas ASB, CSD and ESF are equal. From this it is clear that when planet is nearest to sun, its velocity is maximum and when it is farthest from the sun, its velocity is minimum.

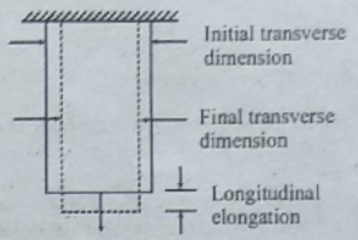
The area swept out by the line joining to the planet to sun per second is called areal velocity. Therefore the second law may also be expressed as

The areal velocity of the planet around the sun always remains constant.

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Q.13. Give the definition of primary strain, secondary strain and Poisson's ratio. **(2019)**

Ans. If a body is subjected to a load its length changes; ratio of this change in length to the original length is known as linear or primary strain. Due to this load, the dimensions of body in all directions right angles to its line of applications changes; the strians thus produced are called lateral or secondary or transverse strains and are of nature opposite to that of primary strain. For example if the load is tensile, there will be increase in length and corresponding decrease in cross-sectional area of the body. In this case, linear or primary strain will be tensile and secondary or lateral or transverse strain compressive.



The ratio of lateral strain to linear strain is known as Poisson's ratio.

i.e. $$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain or Transverse strain}}{\text{Linear or Primary strain}} = \frac{1}{m}$$

where m is a constant and its value varies between 3 and 4 for different materials.

Q.14. Show that elastic potential energy is given by $\frac{1}{2}$Stress × Strain. **(2019)**

Sol. The work done in streching a wire will b stored as its potential energy. Let us find the work done in streching a wire. Consider a wire of length L and cross section area A, streched by a force F along its length. When the wire is streched the strain and stress both increases proportionaly.

At equilibrium: The Young's modulus of the material of wire, $Y = \frac{FL}{Ax}$

where x is the increase in the length of the wire.

We can write it $F = \frac{YAx}{L}$

So the work done to extend the wire by an additional amount Δx will be:

$$\Delta W = \Delta Fx \quad \Rightarrow \quad \Delta W = \frac{YAx}{L}\Delta x$$

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Therefore the total workdone in extending the wire by an amount ℓ is:

$$W = \int_0^{\ell} \frac{YAx}{L} dx = \frac{YA}{L}\left[\frac{x^2}{2}\right]_0^{\ell} = \frac{1}{2}\frac{YA\ell^2}{L}$$

$$\Rightarrow \quad W = \frac{1}{2}\frac{YA\ell}{L} \times \ell$$

This can be written as:

$$W = \frac{1}{2} F \times \ell \qquad \text{.....(i)}$$

i.e. $W = \frac{1}{2} \times$ streching force $\times$ extension

This work done to deform the body is stored in the body as elastic potential energy.

Eq.(i) can be written as

$$W = \frac{1}{2} \times \frac{F}{A} \times \frac{\ell}{L} \times AL$$ (Multiply and divide AL in numerator and denomenator)

We know that $\frac{F}{A}$ = Stress and $\frac{\ell}{L}$ = Strain

$$W = \frac{1}{2} \times \text{Stress} \times \text{Strain} \times AL$$

AL is the volume of the wire. So work done per unit volume can be written as:

$$\frac{W}{AL} = \frac{1}{2} \times \text{Stress} \times \text{Strain}$$

This is the energy stored per unit volume of the wire.

$\therefore$ The potential energy stored per unit volume of the wire $= \frac{1}{2} \times \text{Stress} \times \text{Strain}$

Q.15. Derive the relation between y, k and η. **(2019)**

Sol. As we know that

$$K = \frac{Y}{3(1-2\sigma)} \qquad \text{.....(i)}$$

$$\eta = \frac{Y}{2(1+\sigma)} \qquad \text{.....(ii)}$$

Eq.(i) can be written as

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$$\sigma = \frac{1}{2} - \frac{Y}{6K} \qquad \text{.....(iii)}$$

Eq.(ii) can be written as

$$\sigma = \frac{Y}{2\eta} - 1 \qquad \text{.....(iv)}$$

From eq.(iii) and (iv) we get

$$\frac{1}{2} - \frac{Y}{6K} = \frac{Y}{2\eta} - 1 \qquad \text{.....(v)}$$

Dividing both sides by Y and multiplying by 6, we get

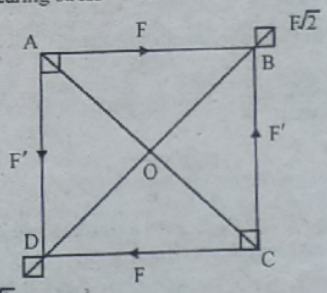
$$\frac{3}{\eta} = \frac{9}{Y} - \frac{1}{K} \qquad \text{.....(vi)}$$

Eq.(vi) gives the relation between Y, K, η.

Q.16. Prove the equivalence of a shear stress to a tensile and compressive stress at right angles to each other and each one is equal to the shear stress. **(2018)**

Ans. Consider a cube ABCD with its face DC fixed. Let L be the length of each side and a the area of each face. Suppose a tangential force F is applied to the face AB as shown in figure to produce a shear 0, then

shearing stress = F/a



It is to be proved that this shearing stress is equivalent to a linear tensile stress along the digonal DB and equivalent compression stress along diagonal AC mutually at right angles.

If there is only one force F acting on the face AB, the the cube should move as a whole in the direction of the force. But as the cube is fixed at the face DC, an equal and opposite force acts on the lower face DC. These two forces, therefore, constitute a couple and tend to rotate it in the clockwise direction.

The cube, however, does not rotate, hence the face DC which is fixed

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exerts an equal and opposite couple by exerting force (say F') on the faces AD and CB as shown

Now moment of clockwise couple = $F \times AD$

and moment of the anti-clockwise couple $= F' \times AB$

But as the cube is in equilibrium.

$\therefore \quad F \times AD = F' \times AB$

or $\quad F = F'$

Hence a tangential force F applied to the face AB gives rise to equal tangential forces acting along all the faces in suitable directions, as shown in figure.

Now, the forces acting along the faces AB and CB give rise to a resultant force $F\sqrt{2}$ along OB and forces acting along the faces AD and CD also give rise to a resultant force $F\sqrt{2}$ along OD. Hence an outward pull is exerted along the diagonal BD at the corners B and D.

Similarly at each of the corners A and C resultant force $F\sqrt{2}$ acts inwards along the diagonal AC.

Thus a tangential force F acting on one of the faces of the cube is equivalent to a force of extension $F\sqrt{2}$ acting outwards along the diagonal BD and a force of compression $F\sqrt{2}$ acting inwards along the diagonal AC.

If a plane is drawn perpendicular to the plane of the paper through the diagonal AC it will have an area

$$= L \times L\sqrt{2} = L^2\sqrt{2} = a\sqrt{2}$$

The force $F\sqrt{2}$ acting along BD will act normally to the area.

$$\therefore \text{ Tensile stress along BD} = \frac{F\sqrt{2}}{a\sqrt{2}} = \frac{F}{a}$$

$$\text{Similarly compression stress along AC} = \frac{F\sqrt{2}}{a\sqrt{2}} = \frac{F}{a}$$

Hence a shearing stress along AB is equivalent to a tensile stress along BD and an equal compression stress along AC at right angles.

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Chapter 4 Mechanics Of Non Rigid Bodies

Stress: -

When a system of external forces act on a body, relative displacements of its various particles take place and consequently the body is deformed. For such deformation, the body offers resistance and internal forces of reaction are developed inside the body. These internal forces, so developed, are equal in magnitude and opposite to the deforming forces and equilibrium is established. The forces of internal reaction tend to restore the body to its original form. On account of the restoring forces, the body recovers its shape or size, when the deforming forces are removed.

The restoring force per unit area, set up inside the body, is called **stress** and it is measured by the magnitude of the deforming force acting on unit area of the body within the elastic limit. Thus, if a force F is acting on a area, A, then stress = F/A.

If the stress is normal to the surface, it is called the normal stress.

Strain: -

When a system of forces or couples acting upon a body causes relative displacements of its various parts, a change in length, volume or shape is produced. The body is then said to be strained.

The change produced in the dimensions of a body by a system of forces or couples, in equilibrium, is called **strain** and it is measured by the ratio of change in length to the original length (longitudinal strain) or change in volume to the original volume (volume strain). If there is a change in shape, the strain (called as shearing strain) is measured by the angle through which a line originally perpendicular to the fixed face is turned.

Thus, the strain is just a ratio or an angle, it is a dimensionless quantity and has no units.

Elasticity: -

Elasticity is that property of matter by virtue of which a body regains the original shape or size on the removal of the deforming forces.

If the body recovers completely its original condition as soon as the deforming forces have been removed, it is said to be perfectly elastic while if it completely retains its modified form, it is said to be perfectly plastic. There are, however, no perfectly elastic or plastic bodies and actual bodies lie between the two extremes.

Hook's Law: -

This is the basic law of elasticity and was established by Robert Hook in 1678. This law states that within elastic limit, the stress is proportional to the corresponding strain, i.e.,

Stress $\propto$ Strain or Stress = E $\times$ Strain

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i.e. $E = \frac{\text{Stress}}{\text{Strain}}$ .....(i)

The constant of proportionality E is called modulus of elasticity.

Young's Modulus: -

Within the elastic limit, the ratio of longitudinal stress to the corresponding strain, produced, is called Young's Modulus for the material of a body and is denoted by the letter 'Y'.

Let L be the original length of a wire of cross-section A and it elongates ℓ by the application of a force F. Then in equilibrium,

stress = F/A and strain = ℓ/L

$\therefore$ Young's Modulus $Y = \frac{F/A}{\ell/L} = \frac{F.L}{A.\ell}$

Bulk Modulus: -

When a solid or a fluid (liquid or gas) is, subjected to change in pressure, its volume changes, but the shape remains unchanged. The force per unit area, applied normally and uniformly to the surface of the body (or pressure) gives the stress and the change per unit volume, the strain. Now, within the limit of elasticity the ratio of uniform and normal stress on the surface of a body to the volumetric strain is called bulk modulus of the material of the body and it is denoted by the letter 'K'.

Let V be the original volume of a body and by the application of a pressure P let the volume change to (V – v). Then

stress = P and volume strain = v/V

$\therefore$ bulk modulus $K = \frac{P}{-v/V} = -\frac{PV}{v}$ .....(i)

The negative sign indicates here, that by the increase of pressure a decrease in volume occurs.

The reciprocal of bulk modulus of a substance is called its compressibility, i.e.,

$$\text{compressibility} = \frac{1}{K} = -\frac{v/V}{P} \quad \text{.....(ii)}$$

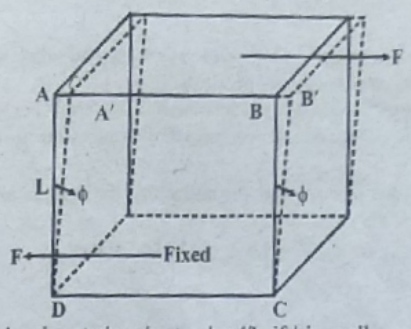
Modulus of Rigidity or Shear Modulus: - **(2006)**

Within the elastic limit, the ratio of tangential stress to the shearing strain is called modulus of rigidity of the material of a body and is denoted by the letter 'η'.

In this case, the shape of a body changes, but its volume remains unchanged. Consider a cube of side L, whose lower edge CD is fixed. A tangential force F is applied in the direction shown over the upper face of area A. A force of reaction of the same magnitude F acts on the lower fixed face in the opposite direction, as shown in figure. These two equal and opposite

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forces, with the lines of action different, constitute a couple. Due to this couple, face AD of the cube is displaced to the position A'D and face BC to the position B'C through an angle ϕ known as the angle of shear or shearing strain. The rotation of the cube by the above mentioned couple is prevented by an opposite couple of equal moment, provided by the constraints.



Thus, shearing strain = ϕ = tan ϕ = ℓ/L, if ϕ is small

where AA' = BB' = ℓ and AD = BC = L.

$$\text{Shearing stress} = \frac{\text{Tangential force}}{\text{Area of the face}} = \frac{F}{A}$$

Modulus of rigidity $\eta = \frac{F/A}{\phi} = \frac{F}{A.\phi} = \frac{F.\ell}{A\ell}$

Poisson's Ratio: - **(2008)**

When a wire is stretched along its length, it is elongated along with a contraction in its diameter. Thus, the length of the wire increases in the direction of the forces, whereas a contraction occurs in the perpendicular direction. This fact is not only true for wire, but for all other bodies under strain.

The ratio of change in dimension to the initial dimension, perpendicular to the direction of the forces, is called lateral strain. Within elastic limit, the lateral strain, is proportional to the longitudinal strain, i.e., the ratio of lateral strain and longitudinal strain is a constant for the material of a body. This constant is known as **Poisson's ratio.**

If β and α are the lateral and longitudinal strains respectively, then

Poisson's ratio σ = β/α.

Consider a wire of length L and diameter D. Let the length of the wire be increased by ℓ under the action of two equal and opposite forces and let a corresponding decrease in diameter be d. Then,

Longitudinal strain α = ℓ/L and Lateral strain β = d/D.

$\therefore$ Poisson's ratio $\sigma = \frac{\beta}{\alpha} = \frac{d/D}{\ell/L} = \frac{L}{\ell}.\frac{d}{D}$ .....(i)

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Expressing in differential form,

$$\sigma = \frac{L}{D}\frac{dD}{dL} \qquad \ldots\ldots(ii)$$

Poisson's ratio is a pure number and has no units and dimensions since it is the ratio of two strains.

Q.1. Obtain the value of Poisson's ratio for a solid substance whose volume cannot be changed by any pressure. **(2016)**

Ans. Poisson's Ratio for an Incompressible Material - For a (hypothetical) material whose volume remains unchanged by changing pressure, we have

$$K = \infty$$

An infinite value of bulk modulus means that the body is incompressible.

putting $K = \infty$ in the relations, i.e. $\frac{Y}{3K} = 1 - 2\sigma$, we get

$$1 - 2\sigma = 0$$

$$\sigma = \frac{1}{2}.$$

Q.2. Derive the relation $y = 3k(1 - 2\sigma)$ where the symbols have their usual meaning. Obtain the value of Poisson's ratio for a solid substance whose volume cannot be changed by any pressure. **(2008, 09, 10, 18)**

Other Related Question -

Q. Prove that $Y/3K = (1 - 2\sigma)$ where the symbols have their usual meaning. **(2012)**

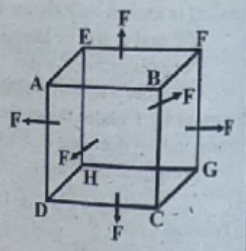
Q. For a homogeneous isotropic solid, derive the relation between y, k and s where the symbols have their usual meaning. **(2014)**

Q. Derive the relation

$Y = 3K(1 - 2\sigma)$. **(2015)**

Ans. Let ABCDEFGH represent a cube of unit side. Let us consider a force F, which acts normally and uniformly on each of its six face in the outward direction. Since each force F is acting on unit area, these forces are representing stresses.

If α is the increase per unit length per unit tension along the direction of the force, then the elongation produced in each of the edges, namely AB, BF and BC, will be $F\alpha$. If β is the contraction produced per unit length per unit stress perpendicular to the edges, then contraction produced perpendicular to each of the edge, namely AB; BF and BC, will be $F\beta$. Thus the side AB (similarly BF and BC) will increase by $F\alpha$ but simultaneously contract by $F\beta + F\beta$ due to the two other pairs of tensile stresses.



