WINSOME

Genius Choice

B.Sc.

II^nd Year

Math - I

According to Latest Syllabus of

ALLAHABAD UNIVERSITY

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WINSOME PUBLICATION
ALLAHABAD-211008

SYLLABUS

Unit I - Definition of a field, Examples R, C, Q and Z of fields, vector space: definition, examples and basic properties, subspaces, subspace generated by subset, Linear dependence and independence of a subset, Basic and dimensions of a vector space, Sum and direct sum of subspaces, Quotient spaces.

Unit II - Linear transformations and isomorphism of vector spaces, Null space, range space, rank and nullity of linear transformation, Fundamental theorem of homomorphism, Rank nullity theorem, Hom (V, M) as a vector space, Dim Hom (V, W) = Dim V. Dim W. Dual space V* and transpose of a linear transformation, Annihilator of a subset of a vector space.

Unit III - Matrix representation of linear transformation, Algebra of matrices, Effect of change of basis of matrix representation, Equivalent and similar matrices, Rank and nullity of a matrix, System of linear equations, their consistency and inconsistency reduction of a matrix to normal form and algorithms to find the inverse (if it exists), Definition of Determinant and statement of properties (without proof), cofactor, Adjoint of a matrix and its properties.

Unit IV - Inner product space: Definition and examples, Cauchy Schwarz inequality, norm of a vector, Motion of angle, Orthogonality, Orthonormal set and basis, Gram Schmidt orthonormalization, Bassel's inequality, Orthogonal complement, Adjoint of a linear transformation (orthogonal Unitary), Hermitian, Skew Hermitian, Symmetric and Skew Symmetric matrices, Invarient Subspaces, Eigenvalues, Eigen vectors, Characteristic polynomials of a linear transformation, Cayley-Hamillton Theorem.

Unit V - Diagonalization of a matrix with distinct eigen values, Orthogonal (Unitary) reduction of real Symmetric (Hermitian) matrices.

Bilinear forms, Equivalence of bilinear forms, matrix, representation of Bilinear forms, Quadratic forms, Classification of Symmetric, Skew Symmetric bilinear forms over R and C, Sylvesters law of inertia. Classification of curves and surfaces represented by equations of 2nd degree.

CONTENTS

S.N.	Details	Page No.
1.	Unit I	1 - 24
2.	Unit II	25 - 54
3.	Unit III	55 - 82
4.	Unit IV	83 - 100
5.	Unit V	101 - 117

Unit I

Q.1. Define basis of a vector space. Prove that a maximal linearly independent subset of a vector space is a basis of the vector spaces. (2014)

Ans. Basis:-

Let V be the vector space over the field F. A subset S of V is called basis of a vector space V over F.

if (i) $L(S) = V$

(ii) S is a linearly independent set.

Let $S = \{v_1, v_2 \ldots\ldots v_n\}$ be a maximal linear independent subset of V. Then S becomes a basis of V, if we prove that $L(S) = V$.

Let $v \in V$ be arbitrary. Since $v, v_1, v_2 \ldots\ldots v_n$ are n+1 vectors in V, they must be linearly dependent and so there exist scalars $\alpha, \alpha_1, \alpha_2, \ldots\ldots \alpha_n$; not all zero.

Such that $\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$ .....(i)

if $\alpha = 0$ then equation (i) becomes -

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$

$\Rightarrow$ $\alpha_1 = 0, \alpha_2 = 0 \ldots\ldots \alpha_n = 0$ ($\because$ S is a linear independent set)

and $\alpha = 0$. This is a contradiction to the fact $\alpha, \alpha_1, \alpha_2, \ldots\ldots \alpha_n$ are not all zero.

Consequently, $\alpha \neq 0 \in F$ and so $\alpha^{-1} \in F$ and $\alpha^{-1}\alpha = 1$. From equation (i), we have

$\alpha v = -\alpha_1 v_1 - \alpha_2 v_2 - \ldots\ldots - \alpha_n v_n$

or $\alpha^{-1}(\alpha v) = -\alpha^{-1}(\alpha_1 v_1) - \alpha^{-1}(\alpha_2 v_2) - \ldots\ldots - \alpha^{-1}(\alpha_n v_n)$

or $V = (-\alpha^{-1}\alpha_1)v_1 + (-\alpha^{-1}\alpha_2)v_2 + \ldots\ldots + (-\alpha^{-1}\alpha_n)v_n$

$\Rightarrow$ $V = L(S)$ and so S is a basis of V containing n vectors.

Q.2. If W is a subspace of a finite dimensional vector space V, prove that dim (V/W) = dim V - dim W. (2012, 14)

Ans. We have to prove that $\dim(V/W) = \dim V - \dim W$

Proof:- Let $\dim W = m$

Let $S = \{a_1, a_2, \ldots\ldots a_m\}$ be a basis of W. Since S is linearly independent subset of V. Therefore S can be extended to form a basis of vector space V (By extension theorem).

Let $S_1 = \{a_1, a_2, \ldots\ldots a_m, b_1, b_2, \ldots\ldots b_n\}$ be basis of V.

$\therefore$ $\dim V = m + n$

consider $S_2 = \{W + b_1, W + b_2, \ldots\ldots, W + b_n\}$

Claim:- S_2 is basis of V/W.

(i) S_2 is linearly independent.

Let $\lambda_1(W + b_1) + \lambda_2(W + b_2) + \ldots\ldots + \lambda_n(W + b_n) = W$

$\Rightarrow W + \lambda_1 b_1 + \lambda_2 b_2 + \ldots\ldots + \lambda_n b_n = W$

$\Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + \ldots\ldots + \lambda_n b_n \in W$ ($\because W + a = W \Leftrightarrow a \in W$)

$\Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + \ldots\ldots + \lambda_n b_n = \upsilon_1 a_1 + \upsilon_2 a_2 + \ldots\ldots + \upsilon_m a_m$

$\Rightarrow (-\upsilon_1)a_1 + (-\upsilon_2)a_2 + \ldots\ldots + (-\upsilon_m)a_m + \lambda_1 b_1 + \lambda_2 b_2 + \ldots\ldots + \lambda_n b_n = 0$

$\Rightarrow \upsilon_1 = \upsilon_2 = \ldots\ldots = \upsilon_m = \lambda_1 = \lambda_2 = \ldots\ldots = \lambda_n = 0_F$

Therefore S_2 is linear independent.

(ii) $V/W = [S_2]$

Let $W + x \in V/W$

$\Rightarrow$ $x \in V$

$\Rightarrow$ $x = \alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_m a_m + \beta_1 b_1 + \beta_2 b_2 + \ldots\ldots + \beta_n b_n$

$\Rightarrow$ $W + x = W + (\alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_m a_m + \beta_1 b_1 + \beta_2 b_2 + \ldots\ldots + \beta_n b_n)$

$\Rightarrow$ $W + x = W + \beta_1 b_1 + \beta_2 b_2 + \ldots\ldots + \beta_n b_n$

$(\because W + \alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_m a_m = W)$

$\Rightarrow$ $W + x = (W + \beta_1 b_1) + (W + \beta_2 b_2) + \ldots\ldots + (W + \beta_n b_n)$

$= \beta_1 (W + b_1) + \beta_2 (W + b_2) + \ldots\ldots + \beta_n (W + b_n)$

$\therefore$ $V/W = [S_2]$

$\therefore$ S_2 is a basis of V/W

$\therefore$ $\dim V/W = n$

$= m + n - m = \dim V - \dim W$

$\Rightarrow$ $\dim V/W = \dim V - \dim W$

Q.3. Define linearly dependent and linearly independent vectors of a vector space over a field. Let $S = \{a_1, a_2, \ldots\ldots, a_n\}$ be a linearly independent subset of a vector space V over a field F and $a \in V$ but $a \in L_s$. Then show that $\{a_1, a_2, \ldots\ldots, a_n, a\}$ will be linearly independent. (2014)

Ans. Linearly dependent and linearly independent vectors of a vector space V(F) :-

Vectors $a_1, a_2, \ldots\ldots, a_n$ are called linearly dependent if $\exists$ scalars $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n \in F$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_n a_n = 0$ in which atleast one $\lambda_i \neq 0_F$. Otherwise vectors are called linearly independent. Let

$\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_n a_n + \beta a = 0$ .....(i) $\lambda_i, \beta \in F$

Claim :- $\beta = 0_F$ in equation (i)

If possible let $\beta \neq 0_F$ in equation (i)

Multiply equation (i) by β^{-1}.