Hence, the sides of the cube become

$AB = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta)$

$BF = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta)$

$BC = 1 + F\alpha - F\beta - F\beta = 1 + F(\alpha - 2\beta)$.

Hence, the final volume of the cube is

$AB \times BF \times BC = [1 + F(\alpha - 2\beta)]^3 = 1 + 3F(\alpha - 2\beta)$

[according to Binomial expansion]

The terms containing higher powers of α and β have been neglected since α and β are very small quantities.

$\therefore$ Change in volume = Final volume - Initial volume

$= 1 + 3F(\alpha - 2\beta) - 1 = 3F(\alpha - 2\beta)$.

$$\text{Volume strain} = \frac{3F(\alpha - 2\beta)}{1} = 3F(\alpha - 2\beta)$$

$$\therefore \quad \text{Bulk modulus } K = \frac{\text{Normal stress}}{\text{Volumetric strain}} = \frac{F}{3F(\alpha - 2\beta)}$$

or $$K = \frac{1}{3(\alpha - 2\beta)} \quad \text{or} \quad K = \frac{1/\alpha}{3(1 - 2\beta/\alpha)}$$

But α is the increase per unit length per unit stress, hence clearly stress = 1 (as the areas of any edge of the cube is unity) and

$$\text{Linear strain} = \frac{\alpha}{1} = \alpha, \quad \therefore \; Y = \frac{1}{\alpha}$$

Also $\beta/\alpha = \sigma$ = Poisson's ratio.

Therefore, $$K = \frac{Y}{3(1 - 2\sigma)} \quad \Rightarrow Y = 3K(1 - 2\sigma)$$

Note:

- Relation connecting γ, η and σ

  $2\eta(1 + \sigma) = Y$

- Relation connecting γ, k and η

  $$\frac{9}{\gamma} = \frac{3}{\eta} + \frac{1}{k}$$

- Relation connecting k, and σ

  $$\sigma = \frac{3k - 2\eta}{2\eta + 6k}$$

- Limiting values of σ are -1 and 0.5 i.e. $-1 < \sigma < 0.5$

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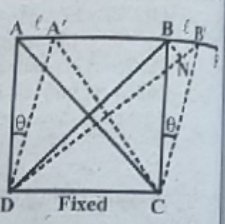
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Q.3. **Prove, $Y = 2\eta(1+\sigma)$.** (2011)

Other Related Questions -

Q. Derive the relation $y = 2\eta(1+\sigma)$ where the symbols have their usual meaning. (2016)

Ans. Let ABCD represent the front face of a cube of side L. A tangential force F is applied on its upper face AB and the bottom face DC is fixed. As a result of this force the cube is sheared to A'B'CD through an angle θ. Then shearing strain is



$$\theta = \frac{AA'}{AD} = \frac{BB'}{BC} = \frac{\ell}{L},$$

where the displacement

$AA' = BB' = \ell$.

Shearing stress

$$T = \frac{F}{\text{Area of the upper face of the cube}} = \frac{F}{L^2}$$

$\therefore$ Coefficient of rigidity $\eta = T/\theta$.

But a shearing stress along AB is equivalent to a tensile stress along DB and an equal compressive stress along AC at right angles to each other.

Let α and β be the longitudinal and lateral strains per unit stress respectively. Then extension along diagonal DB due to tensile stress

$= DB \cdot T \cdot \alpha$

and extension along diagonal DB due to compression stress along AC

$= DB \cdot T \cdot \beta$

Total extension along $DB = DB.T.(\alpha+\beta) = L\sqrt{2}T(\alpha+\beta)$

Let us draw a perpendicular BN on DB'. Then increase in the length of diagonal DB is practically equal to NB'. As θ is very small, therefore angle AB'C is nearly 90° and $\angle BB'N = \angle 45°$.

Thus, $NB' = BB'\cos 45° = \frac{BB'}{\sqrt{2}} = \frac{\ell}{\sqrt{2}}$

$\therefore \quad L\sqrt{2}T(\alpha+\beta) = \frac{\ell}{\sqrt{2}}$ or $T\frac{L}{\ell} = \frac{1}{2(\alpha+\beta)}$.

But $T.\frac{L}{\ell} = \frac{T}{\ell/L} = \frac{T}{\theta} = \eta$,

$\therefore \quad \eta = \frac{1}{2(\alpha+\beta)} = \frac{1}{2\alpha(1+\beta/\alpha)}$

But $\beta/\alpha = \sigma$ and $Y = \frac{\text{Stress}}{\text{Longitudinal strain}} = \frac{1}{\alpha}$

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[$\because$ α is the longitudinal strain per unit stress]

Therefore, $\eta = \frac{Y}{2(1+\sigma)}$

Q.4 Show that value of Poison ratio lies between -1 and 0.5. (2006)

Ans. As we have relation

$3K(1-2\sigma) = 2\eta(1+\sigma)$

where k and η are essential positive quantities, therefore

(i) If Poisson's ratio is positive quantity, then both right hand side and left hand side of above equation must be positive. This is possible when

$$1-2\sigma > 0 \quad \Rightarrow \quad \sigma < \frac{1}{2} \quad \Rightarrow \quad \sigma < 0.5$$

(ii) If σ is a negative quantity then left hand side expression of the above equation will be positive. Then the right hand side expression must also be positive hence

$1+\sigma > 0$ or $\sigma > -1$

Thus the theoretical limiting value of σ are –1 and 0.5 i.e.

$-1 < \sigma < 0.5$

Q.5. A gold wire 0.32 mm in diameter elongates by 0.1 cm when stretching force 330 gm is applied and twists through 1 radian when equal and opposite torques of 145 dyne-cm are applied at its ends. Find the values of Poisson's ratio for gold. (2015)

Ans. When a wire of radius r and length L is elongated by ℓ with a force F, the Young's modulus of its material is given by

$$Y = \frac{F}{\pi r^2}\cdot\frac{L}{\ell}$$

Here, $F = Mg = 0.33 \times 9.81$ newton, $r = 1.6\times10^{-4}$ m, $\ell = 10^{-3}$ m

$$\therefore \quad Y = \frac{0.33\times9.81\times L}{\pi\times\left(1.6\times10^{-4}\right)^2\times10^{-3}}\ n/m^2 \qquad \text{.....(i)}$$

The torque, applied to twist the wire by 1 radian, is given by

$C = \eta\pi r^4/2L = 145\times10^{-7}$ n-m

Whence $\eta = \frac{145\times10^{-7}\times2L}{\pi\times\left(1.6\times10^{-4}\right)^4}$

$$\therefore \quad \text{Poisson's ratio } \sigma = \frac{Y}{2\eta} - 1 = \frac{0.330\times9.81\times L\times\pi\times\left(1.6\times10^{-4}\right)^{-4}}{\pi\times\left(1.6\times10^{-4}\right)^{-2}\times10^{-3}\times2\times145\times2L} - 1$$

$$= 1.429 - 1 = 0.429$$

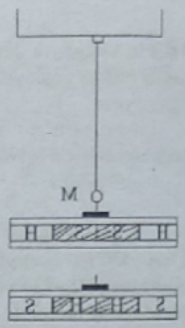
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Q.6 Describe and explain how Maxwell needle can be used to determine dynamically the modulus of rigidity of the material of a wire. (2006)

Ans. Maxwell's needle: -

The method using Maxwell's needle, for the determination of modulus of rigidity of the material of a copper wire, is a dynamical method. Maxwell's needle consists of a long hollow brass tube of length L, suspended horizontally by a vertical copper wire, whose modulus of rigidity is to be determined. The other end of the wire is clamped to a rigid support. Four brass cylinders, two hollow and two solid, of the identical lengths and diameters, can be fitted inside the hollow brass tube. The length of the each cylinder is L/4 and when they are placed in end to end, they just filled the hollow tube completely.

To facilitate the counting of the vibrations of the Maxwell needle, a small plane mirror is attached to the specimen wire and observations are taken with the help of telescope and scale arrangement.



Theory and Method: -

First, the solid cylinders S, S are placed in the inner positions and two hollow cylinders H, H in the outer positions as shown in the figure. The system is allowed to perform torsional oscillation. Let T_1 be the time period of vibration, then

$$T_1 = 2\pi\sqrt{I_1/C} \qquad \text{.....(1)}$$

Where I_1 is the moment of inertia of the suspended system about the copper wire as axis and C the torsional couple per unit twist of the wire. Next, the positions of hollow and solid cylinders are interchanged. Now if I_2 be the moment of inertia of the system about the wire as axis, then the time period is given by

$$T_2 = 2\pi\sqrt{I_2/C} \qquad \text{.....(2)}$$

form eq.(1) and eq.(2) we get

$$C = \frac{4\pi^2(I_2 - I_1)}{T_2^2 - T_1^2}$$

but $$C = \frac{\eta\pi r^4}{2\ell}$$

Where ℓ and r are the length and radius of specimen copper wire respectively

$$\therefore \frac{\eta\pi r^4}{2\ell} = \frac{4\pi^2(I_2 - I_1)}{T_2^2 - T_1^2} \quad \text{or} \quad \eta = \frac{2\pi\ell(I_2 - I_1)}{(T_2^2 - T_1^2).r^4} \qquad \text{.....(3)}$$

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Let I_0, I', and I" be the moments of inertia of the hollow brass case, solid and hollow cylinders respectively about the axis passing through their centers of gravity and perpendicular to their lengths.

As the lengths of each cylinders is L/4, moment of inertia of the needle I_1 about the wire as axis, when the hollow cylinders are placed near the ends of the hollow brass tube, is given by

$$I_1 = I_0 + 2\left[I' + m_1\left(\frac{L}{4.2}\right)^2\right] + 2\left[I'' + m_2\left(\frac{3L}{4.2}\right)^2\right]$$

or $$I_1 = I_0 + 2I' + 2I'' + 2m_1\left(\frac{L}{8}\right)^2 + 2m_2\left(\frac{3L}{8}\right)^2 \qquad \text{.....(4)}$$

Where m_1 is the mass of each solid cylinder and m_2 that for each hollow ones.

Similarly, moment of inertia of the system I_2, when the solid cylinders are at ends, about the copper wire as axis is

$$I_2 = I_0 + 2I' + 2I'' + 2m_1\left(\frac{3L}{8}\right)^2 + 2m_2\left(\frac{L}{8}\right)^2 \qquad \text{.....(5)}$$

Subtracting eq. (5) from eq.(4), we get

$$I_2 - I_1 = (m_1 - m_2)L^2/4 \qquad \text{.....(6)}$$

Substituting the value of $(I_2 - I_1)$ from eq.(6) in eq.(3) we have

$$\eta = \frac{2\pi(m_1 - m_2)L^2\ell}{(T_2^2 - T_1^2)r^4} \qquad \text{.....(7)}$$

The time period T_1 and T_2 in the two cases are calculated with the help of telescope and scale arrangement. The length of tube L and length of copper wire ℓ are measured by meter scale. Masses of the cylinders, m_1 and m_2 are obtained by weighing. The radius of the wire r is measured with the help of a screw gauge at several places on the wire in two perpendicular directions and its mean value is calculated. Thus, substituting the values of the various quantities, in eq (7) we can calculate the modulus of rigidity of the material of the copper wire.

Maxwell's needle has the following two advantages over torsional pendulum: -

(1) In Maxwell's needle, the load suspended at the end of the wire remains the same, since only the positions of solid and hollow cylinders are interchanged. Thus, the couple C per unit twist due to torsional reaction remains constant.

(2) Here, the calculation of $(I_2 - I_1)$ reduces to the determination of the difference between the masses of solid and hollow cylinder and the length of tube can be measured accurately and easily.

Q.7. Show that shearing stress is equivalent to an equal linear tensile stress and an equal compressive stress at right angles to each other.

Other Related Questions -

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Q. Prove the equivalence of a shear stress to a tensile and compressive stress at right angles to each other and each one is equal to the shear stress. (2010, 14, 15)

Ans. Let us consider the front view of a cube ABCD of side 1. The base DC of the cube is fixed and tangential force F is applied on the upper face AB.

$$\text{Shearing Stress} = \frac{F}{L^2}$$

If there is only the force F acting on the upper face AB , then the cube should move as a whole in direction of the force - But cube is fixed at lower face DC. Hence, an equal and opposite force acts on the face DC. These two forces constitute a couple which tends to rotate the cube in clockwise direction.

The face DC which is fixed exert an equal and opposite couple by exerting forces on the faces AD and CB as shown m the figure.

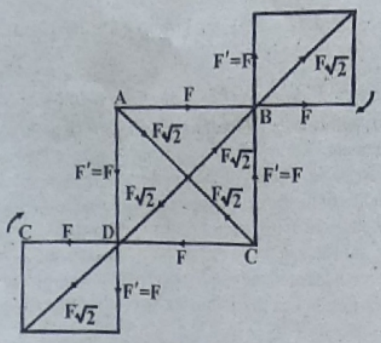
Moment of clock wise couple = F.AD

Moment of Anti clock wise couple = F'.AB

But as the cube is in equilibrium

F.AD = F'.AB

or $F = F'$ $\quad [\because AB = AD = L]$



Therefore , a tangential force F applied to the face AB gives rise to equal tangential forces acting along all the faces particular direction. The forces acting along the faces AB and CB gives rise to a resultant forces $F\sqrt{2}$ along DB and forces acting along the faces AD and CD also give rise to resultant force $F\sqrt{2}$ along BD. Therefore, an outward pull is exerted along the diagonal BD at the corners B and D resulting in its extension.

Similarly, at each of the corners A and C a resultant force $F\sqrt{2}$ acts along the diagonal AC.

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Therefore, a tangential force F acting an one of the faces of the cube is equivalent to a force of extension $F\sqrt{2}$ acting outwards along the diagonal BD and a force of compression $F\sqrt{2}$ acting inwards along the diagonally AC.

If a plane is drawn perpendicular to plane of paper through the diagonals AC and BD, then

$$\text{Area} = L \times L\sqrt{2} = \sqrt{2}L^2$$

So, linear tensile stress along $BD = \frac{F\sqrt{2}}{\sqrt{2}L^2} = \frac{F}{L^2}$

Similarly,

Linear compressive stress along $AC = \frac{F\sqrt{2}}{\sqrt{2}L^2} = \frac{F}{L^2}$

Hence, a sheering stress is equivalent to an equal linear tensile stress and an equal compressive stress at right angles to each other.

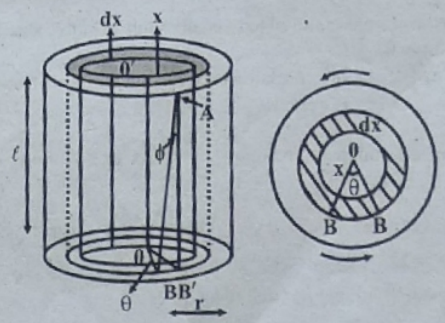
Q.8. Explain, what do you understand by shearing strain? Discuss with the help of a neat diagram how twisting a cylindrical rod gives rise to shearing strain in the rod. How are the angles of shear and twist related? (2007)

Ans. Let us consider a cylindrical rod of length ℓ and radius r. The upper end of the rod is kept fixed and twisting couple is applied at its lower end in a plane perpendicular to its length.