$a = -(\beta^{-1} \odot \lambda_1)a_1 - (\beta^{-1} \odot \lambda_2)a_2 \ldots\ldots - (\beta^{-1} \odot \lambda_n)a_n$

$a = \upsilon_1 a_1 + \upsilon_2 a_2 + \ldots\ldots + \upsilon_n a_n$ $(\because \upsilon_i = \beta^{-1} \odot \lambda_i)$

which is a contradiction because 'a' does not belong to L_s. Therefore our assumption that $\beta \neq 0$ in equation (i) is wrong.

Hence $\beta = 0_F$ in equation (i)

From (i)-

$\Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_n a_n = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \ldots\ldots = \lambda_n = 0_F$ $(\because a_1, a_2, \ldots\ldots a_n$ are linearly independent)

$\Rightarrow \{a_1, a_2, \ldots\ldots, a_n, a\}$ are linearly independent.

Q.4. If $W_1 = \{(x, y, z): x, y, z \in IR$ and $y + 2z = 0\}$ and $W_2 = \{(x, y, z): x, y, z \in IR$ and $x + y + z = 0\}$ are two subspaces of the vector space IR^3 (IR) then find

(i) a basis of W_1 **(ii) a basis of W_2**
(iii) a basis of $W_1 \cap W_2$ and **(iv) a basis of $W_1 + W_2$.** **(2014)**

Ans.(i) $0.x + 1.y + 2.z = 0$

$A = [0, 1, 2]$

$\rho(A) = 1$

Number of linear independent solution $= 3 - 1 = 2$

$y + 2z = 0$

$\Rightarrow \quad y = -2z, x = x, z = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 0.z \\ 0.x + (-2)z \\ 0.x + z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} z$$

$\dim W_1 = 2$

Basis of $W_1 = \{(1, 0, 0), (0, -2, 1)\}$

(ii) $W_2 = \{(x, y, z): x, y, z \in R \text{ and } x + y + z = 0\}$

$x + y + z = 0$

$\Rightarrow \quad x = y - z, y = y, z = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.x + (-1)y + (-1).z \\ 0.x + 1.y + 0.z \\ 0.x + 0.y + 1.z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$\dim W_2 = 2$

Basis of $W_2 = \{(-1, 1, 0), (-1, 0, 1)\}$

(iii) $W_1 \cap W_2 = \{(x, y, z): x, y, z \in R \text{ and } y + 2z = 0, x + y + z = 0\}$

We get $y = -2z, x = z, z = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.x + 0.y + 1.z \\ 0.x + 0.y + (-2).z \\ 0.x + 0.y + 1.z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z$$

$\dim W_1 \cap W_2 = 1$

Basis of $(W_1 \cap W_2) = \{(1, -2, 1)\}$

(iv) We know that

$$\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$$
$$= 2 + 2 - 1 = 3$$

Q.5. Define a vector space. Let V be a vector space over F and q be the zero vector of V, Prove that

(i) $a\theta = \theta \quad \forall a \in F$ **(ii)** $0\alpha = \theta \quad \forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha) = (-a)\alpha \quad \forall \ a \in F$ **and** $\forall a \in V$ **(2015)**

Sol. Vector Space: -

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Let $(F, \oplus, \odot)$ be a field and a system $(V, +, .)$ consisting of

(i) A non-empty set
(ii) An internal binary operation '+' in V
(iii) An external binary operation '.' in V over F, is called a vector space over a field F iff it satisfy the following properties -

(i) $(V, +)$ is an abelian group.

(ii) (a) $\alpha.(\beta.a) = (\alpha \odot \beta).a \quad \forall \alpha, \beta \in F$ and $\forall a \in V$

(b) $1.a = a \quad \forall a \in V$

(iii) (a) $\alpha(a+b) = \alpha.a + \alpha.b \quad \forall \alpha, \beta \in F$ and $a, b \in V$

(b) $(\alpha \oplus \beta).a = \alpha.a + \beta.a \quad \forall \alpha, \beta \in F$ and $a \in V$

(I) $\alpha\theta = \theta \quad \forall \alpha \in F$

Proof - We know that

$0 + 0 = 0$

So $\alpha 0 = \alpha(0+0) = \alpha 0 + \alpha 0$

$\Rightarrow \quad \alpha 0 + 0 = \alpha 0 + \alpha 0$

Hence $0 = \alpha 0$, by cancellation law in the group $(V, +)$.

(ii) $0a = 0 \quad \forall a \in V$

Proof - Since $0 + 0 = 0$

$\Rightarrow \quad 0a = (0+0)a = 0a + 0a$

$\Rightarrow \quad 0a + 0 = 0a + 0a \quad \Rightarrow \quad 0 = 0a$

Hence $0a = 0$, by cancellation law in the group $(V, +)$.

(iii) $\alpha(-a) = -(\alpha a) = (-\alpha)a \quad \forall \alpha \in F$ and $\forall a \in V$

Proof - We know that

$\alpha + (-\alpha) = 0$

$\Rightarrow \quad \{\alpha + (-\alpha)\}a = 0.a = 0$ (by eq.(ii) $0a = 0$)

$\Rightarrow \quad \alpha a + (-\alpha)a = 0$

Hence $(-\alpha)a = -(\alpha a)$

Similarly, $a + (-a) = 0$

$\Rightarrow \quad \alpha\{a + (-a)\} = \alpha 0$

$\Rightarrow \quad \alpha a + \alpha(-a) = 0$ (by eq.(i) $0 = \alpha 0$)

Hence $\alpha(-a) = -(\alpha a)$

Q.6. Prove that the set of non-zero vectors $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ of a vector space V over F is linearly dependent if some $\alpha_k \quad 2 \le k \le n$, is a linear combination of vectors preceding it. **(2015)**

Other Related Questions -

Q. Let V be a vector space over a field F and $S = \{V_1, V_2, \ldots V_n\}$ be a subset of V with each $v_i \neq 0 \ (1 \le i \le n)$. Prove that S is linearly dependent

over F if and only if $\exists k\ (2 \le k \le n)$ such that v_k is linear combination of v_1, v_2, ... v_{k-1}. **(2019)**

Sol. Let $\{\alpha_1, \alpha_2 \ldots\ldots \alpha_n\}$ be linearly dependent. Now, t he Singleton set $\{\alpha_1\}$ is linearly independent. Since $\alpha_1 \neq 0$. Consider $\{\alpha_1, \alpha_2\}$. If it is also linearly dependent, then consider $\{\alpha_1, \alpha_2, \alpha_3\}$. Proceeding thus, Let K be the first positive integer (where $2 \le k \le n$) for which $\{\alpha_1, \alpha_2 \ldots\ldots \alpha_k\}$ is lilnearly independent.

Atmost $k = n$.

Hence $\exists$ scalars $a_1, a_2, \ldots\ldots a_k$ not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_k\alpha_k = 0$. Now $a_k \neq 0$, for if $a_k = 0$, then $\{\alpha_1, \alpha_2 \ldots\ldots \alpha_{k-1}\}$ will be linearly dependent contrary do the definition of k.

Hence $x_k = a_k^{-1}a_1x_1 - a_k^{-1}a_2x_2 \ldots\ldots - a_k^{-1}a_{k-1}x_{k-1}$ which proves that x_k is a linear combination of the preceding vectors $x_1, x_2, \ldots\ldots x_{k-1}$.

Q.7. Show that the vectors (1, 2, 1), (2, 1, 0), (1, -1, 2) from a basis of $\mathbb{R}^3(\mathbb{R})$.

Sol. **(i) S is ℓ.I. -** **(2015)**

Let $\alpha(1,2,1) + \beta(2,1,0) + \gamma(1,-1,2) = 0$

$$\alpha + 2\beta + \gamma = 0 \quad \ldots\ldots\ldots(i)$$

$$2\alpha + \beta - \gamma = 0 \quad \ldots\ldots\ldots(ii)$$

$$\alpha + 0.\beta + 2\gamma = 0 \quad \ldots\ldots\ldots(iii)$$

$$\Rightarrow \quad \alpha = -2\gamma$$

So from eq.(i) and (ii)

$$2\beta - \gamma = 0$$

$$\beta - 5\gamma = 0$$

Solving them we get $\gamma = 0$ And so $\alpha = 0, \beta = 0$.

(ii) $\mathbb{R}^3 = [S]$

Let $(x, y, z) \in \mathbb{R}^3$.

Let $(x, y, z) = \alpha(1,2,1) + \beta(2,1,0) + \gamma(1,-1,2)$

$$(x, y, z) = (\alpha + 2\beta + \gamma, 2\alpha + \beta - \gamma, \alpha + 2\gamma)$$

$$x = \alpha + 2\beta + \gamma \quad \Rightarrow \quad \alpha + 2\beta + \gamma = x$$

$$y = 2\alpha + \beta - \gamma, \quad \Rightarrow \quad 2\alpha + \beta - \gamma = y$$

$$z = \alpha + 2\gamma \quad \Rightarrow \quad \alpha + 2\gamma = z$$

$$\alpha = \frac{\begin{vmatrix} x & 2 & 1 \\ y & 1 & -1 \\ z & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{(-2x + 5y + 2z)}{9}$$

$$\beta = \frac{\begin{vmatrix} 1 & x & 1 \\ 2 & y & -1 \\ 1 & z & 2 \end{vmatrix}}{-9} = \frac{5x+y-3z}{9}$$

$$\gamma = \frac{\begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 1 & 0 & z \end{vmatrix}}{-9} = \frac{x-2y+3z}{9}$$

So, $$(x,y,z) = \frac{(-2x+5y+2z)}{9}(1,2,1) + \frac{(5x+y-3z)}{9}(2,1,0)$$

$$+\frac{(x-2y+3z)}{9}(1,-1,2)$$

Q.8. Prove that corresponding to each sub-space W_1 of a finite dimensional vector space V(F), there exists a subspace W_2 such that V is the direct sum of W_1 and W_2. **(2015)**

Sol. Let $S = \{a_1, a_2, \ldots\ldots a_m\}$ be the basis of W_1. i.e. $\dim W_1 = m$.

Since S is ℓ. I. subset of vector space V. Therefore it can be extended a basis of V.

let $S_1 = \{a_1, a_2, \ldots\ldots a_m, b_1, b_2, \ldots\ldots b_n\}$ be basis of V.

$\therefore$ $\dim V = m+n$

let $S_2 = \{b_1, b_2, \ldots\ldots b_n\}$ such that $W_2 = [S_2]$

Claim - S_2 is ℓ.I.

Since S_2 is a subset of S_1 and S_1 is linearly independent. Therefore S_2 is also ℓ.I. (because a subset of ℓ.I. set isalso ℓ.I.).

Hence S_2 is basis of W_2.

i.e. $\dim W_2 = n$

$\therefore$ $\dim V = \dim W_1 + \dim W_2$.

$\dim V = \dim(W_1 \oplus W_2)$

$\Rightarrow$ $V = W_1 \oplus W_2$

Q.9. If W is a subspace of a vector space V(F), Prove that the set V/W of all cosets $W+\alpha$, where α is an arbitrary element of V, is a vector space over F for the vector addition and scalar multiplication compositions defined as follows:

$$(W+\alpha)+(W+\beta) = W+(\alpha+\beta) \quad \forall \alpha,\beta \in V \text{ and}$$

$$a(W+\alpha) = W + a\alpha \quad \forall a \in F \text{ and } \forall a \in V. \qquad (2015)$$

Sol. We have to show $\left(\frac{V}{W}, +, .\right)$ is a vector space.

(i) $\left(\frac{V}{W}, +\right)$ is a commutative group.

(1) Closure property -

Let $W+\alpha, W+\beta \in \frac{V}{W}, \quad \alpha, \beta \in V$

$$(W+\alpha)+(W+\beta) = W+\alpha+\beta \in \frac{V}{W}$$

$\therefore$ '+' is a internal binary operation.

(ii) Associative property -

Let $(W+\alpha),(W+\beta),(W+\gamma) \in \frac{V}{W}$

$\Rightarrow \quad \{(W+\alpha)+(W+\beta)\}+(W+\gamma)$

$\Rightarrow \quad \{W+\alpha+\beta\}+(W+\gamma)$

$\Rightarrow \quad W+(\alpha+\beta)+\gamma$

$\Rightarrow \quad W+\alpha+(\beta+\gamma)$ (Associative property of V)

$\Rightarrow \quad (W+\alpha)+\{(W+\beta)+(W+\gamma)\}$

(iii) Existence of identity -

Let $W+\alpha \in \frac{V}{W}$

$\because \quad W+(W+\alpha) = (W+0)+(W+\alpha)$

$= W+(0+\alpha)$

$W+(W+\alpha) = W+\alpha$

$\therefore \quad$ Identity $= W$

(iv) Existence of inverse -

$$\forall (W+\alpha), \exists\ W+(-\alpha) \in \frac{V}{W}$$

Such that $(W+\alpha)+(W+(-\alpha)) = W+\alpha+W+(-\alpha)$

$= W+\alpha+(-\alpha) = W+0 = W$

Therefore inverse of each element is exist.

(v) Commutative property -

let $W+\alpha, W+\beta \in \frac{V}{W}$

$\Rightarrow \quad (W+\alpha)+(W+\beta) = W+\alpha+\beta = W+\beta+\alpha \quad (\because (V,+) \text{ is commutative})$

$$= (W + \beta) + (W + \alpha)$$

So $\left(\frac{V}{W}, +\right)$ is an abelian group.

(II) (i) $a.(b.\alpha) \Rightarrow (a \odot b).\alpha \qquad \forall a, b \in F,\ \forall \alpha \in V$

$$\lambda\{\mu(W + \alpha)\} = \lambda(W + \mu\alpha) = W + \lambda(\mu.\alpha)$$

$$= W + (\lambda \odot \mu).\alpha = (\lambda \odot \mu)(W + \alpha)$$

(ii) $1.\alpha = \alpha, \quad \forall \alpha \in V$

$$1.(W + \alpha) = W + 1.\alpha \qquad (\because 1.\alpha = \alpha,\ \alpha \in V)$$

$$= W + \alpha$$

(III) (i) $a.(\alpha + \beta) = a.\alpha + a.\beta \quad \forall\ a \in F,\ \alpha, \beta \in V$

$$\lambda\{(W + \alpha) + (W + \beta)\} = \lambda\{W + (\alpha + \beta)\}$$
$$= W + \lambda(\alpha + \beta) = W + (\lambda\alpha + \lambda\beta) = W + \lambda\alpha + W + \lambda\beta$$
$$= \lambda(W + \alpha) + \lambda(W + \beta)$$

(ii) $(a \oplus b)\alpha = a.\alpha + b.\alpha \qquad \forall a, b \in F, \forall \alpha \in V$

$$(\lambda \oplus \mu).(W + \alpha) = W + (\lambda \oplus \mu).\alpha = W + \lambda\alpha + \mu\alpha$$
$$= W + \lambda\alpha + W + \mu\alpha = \lambda(W + \alpha) + \mu(W + \alpha)$$

So $\frac{V}{W}$ is a vector space.