Let the cylinder consist a large number of the coaxial cylindrical shells and consider one such shell of radius x, thickness dx and length ℓ

From above figure

$\angle BOB = \theta$ and $\angle BAB' = \Phi$



$$\therefore \quad \text{angle} = \frac{\text{arc}}{\text{radius}}$$

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$$\theta = \frac{BB'}{x}$$

$$\Rightarrow \quad BB' = x.\theta \qquad \text{.....(i)}$$

and $$\phi = \frac{BB'}{\ell}$$

$$\Rightarrow \quad BB' = \ell\phi \qquad \text{.....(ii)}$$

From equation (i) and equation (ii), we get

$$x\theta = \ell\phi$$

$$\Rightarrow \quad \phi = \frac{x\theta}{\ell} \qquad \text{.....(iii)}$$

Suppose F is the tangential forces on base of cylindrical shell. then

$$\text{tangential stress} = \left(\frac{F}{\text{area of base of shell}}\right) = \frac{F}{\text{circumference} \times \text{thinkness}}$$

area, of base of shell $= 2\pi x \, . \, dx$

$$\text{tangential stress} = \frac{F}{2\pi x \, dx}$$

Hence the modulus of rigidity of material of rod is given by,

$$\text{Modulus of rigidity } \eta = \frac{\text{tangential force}}{\text{Shear}} = \frac{F}{2\pi x dx.\phi} = \frac{F}{2\pi x dx} \cdot \frac{\ell}{x\theta}$$

where $F = \frac{2\pi\eta\theta}{\ell}.x^2 dx$

This is equal to the couple required to twist the shell of radius x and thickness dx through an angle θ

Integrating the above expression between the limits x = 0 and x = r, we get

$$\text{total twisting couple on the rod} = \int_0^r \frac{2\pi\eta\theta}{\ell}.x^3 dx$$

$$= \frac{2\pi\eta\theta}{\ell}\left(\frac{x^4}{4}\right)_0^r = \frac{\eta\pi r^4}{2\ell}.\theta$$

If $\theta = 1$, then twisting couple per unit twist

$$= \frac{\eta\pi r^4}{2\ell} = C(\text{Say})$$

Where C is known as torsional rigidly of material of the wire.

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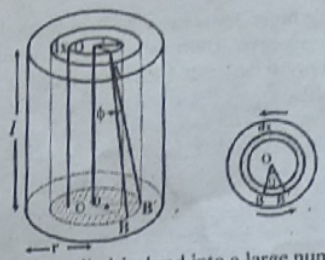
Q.9. Drive the expression for the twisting couple on a cylindrical rod or wire. (2008, 19)

Other Related Questions -

Q. Obtain an expression for the couple required for unit twist of a cylinder. (2016)

Ans. Twisting Couple On Cylindrical Rod: -

Let us a cylindrical rod of length l and radius r be hung vertical with upper end rigidity fixed. Let O' be the lower end. Let the rod be twisted by applying an external (twisting) couple of moment τ at its lower end in the horizontal plane. As the rod is twisted, an elastic restoring couple beings to develop in it. When this couple equals the twisting couple, the rod comes in the equilibrium state. Let ϕ radian be the angle of twist at the lower end of the rod in the equilibrium state.



Let us divide the cylindrical rod into a large number of thin coaxial cylindrical shells and consider one such elementary shell of radius x and thickness dx. The generating line AB on the surface of this shell has been rotated into the position AB'. $\angle BAB'$ is the angle of shear ϕ (say) for this shell. In the curved triangle OBB', we have

$$\text{arc } BB' = \phi \times AB = OB.\theta \qquad \{\text{arc} = \text{angle} \times \text{radius}\}$$

Similarly, in the curved triangle ABB', we have

$$\text{arc } BB' = \phi \times AB = \phi \times l$$

From the last two expressions, we have

$$\theta \times x = \phi \times l \Rightarrow \phi = \frac{x\theta}{l} \qquad \text{.....(i)}$$

Let F be the tangential force acting over the base (shaded) of the elementary shell. It is a circular annulus of area.

$$\pi(x + dx)^2 - \pi x^2 = 2\pi x dx,$$

neglecting the similar term containing $(dx)^2$. Hence

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$$\text{tangential stress} = \frac{\text{force}}{\text{area}} = \frac{F}{2\pi x dx} \qquad \text{.....(ii)}$$

Then, the modulus of rigidity 'η' of the material of the rod is

$$\eta = \frac{\text{tangential}}{\text{shear}} = \frac{F}{2\pi x dx}\frac{\ell}{x\phi} \qquad \{\text{From equation (i) \& (ii)}\}$$

$$F = \frac{2\pi\eta\theta}{\ell}x^2 dx$$

The moment of this force about the axis O'O of the rod is

$$F \times x = \frac{2\pi\eta\theta}{\ell}x^3 dx.$$

This is equal to couple necessary to twist the elementary shell through angle ϕ. The couple τ necessary to twist the total rod will be obtained by integrating the last expression between the limits x = 0 to x = r, thus

$$\tau = \int_0^r \frac{2\pi\eta\theta}{l}x^3 dx = \frac{2\pi\eta\theta}{l}\left[\frac{x^4}{4}\right]_0^r = \frac{2\pi\eta\theta}{l}\frac{r^4}{4}$$

$$\therefore \tau = \frac{\pi\eta r^4}{2l}\theta \qquad \text{.....(iii)}$$

This shows that the couple necessary to twist the rod through ϕ radian is proportional to the twist θ.

Q.10. A hollow circular tube of radii 5 cm and 3cm is melted and recast into a solid rod of the same length. Compare the torsional rigidities in the two cases. Assume the expression of the torsional rigidity. **(2011, 16)**

Ans. If r be the radius of recast solid rod having same length and same mass as the hollow one, then

$$\pi\left(r_1^2 - r_2^2\right)\ell\rho = \pi r^2 \ell\rho$$

where ℓ is the length of rod and ρ the density of material

$$\Rightarrow \quad r_1^2 - r_2^2 = r^2$$

$$\Rightarrow \quad r^2 = \left(5\times10^{-2}\right)^2 - \left(3\times10^{-2}\right)^2 = 16\times10^{-4}\,m^2$$

$$\Rightarrow \quad r = 4\times10^{-2}\,m.$$

Torsional rigidity of the hollow rod $C_1 = \eta\cdot\pi\left(r_1^4 - r_2^4\right)/2L$

Torsional rigidity of the solid rod $C_2 = \eta\pi r^4/2L$.

$$\Rightarrow \quad \frac{C_1}{C_2} = \frac{r_1^4 - r_2^4}{r^4} = \frac{5^4 - 3^4}{4^4} = \frac{625 - 81}{256} = 2.125$$

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Q.11. Two solid cylinders of the same material having lengths $\ell, 2\ell$ and radii r, 2r respectively are joined coaxially. Under a couple applied between the free ends, the shorter cylinder shows a twist 30°. Calculate the twist of the large cylinder. **(2013)**

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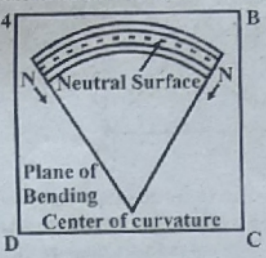
Q. Two solid cylinders of the same material having lengths $\ell, 2\ell$ and radii r, 2r respectively are joined coaxially. Under a couple applied between the free ends, the shorted cylinder shows a twist 30°. Calculate the twist of the large cylinder. **(2015, 17)**

Sol. Torque $T = \frac{\pi\eta r^4\theta_1}{2\ell} = \frac{\pi\eta(2r)^4\theta}{2(2\ell)}$

or $\theta_1 = 8\theta \qquad \therefore \theta = \theta_1\sqrt{8} = \frac{30}{8} = 3.75°.$

Q.12. Define Neutral surface. **(2007)**

Ans. The layer perpendicular to the plane of figure strained. The layers on the upper side are stretched and hence filaments in these layers are stretched. The layers on the lower side are compressed as a result of which the filament in this layer is shortened. Between these upper and lower layer is a surface, which is not strained, and the filaments in this surface do not under go any change in length. The surface is called neutral surface.



Q.13. Prove the equivalence of a shear strain to a tensile and a compressive strain at right angles to each other and each equal to half the shear strain. **(2013)**

Other Related Question -

Q. Prove the equivalence of a shear strain to a tensile and compressive strain at right angles to each other and each equal to half the shear strain. **(2016)**

Sol. Let a tangential force F be applied to the upper face AB of a cube ABCD, of each edge L, with its lower face DC fixed, so that it is sheared

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through a small angle θ into the position $A'B'$, with its perpendicular diagonals DB and AC extended to DB′ and shortened to A′C respectively.

Drop perpendiculars A′M and BN from A′ and B on to AC and DB′ respectively, so that the extension in diagonal DB is NB′ and the compression in diagonal AC is AM.

The shear being small, Δs AMA′ and BNB′ may be assumed to be right-angled isosceles triangles, so that $NB' = BB' \cos 45^\circ = BB'/\sqrt{2}$.

Since AB = L, we have $DB = DN = \sqrt{L^2 + L^2} = L\sqrt{2}$.

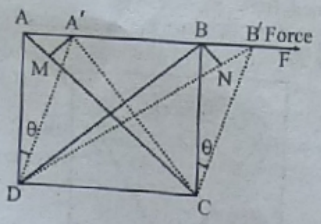
$\therefore$ tensile strain along diagonal D B

$$= \frac{NB'}{DB} = \frac{BB'}{\sqrt{2}\left(L\sqrt{2}\right)} = \frac{BB'}{2L} = \frac{\theta}{2}. \qquad \because \frac{BB'}{L} = \theta.$$

Similarly, compressive strain along diagonal AC

$$= \frac{AM}{AC} = \frac{AA'\cos 45^\circ}{AC} = \frac{AA'}{\sqrt{2}\left(L\sqrt{2}\right)} = \frac{AA'}{2L} = \frac{\theta}{2}.$$

Thus, the simple shear θ is equal to half a tensile and half a compressive strain at right angles to each other.



Q.14. Two cylindercal shafts have the same length and mass and are made of the same material. One is solid, while the other, which is hollow, has an external radius twice the internal radius. Compare their torsional rigidities and the maximum strains produced by equal twisting torques.

Sol. The torsional rigidities C and C′ of the solid and hollow cylinders are given by

$$C = \frac{\pi\eta r^4}{2\ell} \qquad \text{.....(i)}$$

and $$C' = \frac{\pi\eta\left(r_2^4 - r_1^4\right)}{2\ell}$$

where r is the radius of the solid cylinder, and r_2 and r_1 the external and internal radii of the hollow cylinder. But $r_2 = 2r_1$ (given), so that

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$$C' = \frac{\pi\eta\left(15r_1^4\right)}{2\ell} \qquad \text{.....(ii)}$$

From eq.(i) and (ii), we have

$$\frac{C}{C'} = \frac{r^4}{15r_1^4} \qquad \text{.....(iii)}$$

The masses of the two shafts are equal, so that

$$\pi r^2 \ell\rho = \pi\left(r_2^2 - r_1^2\right)\ell\rho$$

where ρ is density of the material of the cylinders.

This gives $r^2 = r_2^2 - r_1^2 = (2r_1)^2 - r_1^2 = 3r_1^2$

or $r = r_1\sqrt{3}$.

Making this substitution in eq.(iii) we get

$$\frac{C}{C'} = \frac{9r_1^4}{15r_1^4} = \frac{3}{5}$$

Now, if ϕ be the twist produced at the free end of a solid cylinder under twisting toruqe τ, then

$$\tau = \frac{\pi\eta r^4}{2\ell}\phi$$

If θ be the (maximum) shearing strain at the surface, then

$$\theta = \frac{r\phi}{\ell} \quad \Rightarrow \quad \phi = \frac{\ell\theta}{r}.$$

Thus we have

$$\tau = \frac{\pi\eta r^4}{2\ell}\left(\frac{\ell\theta}{r}\right) = \frac{\pi\eta r^3\theta}{2} \qquad \text{.....(v)}$$

If θ' be the maximum strain on the hollow cylinder for the same torque τ, then

$$\tau = \frac{\pi\eta\left(r_2^4 - r_1^4\right)}{2\ell}\left(\frac{\ell\theta'}{r_2}\right) = \frac{\pi\eta\left(15r_1^4\right)}{2\ell}\left(\frac{\ell\theta'}{2r_1}\right) \qquad [\because r_2 = 2r_1]$$

$$= \frac{\pi\eta\left(15r_1^3\right)\theta'}{4} \qquad \text{.....(vi)}$$

Comparing eq.(v) and (vi), we obtain

$$\frac{\pi\eta r^3\theta}{2} = \frac{\pi\eta\left(15r_1^3\right)\theta'}{4} \quad \text{or} \quad \frac{\theta}{\theta'} = \frac{15r_1^3}{2r^3}$$

But $r = r_1\sqrt{3}$ from eq.(iv)

$$\therefore \quad \frac{\theta}{\theta'} = \frac{15r_1^3}{2\times 3\sqrt{3}r_1^3} = \frac{5}{2\sqrt{3}} = \frac{1.44}{1}$$

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Q.15. Obtain the expression for the depression of a beam fixed at one end and loaded at the other. How is the expression for the depression modified when weight of cantilever becomes effective. (2005, 09, 11)

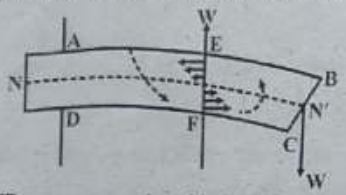
Other Related Questions -

Q. Derive an expression for the depression of loaded end of a cantilever when weight of the cantilever is effective. (2013, 17, 18)

Q. Write short note of Bending moment of a beam. (2018)

Ans. Bending moment: - (2012)

When a horizontal beam is fixed at one end and loaded at other end, a bending is produced due to the moment of load. In this position the beam remains in equilibrium. Provided the limit of elasticity has not been exceeded.



Let ABCD represent a section of the beam fixed rigidly in the wall at AD with the other end BC, loaded with a weight W, such a beam is called Cantilever. Let us consider the equilibrium of a portion BCFE of the beam, cut by a transverse plane EF across it. A force W acts downwards at the end BC, hence an equal and opposite forces acting vertically towards along FE. These two equal and opposite forces constitute a couple. This clockwise couple which bends the beam is called bending couple and the moment of this couple is called bending moment. But the beam is in equilibrium, therefore, there must also be an equal and opposite couple to balance it. The only other forces which act on EBCF are obviously the force due to stretching and compression of the filament of the beam passing across EF. The filaments in the upper portion of the beam are elongated and thus, they must be in tension, the portion of the filaments to the left of EF exerts a pulling force on the portion to the right of EF; similarly a portion of the filaments in the lower half of the beam to the left of EF, because the filaments below the neutral surface are shortened. The distribution of these forces of the section EF has been shown in the figure and their resultant is a couple which balances the bending couple.

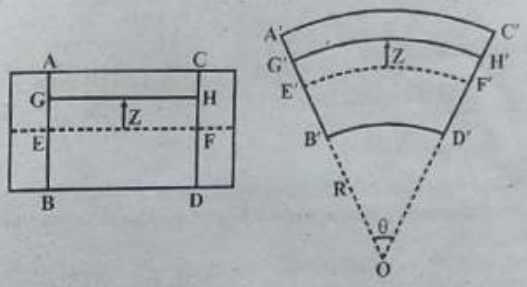
The moment of this balancing couple is called the moment of the resistance to the bending. Since in the position of the equilibrium it is equal and opposite to the bending couple, it is referred as bending moment of the beam.