Q.10. Let S be a generating set containing n elements of a vector spae V over a field F. Prove that any linear independent subset T of V contains at most elements. **(2016)**

Sol. Let V be a vector space over field F and $X = \{x_1, x_2 \ldots\ldots x_n\}$ be a generating subset of V ie. $V = [s]$

$Y = \{y_1, y_2, \ldots\ldots y_m\}$ be linear independent subset of V.

Claim :- $m \le n$

$\because$ $V = [X]$ and $y_1 \in V$

$\therefore$ $y_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n \qquad \alpha_i \in F$

$\Rightarrow$ $(-1)y_1 + \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0 \qquad \ldots\ldots\ldots(i)$

Claim - At least one $\alpha_i \neq 0_F$ in eq.(i)

If possible let all $\alpha_i = 0$ in eq.(i)

$\therefore$ $(-1)y_1 = 0$

$\Rightarrow \quad y_1 = 0$

Which is a contraction because none of the element in a linear independent set is zero.

Therefore our assumption that all $\alpha_i = 0$ in eq.(i) is wrong. Hence $\exists$ atleast one $\alpha_i \neq 0$ in eq.(i)

Let $\alpha_1 \neq o_F \quad \Rightarrow \quad \exists\ \alpha_1^{-1} \in F$

Multiplying eq.(i) by α_1^{-1} we can write x_1 as a linear combination of

$y_1, x_1, \ldots\ldots x_n$

$\therefore \quad V = [x_1]$, where $x_1 = \{y_1, x_2, \ldots\ldots x_n\}$

$\therefore \quad V = [x_1]$ and $y_2 \in V$

$\Rightarrow \quad y_2 = \beta y_1 + \gamma_2 x_2 + \gamma_3 x_3 + \ldots\ldots\ldots + \gamma_n x_n$

$\Rightarrow \quad \beta y_1 + (-1) y_2 + \gamma_2 x_2 + \ldots\ldots\ldots + \gamma_n x_n = 0 \qquad \ldots\ldots\ldots(ii)$

Claim - At least one $\gamma_i \neq o_F$ in eq. (ii)

If possible let all $\gamma_i = o_F$ in eq.(ii)

$\Rightarrow \quad \beta y_1 + (-1)\, y_2 = 0$

$\Rightarrow \quad \{y_1, y_2\}$ is l.d.

$\Rightarrow \quad y = \{y_1, y_2, \ldots\ldots y_n\}$ is l.d. $\quad (\because$ super set of l.d. set also l.d.)

Which is a contradiction because y is linealy independent.

Therefore our assumption that all $V_i = 0$ in equation ..(ii) is wrong. Hence atleast one $\gamma_i \neq o_f$ in eq.(ii)

Let $\gamma_2 \neq o_f$ in equation (ii) $\quad \Rightarrow \quad \gamma_2^{-1} \in f$

Multiply equation (ii) by γ_2^{-1} we can write

X_2 as a linear combination of elements of

$$x_2 = \{y_1, y_2, ---x_3 --- x_n\}$$

$\because \quad v = [x_2]$

Proceeding in this manner a question arise which one of x and y exhausted first.

Claim :- y will be exhausted before X if possible let X exhausted before y.

Using replacement prosses n time we get

$$x_n = \{y_1, y_2 \ldots\ldots y_n\}$$

$$V = [x_n] \quad \text{and} \quad y_{n+1} \in V$$

$\Rightarrow \quad y_{n+1} = \delta_1 y_1 + s_2 y_2 + \ldots\ldots + \delta_n y_n$

$y_1, y_2 \ldots\ldots y_n, y_{n+1}$ is linearly dependent

$\Rightarrow \quad y = \{y_1, y_2, \ldots\ldots y_n, y_{n+1} \ldots\ldots y_m\}$ is l.d. which is a contradiction because y is linearly independent.

Therefore our assumption that $n < m$ wrong then $m \leq n$

Which proves that y can have n vectors.

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Q.11. Let V be a finite dimensional vector space over a field F. Let w_1 and w_2 be subspaces of V. Prove that

$$\dim(w_1+w_2)=\dim w_1+\dim w_2-\dim(w_1\cap w_2).$$ **(2013, 16)**

Sol. Let $\dim(w_1\cap w_2)=\ell$

and let $s=\{a_1, a_2 \ldots\ldots a_\ell\}$ be a basis of $w_1\cap w_2$.

since s is linearly independent subset of $w_1\cap w_2$.therefore s is also L.I. subset of w_1 and w_2. Therefore s can be extended to form a basis of w_1 and w_2 (by extension theorem)

Let $s_1=\{a_1, a_2 \ldots\ldots a_\ell, b_1, b_2 \ldots\ldots b_m\}$ be a basis of W_1.

And $s_2=\{a_1, a_2 \ldots\ldots a_\ell, c_1, c_2 \ldots\ldots c_n\}$ be basis of w_2.

$\because$ $\dim w_1=\ell+m$ and $\dim w_2=\ell+n$

Consider $s_3=\{a_1, a_2 \ldots\ldots a_\ell, b_1, b_2 \ldots\ldots b_m, c_1, c_2, \ldots c_n\}$

Claim :- s_3 is basis of w_1+w_2.

(I) s_3 is ℓ.I. -

Let $\alpha_1a_1+\alpha_2a_2+\ldots\ldots+\alpha_\ell a_\ell+\beta_1b_1+\beta_2b_2+\ldots\ldots+\beta_m b_m$

$$+\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n=0 \quad \ldots\ldots(1)$$

$\Rightarrow$ $\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n\in w_2=(-\alpha_1)a_1+(-\alpha_2)a_2-+(-\alpha_n)a_n$

$$+(-\beta_1)b_1+(-\beta_2)b_2+---+(-\beta_n)b_m\in w_1$$

$\Rightarrow$ $\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n\in w_1\cap w_2$

$\Rightarrow$ $\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n$ can be written as a linear combination of element of s.

$\Rightarrow$ $\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n=\delta_1a_1+\delta_2a_2+\ldots\ldots+\delta_\ell a_\ell$ and $\delta_i\in f$

$\Rightarrow$ $(-\delta_1)a_1+(-\delta_2)a_2+\ldots\ldots+(-\delta_\ell)a_\ell+\gamma_1c_1+\gamma_2c_2+\ldots\ldots+\gamma_n c_n=0$

$$\delta_1=\delta_2=\ldots\ldots\delta_\ell=\gamma_1=\gamma_2=---=\gamma_n=o_F$$

$(\because\ a_1, a_2 \ldots\ldots a_\ell, c_1, c_2 \ldots\ldots c_n$ are ℓ.I.)

Putting the value of $\gamma_1, \gamma_2 \ldots\ldots\gamma_n$ in eq.(1) we get be

$$\alpha_1a_1+\alpha_2a_2+\ldots\ldots+\alpha_\ell a_\ell+\beta_1b_1+\beta_2b_2+\ldots\ldots+\beta_m b_m=0$$

$\Rightarrow$ $\alpha_1=\alpha_2\ldots\ldots=\alpha_\ell=\beta_1=\beta_2=\ldots\ldots=\beta_m=0$

$\because$ $(\because\ a_1, a_2 \ldots\ldots a_\ell, b_1, b_2 \ldots\ldots b_n$ are L.I.)

Hence s_3 is linearly independent.

(II) $W_1+W_2=[s_3]$

Let $x_1+x_2 \in W_1+W_2$

$\Rightarrow$ $x_1 \in W_1$ and $x_2 \in W_2$

$\Rightarrow$ $x_1 = \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_\ell a_\ell + \mu_1 b_1 + \mu_2 b_2 + ... + \mu_m b_m$

$x_2 = \theta_1 a_1 + \theta_2 a_2 + ... + \theta_\ell a_\ell + \phi_1 c_1 + ... + \phi_n c_n$

where $\lambda_i, \mu_j, \theta_k \phi_s \in F$

$\therefore$ $x_1 + x_2 = (\lambda_1 \oplus \theta_1) a_1 + (\lambda_2 \oplus \theta_2) a_2 + ... + (\lambda_\ell \oplus \theta_\ell) a_\ell$

$+\mu_1 b_1 + ... + \mu_m b_m + \phi_1 c_1 + ... + \phi_n c_n$

$\therefore$ $W_1 + W_2 = [s_3]$

Therefore s_3 is basis of $W_1 + W_2$.

$\because$ $\dim (W_1+W_2) = \ell + m + n = (\ell + m) + (\ell + n) - \ell$

$\Rightarrow$ $\dim (W_1+W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$

Q.12. Let $w_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\}$ and $w_2 = \{(\lambda, 2\lambda, -3\lambda, 4\lambda) \lambda : \in \mathbb{R}\}$. Prove that w_1 and w_2 are subspaces of $\mathbb{R}^4$ and $\mathbb{R}^4$ is a direct sum of w_1 and w_2. **(2016)**

Sol. Given that

$W_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\}$

and $W_2 = \{(\lambda, 2\lambda, -3\lambda, 4\lambda,) : \lambda \in \mathbb{R}\}$

Therefore

$\dim w_1 = 3$ and $\dim w_2 = 1$

Now

$W_1 \cap W_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\}$

$x_2 = 2x_1$

$x_3 = -3x_1$

$x_4 = 4x_1$

So from condition

$x_1 - 4x_1 - 3x_1 = 0$

$\Rightarrow$ $-6x_1 = 0 \Rightarrow x_1 = 0$

Therefore $x_2 = 0$, $x_3 = 0$ and $x_1 = 0$

Hence $W_1 \cap W_2 = \{0\}$

Now $\dim V = \dim W_1 + \dim W_2 = 3 + 1 = 4$

therefore $\dim V = \dim W_1 + \dim W_2$

and $W_1 \cap W_2 = \{0\}$

Hence $V = W_1 \oplus W_2$

ie, w_1 and w_2 is direct sum.

Q.13. Define subspace generated by a subset of vector space. Find the equa-

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tions of the subspaces of $\mathbb{R}^3(\mathbb{R})$ generated by
(i) $S_1 = \{(1, 1, 2), (2, 3, 5), (3, 5, 8)\}$
(ii) $S_2 = \{(2, 2, 4), (3, 3, 6)\}$
Also obtain the dimension of the linear sum of the subspaces generated by S_1 and S_2. **(2012)**

Sol. The subspace generated by a subset is the smallest subspace containing the subset. It is denoted by $[S]$.

(i) let $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = \alpha(1, 1, 2) + \beta(2, 3, 5) + \nu(3, 5, 8)$$
$$x = \alpha + 2\beta + 3\nu,\ y = \alpha + 3\beta + 5\nu,\ z = 2\alpha + 5\beta + 8\nu$$
$$\alpha = 3x - 2y + k,\ \beta = y - x - 2k,\ \nu = k$$

(ii) $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = \alpha(2, 2, 4) + \beta(3, 3, 6)$$
$$x = 2\alpha + 3\beta,\ y = 2\alpha + 3\beta,\ z = 4\alpha + 6\beta$$
$$\alpha = k,\ \beta = (x - 2k)/3,\ x = y = 2z$$

dimension of $S_1 = 2$
dimension of $S_2 = 1$
dimension of linear sum = $\dim S_1 + \dim S_2 = 2 + 1 = 3$

Q.14. If W is a subspace of a finite dimensional vector space V, prove that there exiss, a subspace, W′ of V such that $V = W \oplus W'$. Is W′ unique? Justify your answer. **(2012)**

Sol. Let $\dim W = m$. Let the set $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W. Since S_1 is a linearly independent subset of V, therefore S can be extended to form a basis of V. Let the set $S = \{\alpha_1, \alpha_2, \ldots \alpha_m, \beta_1, \beta_1, \ldots \beta_\ell\}$ be a basis of V.

Let W′ be the subspace of V generated by the set $S_2 = \{\beta_1, \beta_1, \ldots \beta_\ell\}$

We shall prove that $V = W \oplus W'$

For this we need to show that

$$V = W + W' \text{ and } W \cap W' = \{0\}$$

Let α be any element of V. Then

α = a linear combination of elements of S
= a linear combination of S_1 + a linear combination of S_2
= an element of W + an element of W′

$$V = W + W'$$

Again let $\beta \in W \cap W'$. Then β can be expressed as a linear combination of S_1 and also as a linear combination of S_2.

So we have

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots\ldots + a_m\alpha_m = b_1\beta_1 + b_2\beta_2 + \ldots\ldots\ldots + b_\ell\beta_\ell$$
$$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots\ldots + a_m\alpha_m - b_1\beta_1 - b_2\beta_2 - \ldots\ldots\ldots - b_\ell\beta_\ell = 0$$
$$\Rightarrow \quad a_1 = 0, a_2 = 0, \ldots\ldots\ldots a_m = 0, b_1 = 0, b_2 = 0, \ldots\ldots\ldots b_\ell = 0$$

Since are $\alpha_1, \alpha_1, \ldots\ldots \alpha_m, \beta_1, \beta_2, \ldots\ldots \beta_\ell$ are linearly independent.

$\therefore \quad \beta = 0$ (zero vector)

Thus $W_1 \cap W_2 = \{0\}$

Hence $V = W \oplus W'$

Q.15. Define linearly dependent and linearly independent vector space. Let {a, b, c} be a basis of $\mathbb{R}^3(\mathbb{R})$ and $d = \alpha a + \beta b + \gamma c$, $\alpha, \beta, \gamma \in \mathbb{R}$. Then prove that {a - d, b - d, c - d} will be basis of $\mathbb{R}^3(\mathbb{R})$ if and only if $\alpha + \beta + \gamma \neq 1$.

(2012)

Sol. **Proof** - Vectors $a_1, a_2, \ldots\ldots a_n$ are called linearly dependent if $\exists$ scalars $\lambda_1, \lambda_2, \ldots\ldots\lambda_n \in F$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_n a_n = 0$ in which atleast one $\lambda_i \neq O_F$ otherwise vectors are called linearly independent.

Let $x, y, z \in \mathbb{R}$

$$\therefore \quad x(a-d) + y(b-d) + z(c-d) + z(c-d) = (0,0,0)$$

$$\Rightarrow \quad x[a - (\alpha a + \beta b + \gamma c)] + y[b - (\alpha a + \beta b + \gamma c)] + z[c - (\alpha a + \beta b + \gamma c)] = (0,0,0)$$

$$\Rightarrow \quad x\{(1-\alpha)a - \beta b - \gamma c\} + y\{-\alpha a + (1-\beta)b - \gamma c\} + z\{-\alpha a - \beta b + (1-\gamma x)\} = (0,0,0)$$

$$\Rightarrow \quad \{(1-\alpha)x - \alpha y - \alpha z\}a + \{-\beta + (1-\beta)x - \beta z\}b + \{-\gamma x - \gamma y + (1-\gamma)z\} = (0,0,0)$$

Since a, b, c are linearly independent

therefore $(1-\alpha)x - \alpha y - \alpha z = 0$

$$\left.\begin{array}{ll} \text{or} & (\alpha-1)x + \alpha y + \alpha z = 0 \\ \text{and} & \beta x + (\beta-1)y + \beta z = 0 \\ \text{and} & \gamma x + \gamma y + (\gamma-1)z = 0 \end{array}\right\} \quad \ldots\ldots\ldots(i)$$

{a - d, b - d, c - d} will be linearly independent iff values of x, y, z will be zero, or system (i) has only trivial solution or iff coefficient matrix of system (i) is non-singular or iff

$$\begin{vmatrix} \alpha-1 & \alpha & \alpha \\ \beta & \beta-1 & \beta \\ \gamma & \gamma & \gamma-1 \end{vmatrix} \neq 0$$

or iff

$$\begin{vmatrix} \alpha+\beta+\gamma-1 & \alpha+\beta+\gamma-1 & \alpha+\beta+\gamma-1 \\ \beta & \beta-1 & \beta \\ \gamma & \gamma & \gamma-1 \end{vmatrix} = 0$$

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or iff $$(\alpha+\beta+\gamma-1)\begin{vmatrix} 1 & 1 & 1 \\ \beta & \beta-1 & \beta \\ \gamma & \gamma & \gamma-1 \end{vmatrix} \neq 0$$

or iff $\alpha+\beta+\gamma-1 \neq 0$

Q.16. If W is a subspace of a finite dimensional vector space V over a field, prove that:

dim V/W = dim V - dim W,

and hence find the total number of cosets of W in V when W is a subspace of V over the field $\mathbb{Z}_3$ and dim V = 5, dim W = 3. (2012)

Ans. First Part: -

Go through Q.2.