Expression for the bending moment: -

Let us consider a small portion of the beam bounded by two transverse sections AB and CD close to each other. After bending as in figure, AC is elongated to A'C' and BD is shortened to B'D'. Line EF represents the

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neutral surface neither stretched nor shortened. Let the small portion, under consideration, be bend in the form of circular arc, subtending an angle θ at the center of curvature O. Let R be the radius of curvature of neutral surface E'F'. Let us consider a filament GH at a distance z from EF in the unbent position of the beam. After bending its position is shown by G'H'.



Now $G'H' = (R + z)\theta$

before bending $GH = EF$

$E'F' = EF$ (for neutral surface)

$\therefore \quad GH = E'F'$

But $E'F' = R\theta$

change in the length of filament $= G'H' - GH = (R + z)\theta - R\theta = z\theta$

$$\text{Strain} = \frac{\text{change in length}}{\text{initial length}} = \frac{G'H' - GH}{GH} = \frac{z\theta}{R\theta} = \frac{z}{R} \quad \text{.....(1)}$$

If PQRS represents a section of the beam at right angles to its length and the plane of the bending then, clearly, the forces acting on the filaments are perpendicular to this section.

The line MN lies on the neutral surface. Let the breadth of the section be PQ = b, and its depth QR = d. The force producing elongations and contractions in filaments, act perpendicular to the upper and lower halves, PQMN and MNRS respectively, of the section PQRS and their directions being opposite to each other. Now consider a small area δa of the section PQRS about a point A, distance z from the neutral surface. The strain produce in a filament passing through this area will be z/R. As shown above, now

$$Y = \frac{\text{Stress}}{\text{Strain}} \Rightarrow \text{Stress} = Y \times \text{Strain}$$

Hence stress about the point $A = Y \times \frac{z}{R}$

Where Y is Young modulus for the material of the beam.

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Force on the area $\delta a = Y \times \frac{z}{R} \times \delta a$

Moment of this force about the line MN $= \frac{Yz}{R} \times \delta a \times z = \frac{Y.\delta a.z^2}{R}$

The total amount of the moments acting on the filaments in the section PQRS is given by $\frac{\sum Y.\delta a.z^2}{R} = \frac{Y}{R}\sum \delta a.z^2 = \frac{YI}{R}$

where $I = \sum \delta a.z^2$ is called geometrical moment of inertia.

Thus bending moment $= \frac{YI}{R}$

In case of a beam of rectangular cross section of breadth b and depth d, the area of cross section is bd and (radius of gyration)^2 is equal to $d^2/12$. Hence

$$I = \text{area of cross section} \times \frac{d^2}{12} = bd \times \frac{d^2}{12} = \frac{bd^3}{12}$$

Q.16. Find the ratio of weights required to product the same depression in two rods of the same material and of same length having circular and square cross-section respectively. **(2005)**

Other Related Questions -

Q. Compare the loads require to produce equal depressions for two beams made of same material and having the same length and weight while one has a circular cross section and the other has a square cross section. **(2014)**

Ans. The depression produced at the free end of a beam damped at the one end is given by $\delta = \frac{W\ell^3}{3YI}$

For rectangular beam of breadth b and depth d. $I = \frac{b.d^3}{12}$

For the square X – section b = d, we get $I = \frac{b^4}{12}$

$$\therefore \quad \delta_1 = \frac{W\ell^3}{3Yb^4/12} = \frac{4W\ell^3}{Yb^4} \qquad \text{.....(i)}$$

For circular section $I = \frac{1}{4}\pi r^4$. Therefore depression for the circular section beam of same length,

$$\delta_2 = \frac{WI^3}{3y\pi r^4/4} = \frac{4WI^3}{3y\pi r^4} \qquad \text{.....(ii)}$$

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From equation (i) and (ii), we have

$$\frac{\delta_1}{\delta_2} = \frac{3\pi r^4}{b^4} \qquad \text{.....(iii)}$$

Since the cross-sectional area are equation $b^2 = \pi r^2$

Then from equation (iii), we get

$$\frac{\delta_1}{\delta_2} = \frac{3\pi r^4}{\pi^2 r^4} = \frac{3}{\pi}$$

Since $3 < \pi$, we get $\delta_1 < \delta_2$

From above we show that the depression of square section beam is less then that of the circular cone. Than that of the circular cone. Hence It is more staff. The ratio of their depression $3/\pi$.

Q.17. Compare loads required to produce equal depression for two beams, made of the same material and having the same length and weight with the only difference that while one has circular cross-section, the cross-section of the other is square. **(2010)**

Ans. The depression δ at the middle point of a beam of length ℓ of circular cross-section of radius r, is given by

$$\delta = \frac{W_1\ell^3}{12Y\pi r^4} \qquad \text{.....(1)}$$

where W_1 is the weight loaded.

If in case of rectangular cross-section beam the weight loaded be W_2, then the depression δ', is given by

$$\delta' = \frac{W_2\ell^3}{4d^3Y}$$

since the two beams have the same length and are made of same material.

But for square cross-section b = d

$$\therefore \quad \delta' = \frac{W_2\ell^3}{4bd^4Y} \qquad \text{.....(2)}$$

The two beams have equal weights and made of the same material, therefore their masses and hence volumes must be equal, i.e.,

$\pi r^2\ell$ or $b^2 = \pi r^2$

Substituting the value of b^2 in (2), we get

$$\delta' = \frac{W_2\ell^3}{4\pi^2 r^4 Y} \qquad \text{.....(3)}$$

But depression is the same in both cases, i.e., $\delta = \delta'$. Therefore equating (1) and (3), we get

$$\frac{W_1\ell^3}{12Y\pi r^4} = \frac{W_2\ell^3}{4\pi^2 r^4 Y}$$

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or $\frac{W_1}{W_2} = \frac{3}{\pi}$, which is the required ratio.

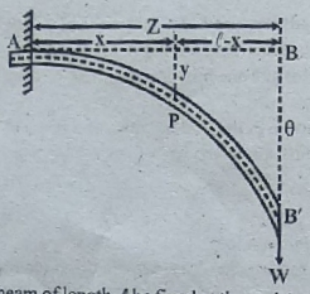
Q.18. Obtain an expression for the depression of a beam fixed at one end and loaded at the other end. How is the expression for the depression modified when weight of cantilever becomes effective. (2008)

Other Related Questions -

Q. A horizontal cantilever is loaded at its free end. Obtain an expression for depression as a function of distance from the fixed end. (2015)

Ans. The depression Y in a cantilever : -

A horizontal beam fixed at one end and loaded at the free end, is called a cantilever.



Let us consider a beam of length ℓ be fixed at the end A and loaded with a load W at B.

Let the end B deflected to the position B′ under the action of the load W.

Suppose, a section of the beam as at P, at distance x from the end A. The moment of bending couple due to the load $W = W(\ell - x)$

$$\text{The bending moment} = \frac{YI}{R}$$

where Y is the Young's modulus of the material of the beam, I the geometrical moment of Inertia of the section and the R the radius of curvature of neutral axis at the section

Thus $\frac{YI}{R} = W(\ell - x)$ .....(i)

The radius of curvature R is given by the equation.

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

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Where y is the depression of the beam at a distance x from the fixed end. Since y is small hence $\left(\frac{dy}{dx}\right)^2$ can be neglected in comparison to 1. Hence

$$R = \frac{1}{\frac{d^2y}{dx^2}}$$

Putting the value of R in equation (i), we have

$$YI\frac{d^2y}{dx^2} = W(\ell - x) \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{W}{YI}(\ell - x) \text{ .....(ii)}$$

The value of depression of the free end M at $x = \ell$.

Integrating equation (ii), we get

$$\int \frac{d^2y}{dx^2}dx = \int \frac{W}{YI}(\ell - x)dx + C_1$$

$$\frac{dy}{dx} = \frac{W}{YI}\left(\ell x - \frac{x^2}{2}\right) + C_1. \quad \text{.....(iii)}$$

where C_1 is constant of integration.

By condition is that at $x = 0$

$$\frac{dy}{dx} = 0$$

Putting these value in equation (iii), we get

$C_1 = 0$

From equation (iii), we get

$$\frac{dy}{dx} = \frac{W}{YI}\left(\ell x - \frac{x^2}{2}\right)$$

Again, Integrating

$$y = \frac{W}{YI}\left(\frac{\ell x^2}{2} - \frac{x^3}{6}\right) + C_2 \quad \text{.....(iv)}$$

where C_2 is integrating constant

But at $x = 0, y = 0$,

Putting these value in equation (iv), we get

$C_2 = 0$

Hence, the expression for the depression y at a distance x from the fixed end B.

$$y = \frac{W}{YI}\left(\frac{\ell x^2}{2} - \frac{x^3}{6}\right) \quad \text{.....(v)}$$

At the free end

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$x = \ell$ and let $y = \delta$, then

$$\delta = \frac{W}{YI}\left(\frac{\ell^3}{2} - \frac{\ell^3}{6}\right)$$

$$\delta = \frac{W\ell^3}{3YI} \qquad \text{.....(vi)}$$

For a beam of rectangular cross-section of breadth b and depth d.

$$I = \frac{bd^3}{12}$$

From eq.(vi), we get

$$\delta = \frac{4W\ell^3}{Ybd^3}$$

For a beam of circular cross-section of radius r

$$I = \frac{\pi r^4}{4}$$

$$\therefore \quad \delta = \frac{4W\ell^3}{3\pi r^4 Y}$$

Q.19. Explain why the griders are manufactured with their section in the form of I. **(2008, 18)**

Ans. Griders standing on pillars at their ends support load, so that the grider suffers bending. Its middle past is depressed and the neural surface lying in the middle of the grider experiences no strain.

The filaments in the lower past of the griders suffers extension while those in the upper past suffer contraction. These extensions and contractions increase towards the outermost filaments. In comparison to inner part of the grider should be strong than the inner parts. Thus the inner parts are made of small width whole the upper and lower surface are given I shape. In this way, without affecting the strength of the grider most of the material is saved.

Q.20. A cantilever loaded at its free end with a mass M is depressed slightly and then released. Show that the time period for the transverse oscillations is given by

$$T = 2\pi\sqrt{\frac{M\ell^3}{3YI}}$$

Assume the expression of bending moment. **(2011)**

Ans. Consider a rod of length ℓ, clamped at one end and loaded at its free end with mass M. In the equilibrium position, the deflection of the free end will be given by equation. So that,

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$$W = Mg = \frac{3YI}{\ell^3}\delta \qquad \text{.....(i)}$$

Let the mass M be slightly depressed and then released. Now, the cantilever will execute transverse vibrations. If y be the displacement of the free end of the cantilever at any time from the equilibrium position. Then a restoring force F will act opposite to W.

Therefore,

$$W - F = \frac{3YI}{\ell^3}(\delta + y) \qquad \text{.....(ii)}$$

Subtracting equation (i) from equation (ii), we get

$$F = -\frac{3YI}{\ell^3}y \qquad \text{.....(iii)}$$

If d^2y/dt^2 be the acceleration produced in the mass M at that instant at the free end, then the force F is $M \cdot d^2y/dt^2$.

Therefore, the equation of motion is

$$M\frac{d^2y}{dt^2} + \frac{3YI}{\ell^3}y = 0 \text{ or } \frac{d^2y}{dt^2} + \frac{3YI}{M\ell^3}\cdot y = 0 \qquad \text{.....(iv)}$$

This equation represents a S.H.M., whose periodic time is given by

$$T = 2\pi\sqrt{\frac{M\ell^3}{3YI}}. \qquad \text{.....(v)}$$

Q.21. Write short note on Reynold number and its application. **(2019)**

Sol. In fluid mechanics, the Reynolds number (Re) is a dimensionless number that gives a measure of the ratio of inertial forces to viscous forces and consequently quantifies the relative importance of these two types of forces for given flow conditions.

This is proportional to the ratio of inertial forces and viscous forces (forces resistant to change and heavy and gluey forces) in a fluid flow. The Reynolds number is used to study fluids as they flow. The Reynolds number determines whether a fluid flow is laminar or turbulent.

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Chapter 5 Fluid Mechanics

Q.1 What do you mean by ideal fluid.

Ans. Ideal Fluid: -

An ideal fluid is one which has no viscosity and surface tension and is incompressible. In true sense no such fluid exist in nature.

Q.2 Write the equation of continuity for a liquid flow and discuss the physical meaning of this equation. **(2007)**

Other Related Questions -

Q. Write short note on Equation Of Continuity. **(2010, 15)**

Ans. Equation Of Continuity: - **(2011)**

Let us assume area A_1 at P perpendicular to the direction of steady flow of the liquid and draw the streamlines through the boundary of A_1 so that PQ is obtained. This is known as tube of flow. Since the sides of such a tube are everywhere in the direction of flow of the liquid, nθ liquid crosses the sides. Actually the liquid moves inside a tube of flow just as if the tube has solid walls. The liquid are mobile and perfectly incompressible. Thus the same amount of liquid flows across every section of the tube in a given time, which is volume of the liquid passing through area A_1 is equal to that passing through the area A_2.

Let the velocity of flow at P be v_1 and at Q be v_2. Therefore volume of the liquid entering per second or rate of flow of liquid at point $P = A_1v_1$ and volume of the liquid leaving per second at $Q = A_2v_2$.

If ρ_1 and ρ_2 are the densities of the liquid at two sections then mass of the liquid entering per second at $P = A_1v_1\rho_1$ and mass of the liquid leaving per second at $Q = A_2v_2\rho_2$

If there is no source or sink within the volume of the liquid between the section P and Q then the masses of the liquid crossing the areas A_1 and A_2 per second must be the same.

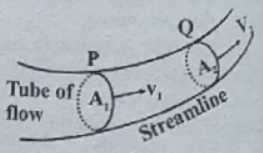
$$A_1v_1\rho_1 = A_2v_2\rho_2$$

Since liquid is incompressible therefore

$$\rho_1 = \rho_2$$

Hence above equation becomes

$$A_1v_1 = A_2v_2 \quad \Rightarrow \boxed{Av = \text{constant}}$$



Thus the volume of the liquid entering the section P is equal to that leaving the section Q.

This equation is known as equation of continuity for the steady flow of liquid. This equation implies that where the cross section of the tube is small the velocity is greater and vice-versa.

Q.3 Write short note on streamline flow. **(2007, 12)**

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Ans. Streamlined Flow: -

When a liquid flows such a way that each particle of the liquid passing a point moves along the same path & has the same velocity as its preceding particles the flow of the liquid is called streamlined flow or orderly flow. In streamlined flow, the velocity at every point in the liquid remains constant in magnitude as well as in direction. In this case a line along which a particle of the liquid moves and the direction of which at any point gives the direction of flow of liquid at that point is called as streamline flow.

A streamline is a curve such that at any instant the tangent to it at a point gives the direction of flow of the liquid at that point. A streamline may be straight or curved according to the lateral pressure on it is the same or different.

In streamline flow, the energy needed to drive the liquid is used up only in overcoming the viscous drag between the layers. The flow of liquid remains streamline only as long as the velocity does not exceed to a certain value called its critical velocity. Beyond its critical velocity the flow of liquid ceases to be streamline but becomes zigzags in character and the path and the velocity of liquid particles changes continuously. So this unsteady motion of liquid known as turbulent flow.



Streamline Flow

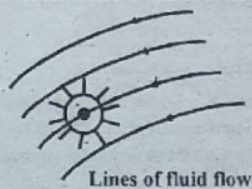
Q.4 Explain the meaning of terms rotational and irrotational flow. **(2007)**

Other Related Questions -

Q. Explain rational and irrotational flow of liquid. **(2012)**

Ans. Irrotational flow: -

If an element of fluid at each point has no net angular velocity about that point, the fluid flow i.e. known as irrotational flow.