Second Part: -

Given that dim V = 5, dim W = 3

Total number of cosets of subspace W in the vector space V = Total number of element in nquotient space.

$$= (\text{No. of elements in a filed } Z_3)^{\dim \frac{V}{W}}$$

$$= (3)^{\dim V - \dim W} = (3)^{5-3} = 3^2 = 9$$

Q.17. Show that the vector space $\mathbb{R}^3(\mathbb{R})$ is the linear sum of a vector space. If $S = \{a_1, a_2, \ldots\ldots a_n\}$ is a subset of a vector space V over a field F, then prove that the following statement are equivalent:

I : S is linearly independent and S generates V.

II : S is a maximal linearly independent and S generates V. (2013)

Sol. Given that $W_1 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x - y + z = 0\}$

and $W_2 = \left\{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } \frac{x}{2} = \frac{y}{1} = \frac{z}{1}\right\}$

let $(x, y, z) \in \mathbb{R}^3(\mathbb{R})$

$\Rightarrow$ $(a, b, c) + (a', b', c') = (x, y, z)$

or $a + a' = x, b + b' = y, c + c' = z$

since $a - b + c = 0$ and $a' = 2b' = 2c'$

$c - b + a' = x$

$z - c' - y + b' + a' = x$

$a' + b' - c' = x + y - z$

$2b' + b' - b' = x + y - z$

$$b = \frac{x+y-z}{2}$$

$a = x - (x + y - z) \Rightarrow a = -y + z$

$\Rightarrow \quad a' = x + y - z$

$$b = y - b' = y - \left(\frac{x+y-z}{2}\right) = \frac{-x+y+z}{2}$$

$$c' = \frac{x+y-z}{2}$$

$$c = z - c' = z - \left(\frac{x+y-z}{2}\right) = \frac{-x-y+3z}{2}$$

$$(x, y, z) = (a, b, c) + (a', b', c')$$

$$= \underbrace{\left(-y+z, \frac{-x+y+z}{2}, \frac{-x-y+3z}{2}\right)}_{\in W_1} + \underbrace{\left(x+y-z, \frac{-x+y+z}{2}, \frac{-x-y+3z}{2}\right)}_{\in W_2}$$

Let $(a, b, c) \in W_1 \cap W_2$

$$a - b + c = 0 \qquad \frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$\Rightarrow \quad a - \frac{a}{2} + \frac{a}{2} = 0 \qquad a = 2b = 2c$

$\Rightarrow \quad a = 0 \qquad \Rightarrow b = 0$ and $c = 0$

$(a, b, c) = (0, 0, 0)$

$\Rightarrow \quad W_1 \cap W_2 = \{0\}$

Q.18. Define maximal linearly independent subset of a vector space. If S = $\{a_1, a_2, \ldots\ldots a_n\}$ is a subset of a vector space V over a field F. Then prove that the following statement are equivalent:

I : S is linearly independent and S generates V.

II : S is a maximal linearly independent subset of V. **(2013)**

Ans. Maximal linearly independent subset of a vector space: -

A subset S of a vector space V over F is called maximal linearly independent subset of the vector space iff it satisfy the following conditions -

(1) S is linearly independent

(2) A proper super set of S is linearly independent.

I $\Rightarrow$ II

Let $S = \{a_1, a_2, \ldots\ldots a_n\} \subseteq V$ be a basis of V i.e. S is linearly independent and S generates V.

$\Rightarrow \quad V = [S]$

Let $\quad x \in V$

$\therefore \quad x = \alpha_1 a_1 + \alpha_2 a_2 \ldots\ldots\ldots + \alpha_n a_n \quad : \quad \alpha_i \in F \quad \ldots\ldots\ldots(i)$

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If possible let x has another expression i.e.

$$x = \beta_1 a_1 + \beta_2 a_2 + \ldots\ldots\ldots + \beta_n a_n \quad : \quad \beta_i \in F \quad \ldots\ldots\ldots(ii)$$

From eq.(i) and (ii)

$$\alpha_1 a_1 + \alpha_2 a_2 \ldots\ldots\ldots + \alpha_n a_n = \beta_1 a_1 + \beta_2 a_2 + \ldots\ldots\ldots + \beta_n a_n$$

$\Rightarrow$ $(\alpha_1 - \beta_1)a_1 + (\alpha_2 - \beta_2)a_2 + \ldots\ldots\ldots (\alpha_n - \beta_n)a_n = 0$

$\therefore$ $a_1, a_2, \ldots\ldots\ldots a_n$ are linearly independent.

$\Rightarrow$ $\alpha_1 - \beta_1 = o_F \ldots\ldots\ldots\ldots\ldots, \alpha_n - \beta_n = o_F$

$\Rightarrow$ $\alpha_1 = \beta_1 \ldots\ldots\ldots\ldots \alpha_n = \beta_n$

Therefore expression is unique.

II $\Rightarrow$ I : -

Let each $x \in V$ can be uniquely expressed as a linear combination of S.

Claim: - S is linearly independent.

Let $\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots\ldots\ldots + \lambda_n a_n = o_F a_1 + o_F a_2 + \ldots\ldots o_F a_n$

$\Rightarrow$ $\lambda_1 = o_F$ ($\because$ Expression is unique)

.

.

$\lambda_n = o_F$

S is linearly independent.

Therefore S is a basis of V i.e.

S is linearly independent and S generates V.

Q.19. Let $S_1 = \{a_1, a_2, \ldots\ldots a_n\}$ be lilnearly independent subset of a vector space V over a field F and $a \in V$ but $a \notin L_{S_1}$.

Then show that the set $S_2 = \{a, a_1 + \lambda_a, a_2 + \lambda_a, \ldots\ldots, a_n + \lambda_a\}$ where $\lambda \in F$ will be linearly independent. **(2013)**

Sol. $\because$ $\{a_1, a_2, \ldots\ldots\ldots, a_n\}$ is linearly independent and $a \notin L_S$. Therefore $\{a_1, a_2, \ldots\ldots\ldots, a_n, a\}$ is also linear independent.

Let

$$\mu_1 a + \mu_2(a_1 + \lambda a) + \mu_3(a_2 + \lambda a) + \ldots\ldots\ldots + \mu_n(a_n + \lambda a) = o_F$$

$\Rightarrow$ $\mu_1 a + \mu_2 a_1 + \mu_3 a_2 + \ldots\ldots\ldots + \mu_n a_n + \mu_2\lambda_a + \mu_3\lambda_a + \ldots\ldots + \mu_n\lambda_a = 0$

$\Rightarrow$ $\mu_1 a + \mu_2 a_1 + \mu_3 a_2 + \ldots\ldots\ldots + \mu_n a_n + \lambda a(\mu_2 + \mu_3 + \ldots\ldots\ldots + \mu_n) = 0$

$\Rightarrow$ $[\lambda(\mu_2 + \mu_3 + \ldots\ldots\ldots + \mu_n) + \mu_1]a + \mu_2 a_1 + \ldots\ldots\ldots + \mu_n a_n = 0$

$\Rightarrow$ $\mu_2 = 0 = \mu_3 = 0 \ldots\ldots\ldots = \mu_n = 0$

and $\lambda(\mu_2 + \mu_3 + \ldots\ldots\ldots + \mu_n) + \mu_1 = 0$

{as $\{a, a_1, a_2, \ldots\ldots a_n\}$ are linearly independent}

$\Rightarrow$ $\lambda.0 + \mu_1 = 0 \quad \Rightarrow \quad \mu_1 = 0$

$\Rightarrow$ $\mu_1 = \mu_2 = 0 = \mu_3 = 0 \ldots\ldots\ldots = \mu_n$

Hence $S_2 = \{a, a_1 + \lambda a, a_2 + \lambda a, \ldots\ldots\ldots, a_n + \lambda a\}$ are linearly independent.