Lines of fluid flow

Let us consider an element of fluid as a small wheel placed in the flowing fluid if fluid flow without rotating on its axis. The flow is Irrotational.

Rotational Flow: -

If alone with the linear motion the wheel rotates also about its axis, the motion of fluid is rotation.

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Rotational flow includes vertex motion such as eddies whirlpools. In rotation motion the velocity vector $(\vec{v})$, varies in transverse direction.

Example :-

The flow of water in a circle along a circular ring space tube is rotational, since although the speed is constant at all points and also with respect to time, the velocity is changing transversely due to transverse acceleration towards the center of ring.

Q.5. Derive Einstein's velocity addition law. **(2016)**

Ans. The Einstein's Velocity Addition Formula: -

All the speeds considered here represent motion along the x and x′ axes, which are parallel to each other. If this is understood, we can speak of the "addition of velocities." The specical relativistic addition of nonparallel velocities is more complicated.

Let us consider two observers, the rocket pilot and alien. We define S and S′ as alien's and pilot's reference frame, respectively, and x and x′ as axes parallel to the direction of motion. The pilot's (S′) sees a light flash moving along the x′ axis with a velocity c, and using Galilean realtivity, predicts that the velocity of the light signal as seen in S will be

$$u = c + v \qquad \text{.....(i)}$$

This is inconsistent with Einstein's second postulate and with the speed that the alien actually observes.

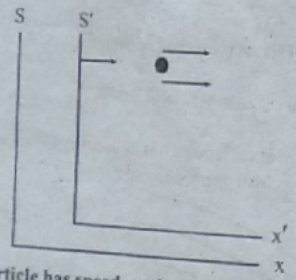


Fig. - A particle has speed u relative to S and speed u′ relative to S′ velocities u, u′ and v all are directed parallel to the x and x′ axes.

We know that the Lorentz transformation equations and derive a new velocity addtion law.

As we know that

difference of position $\Delta x = \dfrac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}$ .....(ii)

and difference of times $\Delta t = \dfrac{\Delta t' + (v/c^2)\Delta x'}{\sqrt{1 - v^2/c^2}}$ .....(iii)

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Then dividing eq.(ii) by eq.(iii), we get

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + (v/c^2)\Delta x'} = \frac{(\Delta x'/\Delta t') + v}{1 + (v/c^2)(\Delta x'/\Delta t')} \qquad \text{.....(iv)}$$

Since $\Delta x/\Delta t = u$ and $\Delta x'/\Delta t' = u'$, then eq.(iv) becomes

$$u = \frac{u' + v}{1 + u'v/c^2}$$

Q.6. Derive Euler's equations of motion. Use these to obtain Bernoulli's theorem for steady rotational flow of a liquid. Where is the total energy per unit mass of the fluid constant at all points in the liquid? (2006, 10, 11, 14, 17)

Other Related Questions -

Q. Derive the Euler's equations of flow. Show that for steady irrotational flow of a fluid, the total energy of the fluid per unit mass is constant at all points in the fluid. **(2013)**

Q. Derive Euler's equations of flow. **(2015)**

Q. Derive Euler's equation of motion. **(2016)**

Q. State Bernoullis theorem. **(2019)**

Q. Derive Euler's equations of flow, show that for steady irrotational flow of fluid, the total energy of the fluid per unit mass is constant at all points in the fluid. **(2019)**

Ans. Bernoulli's Theorem: - **(2012, 18)**

This theorem states that the total energy of an incompressible, non viscous fluid in steady flow remains constant throughout the flow.

Suppose that an Infinitesimally portion AB of the tube of length $d\ell$ and area X – section da. If ρ be the density of the fluid in portion AB then

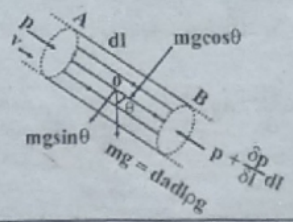
mass M = da$d\ell$. ρ and weight w = mg = da. $d\ell$. ρg

It has two components

(a) mg cosθ = da. $d\ell$ρ.g cosθ along the direction of flow

(b) mg sinθ = da.dl.g sinθ, perpendicular to the flow.

If p be the pressure on the face A, the pressure on face B will be $P + \dfrac{\partial P}{\partial \ell}.d\ell$ in the direction.



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Force acting on face A in forward direction = P.da and force acting on face B in the backward direction $= \left(P + \frac{\partial P}{\partial \ell}.d\ell\right).da$.

$\therefore$ Net force on the mass of the small portion of the fluid is

$$F = P.da - \left(P + \frac{\partial P}{\partial \ell}d\ell\right)da + da.d\ell.\rho g\cos\theta$$

or let v be the velocity of fluid as It enters the tube of fluid at A. Then its acceleration $\frac{dv}{dt}$.

Force acting on the mass da.dlρ of the fluid in the tube is given as

F = m(mass).a(acceleration)

$$F = (da.dl.\rho)\frac{dv}{dt} \qquad \text{.....(ii)}$$

Velocity is the function of both distance ℓ and time t

$\Rightarrow$ $v = f(\ell, t)$

or $$dv = \frac{\partial v}{\partial \ell}d\ell + \frac{\partial v}{\partial t}.dt$$

$\therefore$ $$\frac{dv}{dt} = v\frac{\partial v}{\partial \ell} + \frac{\partial v}{\partial t}$$

putting the value of $\frac{dv}{dt}$ in equation (ii)

$$F = (da.d\ell.\rho)\left(v\frac{\partial v}{\partial \ell} + \frac{\partial v}{\partial t}\right) \qquad \text{.....(iii)}$$

If h is the vertical height of the tube AB

$\Rightarrow$ $$\cos\theta = -\frac{\partial h}{\partial \ell}$$

From equation (i) and equation (ii), we get

$$da.d\ell\rho\left(v\frac{\partial v}{\partial \ell} + \frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial \ell}d\ell.da - da.d\ell\rho g\frac{\partial h}{\partial \ell}$$

or $$\rho\left(v.\frac{\partial v}{\partial \ell} + \frac{\partial v}{\partial t}\right) = -\frac{\partial P}{\partial \ell} - \rho g\frac{\partial h}{\partial \ell}$$

Which is called Euler's equation.
For steady or stream line flow.

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$$\frac{\partial v}{\partial t} = 0$$

so that

$$\rho.vdv + dp + \rho g\,dh = 0 \qquad \text{or} \qquad vdv + \frac{dp}{\rho} + gdh = 0$$

On Integrating, we get

$$\frac{1}{2}v^2 + \int\frac{dP}{\rho} + gh = \text{Constant}$$

Which is Bernoulli's equation
If the fluid is incompressible, then ρ is constant we get

$$\frac{1}{2}v^2 + \frac{P}{\rho} + gh = \text{Constant}$$

Multiplying by ρ on above equation, we get

$$P + \frac{1}{2}\rho v^2 + \rho gh = \text{Constant}$$

This is the Bornollis equation in term of pressure

Each term denotes the energy per unit of the liquid, Therefore the sum of pressure energy, kinetic and gravitation energy of the flow non – viscous liquid is constant.

This is the Barnoulli's Theorem.

Q.7. Show that for steady irrotational flow of a fluid, the total energy of the fluid per unit mass is constant at all points in the fluid. What is the velocity potential? (2015)

Other Related Questions -

Q. Use this to obtain Bernoulli's theorem for steady irrotational flow of a liquid and show that the total energy per unit mass of the fluid for such flow is constant at all points in the liquid. For which value of velocity the concept of velocity potential exists? (2016)

Ans. A velocity potential is a scalar potential used in potential flow theory. It was introduced by Joseph-Louis Lagrange in 1788.

It is used in continuum mechanics, when a continuum occupies a simply-connected region and is irrotational. In such a case,

$$\nabla \times u = 0$$

where u denotes the flow velocity. As a result, u can be represented as the gradient of a scalar function ϕ:

$$u = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial k}k.$$

ϕ is known as a velocity potential for u.

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A velocity potential is not unique. If a is a constant, or a function solely of the temporal variable, then $\phi + a(t)$ is also a velocity potential for u. Conversely, if Ψ is a velocity potential for u then $\Psi = \phi + b$ for some constant, or a function solely of the temporal variable b(t). In other words, velocity potentials are unique up to a constant, or a function solely of the temporal variable.

If a velocity potential satisfies Laplace equation, the flow is incompressible; one can check this statement by, for instance, developing

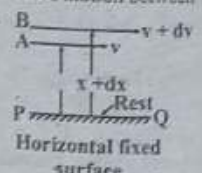
and using, thanks to the Clairaut-Schwarz's theorem, the commutation between the gradient and the laplacian operators.

Unlike a stream function, a velocity potential can exist in three-dimensional flow.

Q.8. Write short note on viscosity.

Ans. When a liquid moves slowly and steadily over a fixed horizontal surface, i.e., when its flow is streamline, every layer of the liquid moves parallel to the fixed surface. Its layer in contact with the fixed surface is at rest and the velocity of every other layer increases uniformly and continuously as we move upwards, i.e., the greater the distance of a layer from the fixed surface, the greater its velocity. It is to be remembered that in a streamline flow, the velocity of the liquid is constant at any point.

Now consider the motion of two adjacent layers, say A and B, of the liquid at distances x and x + dx above the fixed horizontal surface, which are moving with velocities v and v + dv respectively. As the upper layer B is moving faster than the lower layer, the upper layer tends to increase the velocity of the lower layer, while the lower layer tends to decrease the velocity of the upper layer. The two layers together tend to destroy their relative motion as if there is some backward dragging force acting tangentially on the layers. Thus, to maintain the relative motion an external force must be applied to overcome this backward drag. This external force is provided by the pressure difference. In absence of any such outside force, the relative motion between the layers is destroyed and flow of the liquid ceases.



This property by virtue of which a liquid oppose the relative motion between its different layers is known as viscosity and the tangential force that tends to destroy the relative motion is called viscous force. Viscosity is also called the internal friction of the liquid.

If the two layers of liquid are separated by a distance dx and are moving with a relative velocity dv, the quantity dv/dx is called velocity gradient. According to Newton, the tangential viscous force F, acting on a layer is proportional to its area A and velocity gradient dv/dx, i.e.,

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$$F \propto -A\frac{dv}{dx} \text{ or } F = -\eta A\frac{dv}{dx} \quad \ldots\ldots(i)$$

where constant η is called the coefficient of viscosity of the liquid. Negative sign simply indicates that viscous force acts opposite to the direction of flow. It is obvious that the external force required to maintain the relative velocity dv between the two layers will also be equal to F, but opposite in sign, i.e., $F = \eta A\,.\,dv/dx$. If A = 1 and dv/dx = 1, then

$F = \eta$.

Hence the coefficient of viscosity of a liquid may be defined as the tangential force per unit area of a layer, required to maintain unit velocity gradient normal to the direction of flow.

S.I. unit of coefficient of viscosity is Newton-sec/m^2.

The C.G.S. unit of viscosity is dyne-sec/m^2 which is often called poise, in honour of Poiseuille, whose work on viscosity is important. The 100th part of 1 poise is called centipoises and 1 poise is equal to 0.1 S.I. units.

Viscosity in liquids corresponds to solid friction in the respect that it opposes the relative motion between two layers. The viscous force between two layers of a liquid depends upon (i) the area of the layers, (ii) the relative velocity between them and, (iii) the distance apart between them, while solid friction is independent of the area of contact of the solid surfaces and the relative velocities between them.

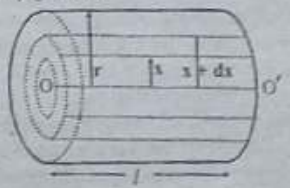
Q.9. Write short note on Poiseulli's equation. (2005, 07, 08, 13, 14, 16)

Other Related Questions -

Q. For streamline flow of liquid in a capilary obtain an expression for velocity at a given distance from the axis and show that this is parabolic. (2011)

Q. Write the short note on Poiseuille's Equation for flow of liquid through a capillary tube. (2017)

Ans. Consider a cylindrical layer of the liquid, coaxial with the tube of radius x. Let the velocity of the liquid at a distance x from the axis of the tube be v so that dv/dx is the velocity gradient.



Let us now consider the cylinder of radius x, the liquid outside this cylinder is moving less rapidly than inside it and thus exerts on the surface of the cylinder a retarding viscous force F_1. In accordance with the Newton's law of viscous flow.

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$$F = -\eta A \frac{dr}{dx}$$

Where η is the coefficient of viscosity, and A the surface area of the cylinder i.e. $A = 2\pi xl$

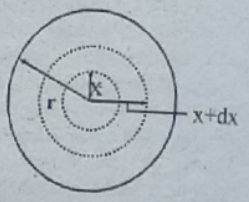
So that $F = -\eta.2\pi xl(dv/dx)$

Since p is the difference of pressure between the two ends of the capillary tube, the force deriving the forward $= p \times \pi x^2$

For steady flow, there must be no acceleration of the liquid i.e., the forward force must be equal to the retarding viscous force,

Hence $p \times \pi x^2 = -\eta.2\pi xl(dv/dx)$

$$dv = -\frac{p}{2\eta l} x dx$$



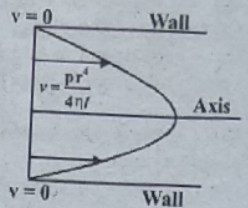
or Integrating, we get $v = -\frac{p}{2\eta l}\frac{x^2}{2} + c$

Where c is a constant of integration

Where x = r, v = 0 (Since velocity of the layer in contact with fixed surface is zero)

$$\therefore \quad 0 = -\frac{p}{2\eta l}\frac{r^2}{2} + c$$

$$c = \frac{p}{2\eta l}\frac{r^2}{2}$$



Hence $v = -\frac{p}{2\eta l}\cdot\frac{x^2}{2} + \frac{p}{2\eta l}\cdot\frac{r^2}{2}$

$$v = \frac{p}{4\eta l}\left(r^2 - x^2\right)$$

This expression gives us the velocity distribution or velocity of the liquid at a distance x from axis of the tube. The velocity distribution curve of the advance liquid in the tube is parabola. The velocity increases from 0 at the walls (x = r) of the tube to a maximum at its center (x = 0).

Let us now consider that layer, having inner radius x, has the thickness dx so that volume dV of the liquid flowing per second through it is given by dV = velocity x cross-sectional area of the cylindrical layer of thickness dx

$$= v \times 2\pi x\ dx = \left(\frac{p}{4\eta l}\right)\left(r^2 - x^2\right)2\pi x\ dx$$

Therefore, the total volume flowing per second through the tube is

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$$V = \int_0^r \frac{p}{4\eta l}\left(r^2 - x^2\right)2\pi x\ dx = \frac{\pi p}{2\eta l}\int_0^r \left(xr^2 - x^3\right)dx$$

$$= \frac{\pi p}{2\eta l}\left[\frac{x^2 r^2}{2} - \frac{x^4}{4}\right]_0^r = \frac{\pi p}{2\eta l}\left[\frac{r^4}{2} - \frac{r^4}{4}\right]$$

$$V = \frac{\pi p r^4}{8\eta l}$$

This is known as poiseuille's formula.