Q.20. Let V be a finitely generated vectors space over a field F. Prove that V contains a basis over F. **(2017)**

Sol. Let V be a finitely generated vector space over field F then ∃ finite subset

$$S = \{a_1, a_2, \ldots\ldots a_n\} \subseteq V \text{ such that}$$
$$V = [S] \text{ or } V = L(S)$$

∵ S is finite subset of V without loss of generately we can assume that non of the vector in S is zero vectors.

Now, if S is linearly independent then S will be a basis of V and then we are done. Assume that, if S is linearly dependent then ∃ a maximal linearly independent subset of S′of S.

i.e. $S' = \{a_1, a_2, \ldots\ldots a_r\}$ is maximal linearly independent subset of S containing r vectors ($r < n$) of S and any $a_i \in S$ but $a_i \notin S'$.

Now, since $\{a_i, a_1, \ldots\ldots a_r\}$ are linearly independent and $\{a_1, \ldots\ldots, a_r\}$ is maximal linearly independent then ∃ scalar $\alpha_i, \alpha_1, \ldots\ldots, \alpha_r \in F$ such that

$$\alpha_i a_i + \alpha_1 a_1 + \ldots\ldots\ldots + \alpha_r a_r = 0 \qquad \ldots\ldots\ldots(i)$$

where at least one α's $\neq 0$

$\Rightarrow$ $a_i \neq O_F$

otherwise if $\alpha_i = O_F$ then by eq.(i)

$$\alpha_1 a_1, \ldots\ldots\ldots, \alpha_r a_r = 0$$

where at least one α's $\neq 0$.

then S′ is linearly independent which is a contradiction to $a_i \neq 0$.

and hence α_i will be linear combination of $\{a_1, \ldots\ldots\ldots, a_r\}$ i.e. element of S′.

$\Rightarrow$ $a_i = \alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots\ldots + \alpha_r a_r$.

$\Rightarrow$ $V = [S']$

Hence S′ is a basis of V.

Q.21. Prove that $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \, x_2 + x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^5$. Determine $\dim_{\mathbb{R}}^{W}$ and a subspace U of subspace U of $\mathbb{R}^5$ such that $\mathbb{R}^5$ $W \oplus U$. **(2017)**

Sol. Given that

$$W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \, x_2 + x_3 + x_4 = 0\}$$

Let $A, B \in W$

i.e. $A = (a_1, a_2, a_3, a_4, a_5)$ and $B = (b_1, b_2, b_3, b_4, b_5) \in W$ and $\alpha, \beta \in \mathbb{R}$

∵ $A \in W$ and $B \in W$

then $a_2 + a_3 + a_4 = 0$ and $b_2 + b_3 + b_4 = 0$

$\Rightarrow$ $\alpha(a_2 + a_3 + a_4) = 0$ and $\beta(b_2 + b_3 + b_4) = 0$

$\Rightarrow$ $(\alpha a_2 + \beta b_2) + (\alpha(a_3) + \beta(b_3)) + (\alpha(a_4) + \beta(b_4)) = 0$

$\Rightarrow$ $(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3, \alpha a_4 + \beta b_4, \alpha a_5 + \beta b_5) \in W$

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$\Rightarrow \quad \alpha A + \beta B \in W$

i.e. $\forall\ A, B \in W \quad \Rightarrow \quad \alpha A + \beta B \in W$

Hence, W is a subspace of $\mathbb{R}^5$.

Clearly,

Let $x_2 = -x_3 - x_4$

Take $x_2 = 0, x_3 = 1, x_4 = -1$

$x_2 = 1, x_3 = -1, x_4 = 1$

$x_2 = -1, x_3 = 1, x_4 = 0$

$x_2 = -1, x_3 = 0, x_4 = 1$

$x_2 = 1, x_3 = 0, x_4 = -1$

Let $S = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0), (-1, 0, 1), (1, 0, -1)\}$

which is linearly dependent because $(1, 0, -1) = -(-1, 0, 1)$

Let $S_1 = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0), (-1, 0, 1)\}$

No one is scalar multiple of other .

Hence, dim W = 4

Given, $\mathbb{R}^5 = W \oplus V$

$\Rightarrow \quad 5 = 4 + \dim U \quad \Rightarrow \quad \dim U = 1$

Thus, the subspace U is generated by (1, 0, 0, 0, 0).

Q.22. Prove that union of two subspaces of a vector space is a subspace of the vector space if and only of one of them is contained in the other. Give an example to show that the union of two subspaces of a vector space may not be a subspace of the vector space. (2018)

Sol. Let w_1 and w_2 be subspace of vector space V(F) and let $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$, then we have to show that $w_1 \cup w_2$ is subspace.

$\because \quad w_1 \cup w_2 = w_1 \quad \{\text{If } w_2 \subseteq w_1\}$

And $\quad w_1 \cup w_2 = w_2 \quad \{\text{If } w_1 \subseteq w_2\}$

Thus $w_1 \cup w_2$ is a subspace of vector space V(F).

Conversely -

Let $w_1 \cup w_2$ is a subspace of vector space V(F) then we have to show that $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$.

Let, if possible $w_1 \nsubseteq w_2$ or $w_2 \nsubseteq w_1$.

let $a \in w_1 - w_2$ and $b \in w_2 - w_1$

$\Rightarrow\ a \in w_1$ and $a \notin w_2$ and $b \in w_2$ and $b \notin w_1$

If $\quad a \in w_1 \quad \Rightarrow \quad a \in w_1 \cup w_2$

And $\quad b \in w_2 \quad \Rightarrow \quad b \in w_1 \cup w_2$

$a, b \in w_1 \cup w_2 \Rightarrow a+b \in w_1 \cup w_2$ $\{\because w_1 \cup w_2$ is subspace

$\Rightarrow$ $a, b \in w_1 \cup w_2 \Rightarrow a+b \in w_1$ or $a+b \in w_2$

Case (I) When $a+b \in w_1$ -

$\because$ $a+b \in w_1$, $a \in w_1$ and $b \notin w_1$

$\Rightarrow$ $a \in w_2$, $-a \in w_2$, and $b \notin w_1$

$\Rightarrow$ $-a+(a+b) \in w_1$ and $b \notin w_1$

$\Rightarrow$ $(-a+a)+b \in w_1$ and $b \notin w_1$ (associativity)

$\Rightarrow$ $0+b \in w_1$ and $b \notin w_1$

$\Rightarrow$ $b \in w_1$ and $b \notin w_1$

which is self contraduction or which is not possible.

Case (II) When $a+b \in w_2$ -

$\because$ $a+b \in w_2$, $b \in w_2$ and $a \notin w_2$

$\Rightarrow$ $a+b \in w_2$, $-b \in w_2$ and $a \notin w_2$

$\Rightarrow$ $-b+(a+b) \in w_2$ and $a \notin w_2$

$\Rightarrow$ $a+(-b+b) \in w_2$ and $a \notin w_2$

$\Rightarrow$ $a+0 \in w_2$ and $a \notin w_2$

$\Rightarrow$ $a \in w_2$ and $a \notin w_2$

which is self contraduction or which is not possible which is contracduction to on assumption $w_1 \not\subseteq w_2$ and $w_2 \not\subseteq w_1$ is wrong.