In an actual experiment, the pressure difference p is measured by the difference of the liquid levels in two liquid columns at the two ends of the tube of length *l*. If ρ is density of the liquid, flowing through the tube, then

$p = h\rho g$

Limitations of Poiseuille's formula: -

(i) This formula holds only for streamlined flow of the liquid

(ii) It is not true if layer of the liquid in contact of the walls of the tube are not at rest, and thus have a relative slipping.

(iii) Poiseuille's formula has been derived on the basis that the pressure difference across the tube is used to overcome the viscous forces and no part of it is spent in giving the kinetic energy to the liquid. Thus, Poiseuille's formula is true for liquid velocities, involving negligible amount of kinetic energy.

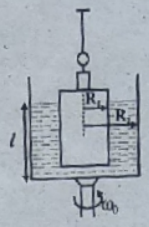
Q.10. Write short note on rotation viscometer. **(2006, 2009)**

And. Rotation Viscometer: -

This is a convenient apparatus for determining the viscosity of more viscous liquid and gases.

Construction: -

It consists of a pair of coaxial cylinders. The inner cylinder is suspended inside the outer cylinder by a phosphor –bronze suspension. The space between the two cylinders is filled with the experimental liquid. The outer cylinder is rotated with a constant angular velocity by an electric motor, the number of rotations being recorded on an automatic counter.



Principle: -

The outer cylinder is set in rotation with a certain angular velocity. The layer of the liquid in contact with the outer cylinder also rotates with the same velocity. The inner layer, however, also rotate but due to viscous drag the angular velocity decreases towards the inner cylinder. The viscous drag transmitted to the inner cylinder exerts a torque upon it which rotates until the

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elastic restoring torque set up in its suspension brings it to equilibrium state. The steady angular deflection of the cylinder is measured. This leads to a determination of the viscous torque. Hence the viscosity of the liquid is also determined.

Q.11. Discuss briefly the flow of liquid in two capillaries connected in series and parallel.

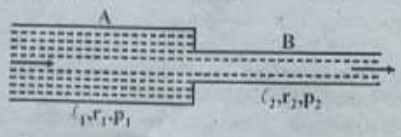
Other Related Questions -

Q. Two capillary tubes of lengths ℓ_1 and ℓ_2 and of radii r_1 and r_2 respectively are connected in series. If the pressures at the two ends are P_1 and P_2, find the pressure at the joint. (2011)

Q. Find the quantitity of fluid passing/sec through the combination of capillaries having ℓ_1, ℓ_2 and radii r_1, r_2 arranged in series. (2012)

Ans. Capillaries in Series and Parallel: -

If two capillaries of length ℓ_1, ℓ_2 and radii r_1, r_2 respectively are connected in series, then obviously the volume flowing per second through them will be the same, i.e.,



$$V = \frac{\pi p_1 r_1^4}{8\eta \ell_1} = \frac{\pi p_2 r_2^4}{8\eta \ell_2} \qquad \text{.....(i)}$$

where p_1 and p_2 are pressure differences across the two capillaries. Now, if p is the pressure difference across the composite tube, then

$p = p_1 + p_2$ or $p_2 = p - p_1$

$$\therefore \quad V = \frac{\pi p_1 r_1^4}{8\eta \ell_1} = \frac{\pi (p - p_1) r_2^4}{8\eta \ell_2} \quad \text{or} \quad \frac{p_1 r_1^4}{\ell_1} = \frac{p r_2^4}{\ell_2} - \frac{p_1 r_2^4}{\ell_2}$$

$$\therefore \quad p_1 = \frac{\dfrac{p r_2^4}{\ell_2}}{\left(\dfrac{r_1^4}{\ell_1} + \dfrac{r_2^4}{\ell_2}\right)} \qquad \text{.....(ii)}$$

$$\therefore \quad V = \frac{\dfrac{\pi r_1^4}{8\eta \ell_1} \cdot \dfrac{p r_2^4}{\ell_2}}{\left(\dfrac{r_1^4}{\ell_1} + \dfrac{r_2^4}{\ell_2}\right)}$$

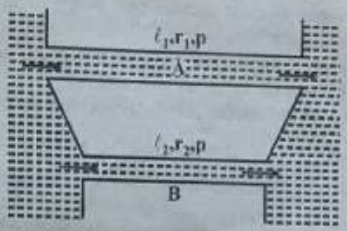
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or $$V = \frac{\pi p}{8\eta} \cdot \frac{r_1^4 r_2^4}{\ell_1 \ell_2 \cdot \dfrac{r_1^4 r_2^4}{\ell_1 \ell_2}\left(\dfrac{\ell_1}{r_1^4} + \dfrac{\ell_2}{r_2^4}\right)}$$

$$= \frac{\pi p}{8\eta}\left(\frac{\ell_1}{r_1^4} + \frac{\ell_2}{r_2^4}\right) \qquad \text{.....(iii)}$$

This expression represents the volume flowing per second through two capillary tubes, connected in series.

If the two capillaries are joined in parallel, i.e., the pressure difference across them is the same, then the volume of the liquid flowing per second through them is given by



$$V = V_1 + V_2 = \frac{\pi p r_1^4}{8\eta \ell_1} + \frac{\pi p r_2^4}{8} = \frac{\pi p}{8\eta}\left(\frac{r_1^4}{\ell_1} + \frac{r_2^4}{\ell_2}\right) \qquad \text{.....(iv)}$$

If we put $R = 8\eta\ell/\pi r^4$, then $V = p/R$ [From Poiseulle's equation]

For series arrangement V $(= R_2)$ [From equation (iii)]

For parallel arrangement $V = p\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)$ [From equation (iv)]

Hence it is clear that here V, p, R are equivalent to current, potential difference and resistance of an electrical circuit.

In case a number of capillaries are connected in series or parallel, then $V = p/(R_1 + R_2 + R_3 + ...)$ in series arrangement,

and $V = p\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + ...\right)$ in parallel arrangement.

Q.12. What is the velocity potential? (2019)

Sol. Velocity Potential: -

A velocity potential is a scalar potential used in potential flow theory. It was introduced by Joseph-Louis Lagrange in 1788.

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It is used in continuum mechanics, when a continuum occupies a simply-connected region and is irrotational. In such a case,

$$\nabla \times u = 0$$

where u denotes the flow velocity. As a result, u can be represented as the gradient of a scalar function ϕ:

$$u = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

ϕ is known as a velocity potential for u.

A velocity potential is not unique. If ϕ is a velocity potential, then ϕ - a(t) is also a velocity potential for u, where a(t) is a scalar function of time and can be constant . In other words, velocity potentials are unique up to a constant, or a function solely of the temporal variable.

If a velocity potential satisfies Laplace equation, the flow is incompressible ; one can check this statement by, for instance, developing $\nabla \times (\nabla \times u)$ and using, thanks to the Clairaut-Schwarz's theorem, the commutation between the gradient and the laplacian operators.

Unlike a stream function, a velocity potential can exist in three-dimensional flow.

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Chapter 6 Special Theory Of Relativity

Q.1 Explain the postulate of special theory of relativity which makes it different from Galilean theory of relativity. (2007, 2009)

Other Related Questions -

Q. Derive the Lorentz transformation equations. Show that these reduce to Galilean transformation for v << c. (2010, 12, 18, 19)

Q. For Lorentz transformations when internal frame S′ moves with respect to inertial frame S along x-direction with velocity v,

(a) Explain why the transformation relations giving x′, y′, z′, t′ in terms of x, y, z and t are linear.

(b) Prove that y′ = y. (2014)

Q. In Lorentz transformation from an inertial frame S to another Lorentz frame S′, explain why the coordinates x′, y′, z′ and t′ in the frame s′ are linear in the coordinates x, y, z and t in the frame S? (2015)

Q. If inertial frame S′ is moving with respect to inertial frame S with velocity v along the x-axis and the two origins and corresponding axes of the two frames coincide at t = t′ = 0, prove that y′ = y and $x' = a_{11}(x - vt)$, where a_{11} is a constant. (2015)

Q. Write short note on Lorentz transformations. (2016)

Ans. Special Theory of Relativity: -

Explanation : -

(i) Einstein ignored the idea of absolute frame of reference attached to ether, by saying that there cannot be any unique universal frame of reference only in which the laws of electro dynamics would be valid all the natural law of physics should be invariant in all inertial frame of reference. The description of physical event is always relative and may vary from one inertial frame to another, even the value of physical quantities may change from frame to frame.

(ii) Michelson - morley experiment shows that the velocity of light is not effected by the motion of surface. Einstein came to conclusion on the basis of all experiment about the velocity of light. It remains the same in all directions in all inertial frames of reference and does not depend upon the motion of the source and of the observer.

Postulate of Special Theory of Relativity : -

According to classical mechanics and galilean transformations, the observed velocity of light should have depended on the velocity of observer.

The two fundamental postulales of speed theory of relativity are given below.

(i) All the laws of physics have the same form for all coordinate system moving with constant relative speed to one another.

(ii) The speed of light in vaccum is the constant in every inertial system.

This means that the speed of light does not depend upon the direction of propagation and relative velocity between the source and observer.

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Lorentz Transformations: -

Let S and S' be the two internal frames of reference. S' is moving along positive direction of X-axis with velocity v relative to the frame S. Let t and t' be the times recorded in two frames. For our convience, we will assume that the origins O and O' of the two co-ordinate systems coincide at $t = t' = 0$. Now suppose that a source of light is situated at $t = 0$. When the light reaches at the point P, let the positions and times, measured by teh observers O and O', be (x, y, z, t) and (x', y', z', t') respectively. If the velocity of light is c, then the time required by the light signal in traversing the distance OP in frame S is

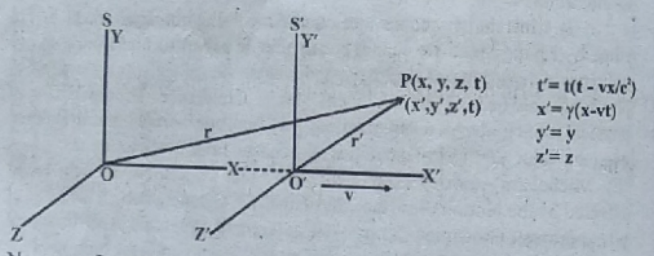
$$t = \frac{OP}{c} = \frac{\left(x^2 + y^2 + z^2\right)^{1/2}}{c}$$

$$\therefore \quad x^2 + y^2 + z^2 = c^2t^2 \qquad \text{.....(i)}$$

According to the special theory of relativity, the velocity of light will also be c in the second frame S'. Hence, in frame S', the time required by the light signal in travelling the distance O'P, is

$$t' = \frac{OP'}{c} = \frac{\left(x'^2 + y'^2 + z'^2\right)^{1/2}}{c}$$

$$\therefore \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad \text{.....(ii)}$$



Now, transformnation eqution relating x, y, z, t and x', y', z', t' should be such that eq.(i) transforms to eq.(ii). The Galilean transformations connect the measurements in the two frames according to the following equations:

$$x' = x - vt;\ y' = y;\ z' = z;\ t' = t \qquad \text{.....(iii)}$$

Substituting these values in eq.(ii), we get

$$(x - vt)^2 + y^2 + z^2 = c^2t^2$$

or $\quad x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 \qquad \text{.....(iv)}$

This equation is certainly not in agreement with eq.(i), becausae it contains an extra term $(-2xvt + v^2t^2)$. Thus the Galilean transformation fails. If the principleof the constancy of the speed of light is valid in all frames, there should exist

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some transformation which reduces to the Galilean for $v/c \to 0$ which transforms $x'^2 + y'^2 + z'^2 = c^2t'^2$ into $x^2 + y^2 + z^2 = c^2t^2$.

We see that in eq.(i) and (iv), the terms of y and z are in agreement. Hence we can say $y' = y$ and $z' = z$. The extra term $(-2xvt + v^2t^2)$ indicates that the transformation in x and t should be modified so that this extra term is cancelled.

We note that for observer O, the distance $OO' = vt$ and therefore when $x' = 0$ (point O'), $x = vt$. This suggest the transformation $x' = \alpha(x - vt)$ because only then for $x' = 0$, $x = vt$. Since t' is different from t and may be depending on x, so that in general we may also assume that $t' = \alpha'(t + fx)$. Here α, α' and f are constants, to be determined (for Galilean transformations $\alpha = \alpha' = 1$ and $f = 0$). Now, substituting for x', y', z' and t' in eq.(ii), we get

$$\alpha^2(x - vt)^2 + y^2 + z^2 = c^2\alpha'^2(t + fx)^2$$

or $\quad \alpha^2(x^2 - 2vxt + v^2t^2) + y^2 + z^2 = c^2\alpha'^2(t^2 + 2fxt + f^2x^2)$

or $\quad x^2(\alpha^2 - f^2\alpha'^2c^2) - 2xt(\alpha^2v + fc^2\alpha'^2) y^2 + z^2 = \left(\alpha'^2 - \frac{\alpha^2v^2}{c^2}\right)c^2t^2 \qquad \text{....(v)}$

This result, obtained by applying transformations from S' to S, must be identical to eq.(i). Therefore,

$$\alpha^2 - f^2\alpha'^2c^2 = 1;\ \alpha^2v + fc^2\alpha'^2 = 0;\ \alpha'^2 - \frac{\alpha^2v^2}{c^2} = 1$$

Solving these three equations, we get

$$\alpha = \alpha' = 1/\sqrt{1 - v^2/c^2} \text{ and } f = -v/c^2$$

$$\therefore \quad x' = \alpha(x - vt) = (x - vt)\sqrt{1 - v^2/c^2}$$

and $\quad t' = \alpha'(t + fx) = \left(t - vx/c^2\right)/\sqrt{1 - v^2/c^2}$

Thus, the new transformation equations, which are in agreement with the invariance of velocity of light c, are

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}};\ y' = y;\ z' = z;\ t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{.....(vi)}$$

These equations are known as **Lorentz transformations**, because they were first obtained by Dutch physicist H. Lorentz.

We see that when $v << c$ i.e. $v/c \to 0$, we get the Galilean transformations from the Lorentz transformations. in most of the cases, which we encounter on earth, c is a velocity very large compared with the great majority of velocities i.e. $v << c$ so that the results of Lorentz transformations do not differ to any

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great extent from those of the Galilean transformations; but from a theoretical point of view the Lorentz transformations represent a most profound conceptual change specially with regard to space and time.

For our convience, sometimes we put

$\beta = v/c$ and $1/\sqrt{1-v^2/c^2} = 1/\sqrt{1-\beta^2} = \gamma$

Hence the transformations are

$$x' = \gamma(x - vt); y' = y; z' = z; t' = \gamma\left(t' - \frac{vx}{c^2}\right) \quad \text{.....(vii)}$$

In the deduction of these equations, we assumed that frame S′ is moving in positive X-direction with velocity v relative to the frame S. But if we say that frame S is moving with - v velocity relative to S′ along negative X - direction, then the transformations are

$$x = \gamma(x' - vt'); y = y'; z = z'; t = \gamma\left(t' - \frac{vx'}{c^2}\right) \quad \text{.....(viii)}$$

These are inverse transformations.

Galilean Transformation: -

The equations which provide the relationship between the coordinates of two reference system are called transformation equations. The transformation of coordinates of particle from are inertial frame to another is known as galilean transformation.

If we assume that

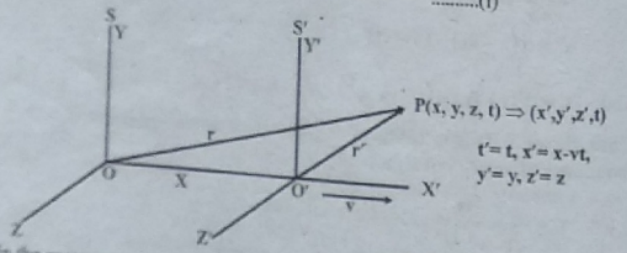
(i) The origins of the two frames coincides at t = 0

(ii) The coordinate axes of the second frame are parallel to that of the first.

(iii) The velocity of second frame relative to the first is v along X- axis , then

Position vectors of a particle at any instant t are related by the equation

$\vec{r}' = \vec{r} - \vec{v}t$ .....(i)



In the component form, the coordinates are related by the equations.

$x' = x - vt, y' = y, z' = z$ .....(ii)

equation (i) or equation (ii) indicate the transformation of the coordinates from one inertial frame to another. They are called Galilean transformation. Equation (i) and equation (ii) depends on the relative motion of two frame of reference,

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but if also depends upon the nature of time and space therefore it is assumed that the time t is independent of any particular frame of reference. If t and t′ be the times recorded by the observer O and O′ of an event occuring at p, then t = t′

Then Galilean transformation equation are

$x' = x - vt; y' = y; z' = z; t' = t$

Q.2 Describe briefly consequence of lorentz transformations regarding the length of bodies and time intervals between given events.

Other Related Questions -

Q. Write short notes on time dialation and length contraction. (2019)

Ans. Consequences of Lorentz Transformations: -

(a) Length Contraction: - (2012)

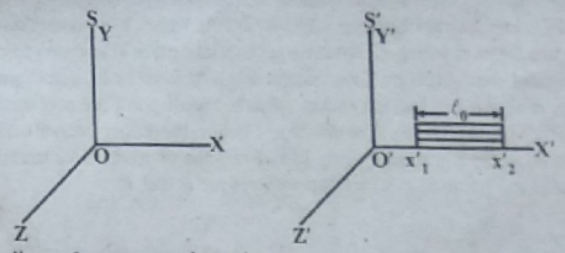
For the measurement of the length of an object in motion, relative to an observer, the positions of the two end points must be recorded simultaneously. Consider a frame S relative to which a rod is moving with velocity v along the X-axis. Let us associate a frame S′ with the rod so that the rod is at rest in S′. If in this frame, the X-coordinates of the ends of the rod are x_1' and x_2' then

$\ell_0 = x_2' - x_1'$.

This length ℓ_0 has been measured by a stationary observer relative to the rod and is called the proper length of the rod. It is not essential that the observer O′ should measure the positions of the end points of the rod simultaneously, because the rod is at rest relative to him.

Now, if the X-coordinates of the end points of that rod in frame S are measured to be x_1 and x_2 at the same time t, then in this frame the observed length of the rod is

$\ell = x_2 - x_1$



According to Lorentz transformations

$x_1' = \gamma(x_1 - vt)$ and $x_2' = \gamma(x_2 - vt)$

$\therefore \quad x_2' - x_1' = \gamma(x_2 - x_1)$ or $\ell_0 = \gamma\ell$

Thus $\ell = \ell_0\sqrt{1 - v^2/c^2}$ $\quad \left(\because \gamma = 1/\sqrt{1 - v^2/c^2}\right)$

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Since the factor $\sqrt{1-v^2/c^2}$ is smaller than unity, we have $\ell < \ell_0$. This means that the length of the rod (ℓ) as measured by an observer relative to which the rod is in motion, is smaller than its proper length. Such a contraction of length in the direction of motion relative to an observer is called Lorentz Fitzgerald contraction. But there will be no change in length in the perpendicular direction of motion.

(2) Time Dilation : Suppose that a frame S′ is moving along X-axis with velocity v relative to S′. Now, if a clock, being at rest in the frame S′, measures the times t_1' and t_2' of two events occurring at a fixed position x′ in this frame, then the interval of time between these events is

$\Delta t' = t_2' - t_1'$.

If t_1 and t_2 are the times of the same events, noted by a clock at rest in frame S, then

$\Delta t = t_2 - t_1$

Now, according to Lorentz transformations, we have

$t_1 = \gamma (t_1' + vx'/c^2)$ and $t_2 = \gamma (t_2' + vx'/c^2)$

$\therefore \quad t_2 - t_1 = \gamma (t_2' - t_1)$ or $\Delta t = \gamma \Delta t'$

Thus $$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} \qquad \text{.....(i)}$$

Since the factor $1/\sqrt{1-v^2/c^2}$ is greater than 1.

$\therefore \quad \Delta t > \Delta t'$ or $(\Delta t)_{motion} > (\Delta t)_{rest}$

Thus, the interval, measured in the frame S, is longer than the time interval in the frame S′, in which the events are occurring at a certain point x′. This effect is called time dilation. This means that to a stationary observer the moving clock will appear to go slow.

In consequence we may say that the two events appear to take a longer time when they occur in a body in motion relative to the observer than when the body is at rest relative to the observer.

The time interval between any two events which has been measured relative to a frame in which the same two events take place at a certain point, is minimum and is called proper time interval and is denoted by Δt_0. For example, an observer sitting in a moving rocket, takes the readings of the needle of his clock at 9 A.M. and 10 A.M., then this $\Delta t_0 = 1$ hour is the proper interval of time.

Now, if Δt_0 is the decay half life of mesons or radioactive matter as measured in the frame S′ in which the particles are at rest, then

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \qquad \text{.....(ii)}$$

is the decay half life observed in a frame S in which the particles are moving with velocity v.

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Q.3 The space-time coordinates of two events in frame S are (0, 0, 0, 0) and (5c, 0, 0, 4) respectively. Obtain the space-time interval between them. What should be the velocity of a frame S relative to S in which the two events may appear to occur simultaneously. (2009)

Ans. We have $S_{12}^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = 25c^2 - 16c^2$

$S_{12}^2 = 9c^2$

$S_{12} = 3c$

Space – time Interval $S_{12} = 3c$

Which is space like

Let the velocity of frame S′ relative to S be v so that the two events are simultaneous (ie $\Delta t' = 0$) in the frame S′. Therefore,

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 0$$

$$\Delta t = \frac{v\Delta x}{c^2}$$

$$4 = \frac{v \times 5c}{c^2} \qquad \Rightarrow \qquad v = 0.8\,c$$

Q.4 Write short note on the relativity of simultaneity of events. (2008)

Other Related Questions -

Q. Write short note on simultaneity in relativity theory. (2010, 12, 15)

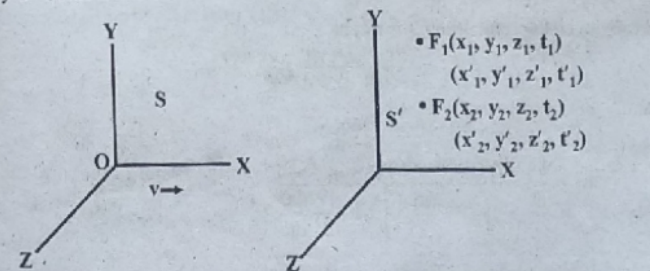
Ans. Simultaneity: -

"If two events occur at the same time in a frame they are said to be simultaneous."

Let us assume that frame S′ is moving relative to frame S along x-axis with velocity v. let two events occur simultaneously in frame S at point F_1 and F_2 with coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) respectively.

Hence, $t_1 = t_2$

If t_1' and t_2' are the corresponding times of the same two events as measured by O′ of the frame S′.



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Hence from Lorentz-transformation

$$t_1' = \frac{t_1 - vx_1/c^2}{\sqrt{1-v^2/c^2}} \text{ and } t_2' = \frac{t_2 - vx_2/c^2}{\sqrt{1-v^2/c^2}}$$

Therefore, $t_2' - t_1' = \frac{(t_2 - t_1)}{\sqrt{1-v^2/c^2}} - \frac{(v/c^2)(x_2 - x_1)}{\sqrt{1-v^2/c^2}}$

Since $t_2 - t_1 = 0$

Hence $(t_2' - t_1') = -\frac{(v/c^2)(x_2 - x_1)}{\sqrt{1-v^2/c^2}}$

We observed that $t_2' - t_1' \neq 0$. This means F_1 and F_2 which are simultaneously for O in frame S are not simultaneous for the observer O' of the frame S' moving with velocity v along x-axis relative to S.

Thus simultaneity is not absolute but relative.

Q.5 Describe briefly relativistic law of addition of velocity.

Other Related Questions -

Q. Inertia frames S is moving with a velocity v with respect of the intertial frame S along the common x-x direction. In frame S' a passenger is moving with a velocity $\vec{u}$, obtain the expression for u_x', u_y' and u_z' while passenger's velocity components in frame S are u_x, u_y and u_z. (2008, 2011)

Q. Write the short note on Relativistic velocity transformation equations. (2017)

Ans. Suppose that the coordinates of a particle in frame S are (x, y, z, t) and in frame S', (x', y', z', t') then the components of its velocity in two frames can be written as

$$u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt},\ u_z = \frac{dz}{dt} \quad \text{(in S)}$$

$$u_x' = \frac{dx'}{dt'},\ u_y' = \frac{dy'}{dt'},\ u_z' = \frac{dz'}{dt'} \quad \text{(in S')}$$

From inverse lorentz transformations

$x = \gamma(x' + vt'), y = y', z = z', t = \gamma(t' + vx'/c^2)$

$\therefore\ dx = \gamma(dx' + vdt'), dy = dy', dz = dz'$

and $dt = \gamma(dt' + vdx'/c^2)$

$$\therefore\ u_x = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \text{ Or } u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$

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$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{dy'/dt'}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \text{ Or } u_y = \frac{u_y'}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}$$

Similarly $u_z = \frac{u_z'}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}$

This is the relativistic law of addition of velocities while in classical mechanics $u_x = u_x' + v$, $u_y' = u_y$ and $u_z' = u_z$

If we take transformation equations as $x' = \gamma(x - vt), y' = y, z' = z$ and $t' = \gamma(t - vx/c^2)$

We can prove that

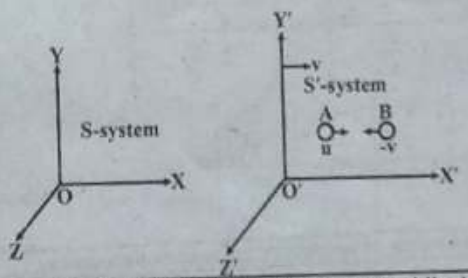
$$\left.\begin{aligned} u_x' &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\ u_y' &= \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\ u_z' &= \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \end{aligned}\right\} \quad \text{.....(v)}$$

Q.6 Prove that

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where symbols have their usual meaning. (2010, 15)

Ans. Variation of Mass with Velocity: -



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Consider two systems S and S'. S' is moving with a constant velocity v relative to the system S, in the positive X-direction. Suppose that in the system S', two exactly similar elastic balls A and B approach each other at equal speeds (i.e., u and - u). Let the mass of each ball be m in S'. They collide with each other and after collision coalesce into one body. According to the law of conservation of momentum,

Momentum of ball A + momentum of ball B = momentum of coalesced mass. Or mu + (-mu) = momentum of coalesced mass = 0. Thus the coalesced mass must be at rest in S' system.

Let us now consider the collision with reference to the system S. Let u_1 and u_2 be the velocities of the balls relative to S. Then,

$$u_1 = \frac{u+v}{1+\frac{uv}{c^2}} \qquad \text{.....(1)}$$

and

$$u_2 = \frac{-u+v}{1-\frac{uv}{c^2}} \qquad \text{.....(2)}$$

After collision, velocity of the coalesced mass is v relative to the system S. Let mass of the ball A travelling with velocity u_1 be m_1 and that of B with velocity u_2 be m_2 in the system S. Total momentum of the balls is conserved. Therefore,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \qquad \text{.....(3)}$$

Substituting for u_1 and u_2 from equations (1) and (2), we have,

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}}\right] + m_2\left[\frac{-u+v}{1-\frac{uv}{c^2}}\right] = (m_1+m_2)v$$

or

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}} - v\right] = m_2\left[v - \frac{-u+v}{1-\frac{uv}{c^2}}\right]$$

or

$$m_1\left[\frac{u+v-v-\frac{uv^2}{c^2}}{1+\frac{uv}{c^2}}\right] = m_2\left[\frac{v-\frac{uv^2}{c^2}+u-v}{1-\frac{uv}{c^2}}\right]$$

or

$$m_1\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1+\frac{uv}{c^2}}\right] = m_2\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1-\frac{uv}{c^2}}\right]$$

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or

$$\frac{m_1}{m_2} = \frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}} \qquad \text{.....(4)}$$

Also,

$$1-\frac{u_1^2}{c^2} = 1-\frac{\left\{\frac{u+v}{c}\right\}^2}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$= \frac{1+\frac{u^2v^2}{c^4}+\frac{2uv}{c^2}-\frac{u^2}{c^2}-\frac{v^2}{c^2}-\frac{2uv}{c^2}}{\left(1+\frac{uv}{c^2}\right)^2} = \frac{\left(1-\frac{u^2}{c^2}\right)-\frac{v^2}{c^2}\left(1-\frac{u^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$\therefore \quad 1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2} \qquad \text{.....(5)}$$

Similarly,

$$1-\frac{u_2^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)^2} \qquad \text{.....(6)}$$

Dividing equation (6) by equation (5),

$$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} = \frac{\left(1+\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}$$

or

$$\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} = \frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}} \qquad \text{.....(7)}$$

From equations (7) and (4), $\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$

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or $$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} \quad \text{.....(8)}$$

Since the L.H.S. and R.H.S. of equation (8) are independent of one another, the above result can be true only if each is a constant. Therefore, $m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$. The constant denoted by m_0 is called the rest mass of the body and corresponds to zero velocity.

Thus, $$m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

In general, if m denotes the mass of a body when it is moving with a velocity v, then, $$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \quad \text{.....(9)}$$

Q.7. Prove that

E $_0c^2 + K$ **(2015)**

Other Related Question -

Q. Derive relativistic mass-energy equivalence equation.

Q. Discuss Einstein's mass-energy relation. (2019)

Ans. Relativistic Mass-Energy Equivalence: -

Energy is defined in terms of work, that is equal to force x distance; but force is the rate of change in momentum. Hence,

$$F = \frac{d(mv)}{dt} \quad \text{.....(i)}$$

The above can be explained classically.

According to the relativity theory, the force is mass variable as well as velocity variable, given by

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

If the body is displaced by a distance dx by applying force F, then increase in kinetic energy is

$dk = Fdx$ .....(ii)

Putting value of F

$$dk = \left(m\frac{dv}{dt} + v\frac{dm}{dt}\right).dx = m\frac{dv}{dt}.dx + v.\frac{dm}{dt}dx$$

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$= mvdv + v^2dm$ .....(iii)

But variation of mass with velocity v is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow m^2 = \frac{m_0^2c^2}{c^2-v^2}$$

$\therefore \quad m^2c^2 - m^2v^2 = m_0^2c^2$

Differentiating both sidces we, get

$c^2 2m\,dm - v^2 2m\,dm - m^2 2v\,dv = 0$ ($\because$ m_0 is constant)

$c^2dm - v^2dm - mv\,dv = 0$

$c^2dm = v^2dm + mv\,dv$ .....(iv)

Comparing eq.(iii) and (iv)

$dk = c^2dm$ .....(v)

If initially the body is at rest and on applying force its velocity is v, and the mass of body increases from m_0 to m. Total K.E. will be given by

$$\int dk = \int_{m_0}^{m} c^2dm$$

Increase in K.E. $k = c^2(m - m_0)$

This is due to the increase in mass from m_0 to m. But the total energy of a moving particle is the sum of its K.E. and energy at rest; then

Total energy $E = k + m_0c^2$ (rest mass energy = m_0c^2)

$= c^2(m - m_0) + m_0c^2$

$E = mc^2$ .....(vi)

This is called Einstein's mass energy equivalence equation. It shows the conversion of mass into energy and energy into mass. When a particle collides with its anti-particle, the total mass is converted into radient energy and conversion of mass into energy takes place. On the other hand, when radiant energy comes near a charged nucleus, particles and anti particles are produced and energy is converted into mass.

Q.8. Explain why we cannot accelerate an electron to a velocity greater than the velocity of light in free space.

Ans. The mass of body moving relative to an deserver increase with increasing velocity according to the equation.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m is the mass at velocity v and m_0 is the rest mass. Thus as the body's velocity v approaches the velocity of light c, the ratio $\frac{v^2}{c^2} \to 1$ and the quantity

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$\sqrt{1-\frac{v^2}{c^2}} \rightarrow$ Hence mass m of the body approaches infinity as v approaches c,. For instance, an electron energized to 10^9ev has an effective mass nearly 200 times its rest mass.

The variation of mass with velocity, therefore, places a limit on the velocity which a material particle can have relative to an observer. As we accelerated the particle. Its mass becomes larger and it becomes increasingly more difficult to accelerate It further. Infact since the mass would become infinite when v = c, an infinite force would be needed to accelerated the particle to the velocity of light. Since no infinite forces are in the universe the velocity of particle can never be made to exceed the velocity of light in free space.

Q.9 Prove the relation $E^2 - P^2c^2 = m_0^2c^4$ where m_0 is the rest mass and p is the momentum. **(2005)**

Ans. Relativistic relation between energy and momentum to the total (rest mass plus kinetic) energy of a particle is given by

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\left(v^2/c^2\right)}} \quad .....(1)$$

And the relativistic momentum is given by

$$P = mv = \frac{m_0v}{\sqrt{1-\left(v^2/c^2\right)}} \quad .....(2)$$

And so $$Pc = \frac{m_0vc}{\sqrt{1-\left(v^2/c^2\right)}} \quad .....(3)$$

Squaring and subtracting eq. (3) from eq. (1) we get

$E^2 - P^2c^2 = m_0^2c^4$

$$E^2 - P^2c^2 = \frac{m_0^2c^4 - m_0^2v^2c^2}{1-\left(v^2/c^2\right)} = m_0^2c^4\frac{1-\left(v^2/c^2\right)}{1-\left(v^2/c^2\right)} = m_0^2c^4 \quad \text{(Proved)}$$

$$E^2 = m_0^2c^4 + P^2c^2$$

$$E = \sqrt{m_0^2c^4 + P^2c^2}$$

This is the relativistic between total energy E and momentum P of a particle.

Q.10 Show that for relativistic particles Kinetic Energy $K = mc^2 - m_0c^2$ where m is relativistic mass and m_0 is rest mass. **(2008, 14, 17)**

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Q. Deduce an expression for the kinetic enerrgy of a relativistic particle. **(2012, 19)**

Ans. Let us consider a force $F = \frac{d}{dt}(mv)$ be acting on particle of mass m so that its kinetic energy increases. The gain in kinetic energy will be equal to the work done on the particle. If the force displaces the particle through a distance dr along its line of action, then small gain in kinetic energy is

$$dE_K = Fdr$$

$$= \frac{d}{dt}(m,v)dr = v\,d\,(m\,v) \qquad \left[\because v = \frac{dr}{dt}\right]$$

If the particle start from rest (v = 0) and acquires velocity v under the action of force, then the kinetic energy, gained by the particle, will be given by.

$$E_k = \int dE_k = \int_0^v vd(mv)$$

$$= \{v.mv\}_0^v - \int_0^0 mv\,dv = mv^2 - \int_0^v \frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}dv$$

$$E_k = mv^2 - \int_0^v \frac{cm_0v}{\sqrt{c^2-v^2}}dv$$

put $c^2 - v^2 = t$ $\qquad (0 - 2v)\,dv = dt \Rightarrow vdv = -\frac{dt}{2}$

$$E_k = mv^2 + \int_{c^2}^{c^2-v^2} \frac{cm_0}{t^{1/2}}\frac{dt}{2}$$

$$= mv^2 + \frac{cm_0}{2}\int_{c^2}^{c^2-v^2} t^{-1/2}dt$$

$$= mv^2 + \frac{cm_0}{2}\left[\frac{t^{1/2}}{1/2}\right]_{c^2}^{c^2-v^2} = mv^2 + m_0c\left[\sqrt{c^2-v^2} - c\right]$$

$$= mv^2 + m_0c^2\left[\sqrt{1-\frac{v^2}{c^2}} - 1\right]$$

$$= \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0c^2\sqrt{1-\frac{v^2}{c^2}} - m_0c^2$$

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$$= \frac{m_0v^2 + m_0c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$= \frac{m_0v^2 + m_0c^2 - m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$E_k = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$E_k = mc^2 - m_0c^2 \qquad \because\ m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

This equation is known as relativistic kinetic energy.

Q.11. Write down the relativistic expression for kinetic energy of a body and show that for small speeds it reduces to the nonrelativistic expression. (2009)

Ans. If m be the relativistic mass of a particle moving with velocity v and m_0 be the rest mass of the body, then

$$\frac{k}{c^2} = m - m_0 \qquad \text{.....(i)}$$

where k is the kinetic energy of the particle we know that

$$\text{Relativistic mass } m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{.....(ii)}$$

from equation (i) and (ii), we get

$$\frac{k}{c^2} = m_0\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right]$$

$$k = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right] \qquad \text{.....(iii)}$$

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Equation (iii) gives the required expression in terms of the rest mass and velocity of the particle.

From equation (iii), kinetic energy of particle

$$k = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots$$

$$\Rightarrow \quad \left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots\ldots\ldots$$

For very small velocities ($v << c$), the quantity v^2/c^2 is very small and its higher term can be neglected. Then

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

$$\therefore \quad k = m_0c^2\left[1 + \frac{v^2}{2c^2} - 1\right]$$

$$k = \frac{1}{2}m_0v^2 \qquad \text{.....(iv)}$$

This expression is known as classical expression for kinetic energy.

Since the rest energy of the particle is given by

$$E_0 = m_0c^2 \text{ and } \frac{v^2}{2} << c^2$$

Hence, kinetic energy is given by equation (iv) less than the rest energy.

Q.12 A spaceship moving away from earth with velocity 0.5 c fires a rocket whose velocity relative to space is 0.5 c directed away from earth and towards the earth. Calculate the velocity of rocket as observed from the earth. (2006)

Ans. Let the velocity of the rocket as observed from the earth be u.

First Case : $u' = 0.5c, v = 0.5c$

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}} \quad \Rightarrow \quad u = \frac{0.5c + 0.5c}{1+\frac{0.5c \times 0.5c}{c^2}}$$

$$u = \frac{c.c^2}{c^2 + 0.25c^2} = \frac{c.c^2}{1.25c^2}$$

$$u = \frac{c}{1.25} \quad \Rightarrow \quad u = 0.8c$$

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Second Case : $u' = -0.5c, v = 0.5c$

$$u = \frac{-0.5 + 0.5c}{1 - \frac{0.5 \times 0.5c}{c^2}} \Rightarrow \quad u = 0$$

Q.13 A beam of particles of half-life 2×10^{-6} sec travels in the laboratory with speed 0.96 times the speed of light. How much distance does the beam travel before the flux falls to ½ times the initial flux? (2006)

Ans. We know that,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where Δt is the observed half life, then

$$\Delta t = \frac{2 \times 10^{-6}}{\sqrt{1 - (0.96)^2}} = \frac{2 \times 10^{-6}}{\sqrt{1 - .9316}}$$

$$= \frac{2 \times 10^{-6}}{\sqrt{.0684}} = \frac{2 \times 10^{-6}}{0.28} \text{ sec}$$

In this observed half-life the flux will fall to ½ times the initial flux.

In this time the distance travelled by the beam is = $v.\Delta t$

$$= 0.96 \times 3 \times 10^8 \times \frac{2 \times 10^{-6}}{0.28} = 2000 \text{m}$$

Q.14 An electron is accelerated from rest through a potential difference of 125 kilovolts. Calculate the final speed of electron. Rest energy of electron is given as 0.5 Mev. (2007)

Ans. According to classical mechanics, the kinetic energy of a particle of mass m_e moving with speed v is $\frac{1}{2} m_e v^2$. When an electron acquires this speed when accelerated through a potential difference V, we have

$$\frac{1}{2} m_e v^2 = eV$$

Where e is the electron charge, we get

$$v = \sqrt{\frac{2eV}{m_e}}$$

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Putting $e = 1.6 \times 10^{-19}$ coulomb

$V = 125$ kilo volts $= 125 \times 10^3$ volts

And $m_e = 9.1 \times 10^{-31}$ kg, we get

$$v = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 125 \times 10^3}{9.1 \times 10^{-31}}} = 2.08 \times 10^8 \text{ m/s}$$

This speed is less than the speed of light and hence acceptable. Here, in the given problem we must use relativistic expression of kinetic energy to calculate the speed of electron. The expression is –

$$K = mc^2 - m_e c^2$$

$$K = \frac{m_e c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_e c^2$$

$$K = m_e c^2 \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right]$$

Where $k = 0.5$ MeV $= 0.5 \times 1.6 \times 10^{-13} = 8 \times 10^{-13}$ Joule

$$\Rightarrow \quad \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 = \frac{8 \times 10^{-13}}{9.1 \times 10^{-31} \times (3 \times 10^8)^2} = 0.09 \times 10^2$$

$$\Rightarrow \quad \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 10 \qquad \Rightarrow \qquad \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{10}$$

$$\Rightarrow \quad v^2 = \frac{99}{100} c^2 = 0.9c^2 \Rightarrow \qquad v = 0.94c$$

Q.15 Show that the momentum of a relativistically moving particle of rest mass m_e and kinetic energy, K_E is given by the expression $p = \sqrt{\frac{K_E^2}{C^2} + 2m_e K_E}$. (2011)

Ans. We have the relation $E^2 - p^2c^2 = m_0^2 c^4$, whence, $E^2 = m_0^2 c^4 + p^2 c^2$.

Or, $E = \left(m_0^2 c^4 + p^2 c^2\right)^{\frac{1}{2}}$

Also, we know that the total energy E = rest energy + kinetic energy $= m_e c^2 + k_E$. Equating the two values of E, therefore, we have

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$$\left(m_0^2c^4 + p^2c^2\right)^{\frac{1}{2}} = m_0c^2 + k_E.$$

Squaring both sides of the equation, we have

$$m_0^2c^4 + p^2c^2 = m_0^2c^4 + 2m_0c^2k_E + k_E^{\ 2} \quad \text{Or} \quad p^2c^2 = k_E^{\ 2} + 2m_0c^2k_E$$

whence, $p^2 = \frac{k_E^{\ 2}}{c^2} + 2m_0k_E$ and, $\therefore p = \sqrt{\frac{k_E^{\ 2}}{c^2} + 2m_0k_E}$ the required relation.

Q.16. Obtain the velocity addition theorem. **(2019)**

Sol. (Relativistic) Addition of velocities or velocity addition theorem: -

The Lorentz transformation equations enable us to transform velocity from one frame of reference to another, in relative motion with respect to it. This leads to a relativitic formula for the addition of velocities. This formula is known as Einstein's velocity addition theorem.

Let S and S′ be the two inertial frames in relative motion, so that S′ moves with a uniform velocity v to the right, along X-axis, relative to S.

Let $\bar{u}$ and $\bar{u}'$ be the velocities of a particle measured in the inertial frames S and S′ respectively. The respective components of these velocities are (u_x, u_y, u_z) and (u'_x, u'_y, u'_z)

$$u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}, u_z = \frac{dz}{dt}$$

and $$u'_x = \frac{dx'}{dt'}, u'_y = \frac{dy'}{dt'}, u'_z = \frac{dz'}{dt'}$$

According to Lorentz transformation equations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}; \; y' = y; \; z' = z \text{ and } t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Taking the differentails of the above transformation equations, we get

$$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}}; dy' = dy; dz' = dz; dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad \frac{dx'}{dt'} = \frac{dx - v.dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

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or $$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \qquad \text{.....(i)}$$

and $$\frac{dy'}{dt'} = \frac{dy\sqrt{1 - \frac{v^2}{c^2}}}{dt - \frac{v}{c^2}dx} = \frac{\frac{dy}{dt}\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}.\frac{dx}{dt}}$$

or $$u'_y = \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}.u_x} \qquad \text{.....(ii)}$$

Similarly $$\frac{dz'}{dt'} = \frac{dz\sqrt{1 - \frac{v^2}{c^2}}}{dt - \frac{v}{c^2}dx} = \frac{\frac{dz}{dt}\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

or $$u'_z = \frac{u_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}u_x} \qquad \text{.....(iii)}$$

Equations (i), (ii) and (iii) represent relativistic law of addition of velocities.

Q.17 Prove that the particle of zero rest mass moves with the speed of light. **(2011)**

Ans. The relativistic energy E of a particle of rest mass m_0 and momentum p is given by

$$E = \left(p^2c^2 + m_0^2c^4\right)^{1/2} \qquad \text{.....(i)}$$

When $m_0 = 0$, $E = pc$ .....(ii)

But $p = v\frac{E}{c^2}$, $\therefore \frac{Ev}{c^2} = \frac{E}{c}$

whence $v = c$ .....(iii)

Thus, a particle of zero rest mass travels with speed of light.

Q.18. Write short note on Michelson-Morley experiment. **(2018)**

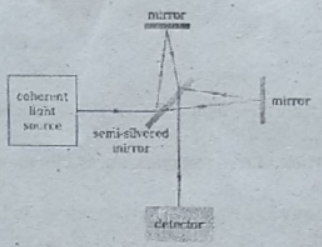
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Ans. The Michelson–Morley experiment was a scientific experiment to find the presence and properties of a substance called aether, a substance believed to fill empty space. The experiment was done by Albert A. Michelson and Edward Morley in 1887.

Since waves in water need something to move in (water) and sound waves do as well (air), it was believed that light also needed something to move in. Scientists in the 18th century named this substance "aether," after the Greek god of light. They believed that aether was all around us and that it also filled the vacuum of space. Michelson and Morley created this experiment to try and prove the theory that aether existed. They did this with a device called an interferometer.

The Earth travels very quickly (more than 100,000 km per hour) around the Sun. If aether exists, the Earth moving through it would cause a "wind" in the same way that there seems to be a wind outside a moving car. To a person in the car, the air outside the car would seem like a moving substance. In the same way, aether should seem like a moving substance to things on Earth.

The interferometer was designed to measure the speed and direction of the "aether wind" by measuring the difference between the speed of light traveling in different directions. It measured this difference by shining a beam of light into a mirror that was only partially coated in silver. Part of the beam would be reflected one way, and the rest would go the other. Those two parts would then be reflected back to where they were split apart, and recombined. By looking at interference patterns in the recombined beam of light, any changes in speed because of aether wind could be seen.



They found that there was in fact no substantial difference in the measurements. This was puzzling to the scientific community at the time, and led to the creation of various new theories to explain the result. The most important was Albert Einstein's special theory of relativity.

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