Hence, $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$.

Thus union of two subspace of vector space V(F) is subspace iff one contain in other.

Example -

(1) Let $w_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$ and $w_2 = \{(0, y, 0) : y \in \mathbb{R}\}$ be subspace of $\mathbb{R}^3(\mathbb{R})$.

$\because$ $(1, 0, 0) \in w_1 \Rightarrow (1, 0, 0) \in w_1 \cup w_2$

And $(0, 1, 0) \in w_2 \Rightarrow (0, 1, 0) \in w_1 \cup w_2$

$(1, 0, 0)+(0, 1, 0) \Rightarrow (1, 1, 0) \notin w_1 \cup w_2$

Thus, union is not subspace.

(2) Let, $V_4(\mathbb{Z}_2)$ be vector space.

Let $w_1 = \{e, a\}$ and $w_2 = \{e, b\}$

$\because$ w_1 and w_2 is a subspace of $V_4(\mathbb{Z}_2)$.

but $w_1 \cup w_2 = \{e, a, b\}$ is not a subspace of $V_4(\mathbb{Z}_2)$.

Thus union is not subspace.

Q.23. Define a subspace of a vector space. If W is a subspace of $\mathbb{R}^3$ generated by $\{(1, 1, -2), (0, -1, 1), (3, -2, -1)\}$ then find the dimension of another sub-

space W′ of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W \oplus W'$. **(2018)**

Subspace: -

Let (V, +, .) be a vector space over filed (F, ⊕, ⊙). A non empty subset W of V(F) is called subspace of vector space V(F) if W also forms V.S. over F.

Example -

(1) Q is subspace of $\mathbb{R}$ (Q).

(2) $\mathbb{R}$ is a subspace of C($\mathbb{R}$).

Let $$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$ $$A = \begin{pmatrix} 1 & 1 & -2 \\ 3 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$ $$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & -1 & 1 \end{pmatrix}$$

$R_2 \to -\frac{1}{5}R_2$ $$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$R_3 \to R_3 + R_2$ $$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is an Echelon form.

Dimension of W = 2.

Hence $\mathbb{R}^3 = W \oplus W'$

$\Rightarrow$ $\dim \mathbb{R}^3 = \dim W + \dim W'$

$\Rightarrow$ $3 = 2 + \dim W'$

$\Rightarrow$ $\dim W' = 3 - 2 = 1$

So dimension of W′ is 1.

Q.24. Let W be a subspace of a vector space V over a filed F. If V is finite dimensional, then prove that **(2018)**

dim(V/W) = dimV - dimW.

Sol. Since W be subspace of vector space V(F). Let $S_1 = (a_1, a_2, ..., a_m)$ be the basis of W then

$\dim W = m$ .....(1)

$\because$ S_1 is linearly independent subset of V $\{S_1 \subset W \subseteq V \Rightarrow S_1 \subseteq V\}$ then by extension theorem S_1 is either basis of V or can be extended to form basis of V.

Let $S_2 = \{a_1, a_2, \ldots\ldots, a_m, b_1, b_2, \ldots\ldots b_m\}$ be basis of V then

$\dim V = m.n$ .....(2)

Now Let $S = \{W + b_1, W + b_2, \ldots\ldots, W + b_n\} \subseteq V/W$

Claim -

S is basis of V/W -

$\because$ $\{W + b_1, W + b_2, \ldots\ldots, W + b_n\} \in V/W$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

Such that

$$\alpha_1(W + b_1) + \alpha_2(W + b_2) + \ldots\ldots + \alpha_n(W + b_n) = W \quad \ldots\ldots(3)$$

$\Rightarrow$ $(W + \alpha_1 b_1) + (W + \alpha_2 b_2) + \ldots\ldots + (W + \alpha_n b_n) = W$

{from scalar multiplication of a cosets}

$\Rightarrow$ $W + \{\alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n\} = W$

{because of property of a cosets}

$\Rightarrow$ $\alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n \in W$

{because property $W + a \Leftrightarrow a \in W = W$ }

$\Rightarrow$ $\alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n \in W$ and S_1 is basis of W.

$\Rightarrow$ $\alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n \in [S_1]$

$\Rightarrow$ $\alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n = \beta_1 a_1 + \beta_2 a_2 + \ldots\ldots \beta_m a_m$

$\Rightarrow$ $(-\beta_1)a_1 + (-\beta_2)a_2 + \ldots\ldots + (-\beta_m)a_m + \alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n = 0$

$\Rightarrow$ $-\beta_1 = -\beta_2 = \ldots\ldots\ldots = -\beta_m = O_F$ and $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = O_F$

{because $a_1, a_2, \ldots\ldots a_m, b_1, b_2, \ldots\ldots b_n$ being basis elements are linearly independent}

$\Rightarrow$ $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = O_F$

Hence S is linearly independent.

(2) V/W = [S]

For this

Let $W + x \in V/W \Rightarrow x \in V$

$\because$ $x \in V$ and S_2 is basis of V.

$\Rightarrow$ $x = \theta_1 a_1 + \theta_2 a_2 + \ldots\ldots\ldots + \theta_m a_m + \phi_1 b_1 + \phi_2 b_2 + \ldots\ldots\ldots + \phi_n b_n$

{where θ's and ϕ's ae unique elements}

$\because$ $W + x = W + \underbrace{(\theta_1 a_1 + \theta_2 a_2 + \ldots\ldots + \theta_m a_m}_{w \in W} + \phi_1 b_1 + \phi_2 b_2 + \ldots\ldots + \phi_n b_n)$

$\Rightarrow$ $W + x = (W + w) + (W + (\phi_1 b_1 + \phi_2 b_2 + \ldots\ldots + \phi_n b_n))$

$\Rightarrow$ $W + x = W + (\phi_1 b_1 + \phi_2 b_2 + \ldots\ldots + \phi_n b_n)$

$\Rightarrow$ $W + x = (W + \phi_1 b_1) + (W + \phi_2 b_2) + \ldots\ldots\ldots + (W + \phi_n b_n)$

$W + x = \phi_1(W + b_1) + \phi_2(W + b_2) + \ldots\ldots\ldots + \phi_n(W + b_n)$

{from defination of scalar multiplication of coset}

$\Rightarrow$ $V/W = [S]$

From (1) and (2)

S is a basis of V/W

then dim V/W = n

$\Rightarrow$ dim V/W = (m + n) - n

$\Rightarrow$ dim V/W = dim V - dim W

Q.25. Show that the vector space $\mathbb{R}^3$ over $\mathbb{R}$ is the sum of its sub-spaces

$$w_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0\} \text{ and } w_2 = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x}{-1} = \frac{y}{1} = \frac{z}{1}\right\}.$$

Is this sum a direct sum? Justify your answer. **(2019)**

Sol. Given that $w_1 = \{(x, y, z)\} \in \mathbb{R}^3 : x + 2y + z = 0$ and

$$w_2 = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x}{-1} = \frac{y}{1} = \frac{z}{1}\right\}$$

Let $(x, y, z) \in \mathbb{R}^3(\mathbb{R})$

$\Rightarrow$ $(a, b, c) + (a', b', c') = (x, y, z)$

or $a + a' = x$, $b + b' = y$, $c + c' = z$

since $a + 2b + c = 0$ and $a' = -b' = -c'$

Then $-2b - c + a' = x$

$\Rightarrow$ $-2(y - b') - (z - c') + a' = x$

$\Rightarrow$ $-2y + 2b' - z + c' + a' = x$

$\Rightarrow$ $a' + 2b' + c' = x + 2y + z$

$\Rightarrow$ $-b + 2b' + b' = x + 2y + z$

$\Rightarrow$ $b' = \dfrac{(x + 2y + z)}{2}$

and $a' - 2a' - a' = x + 2y + z$

$$a' = \frac{(x + 2y + z)}{-2} \text{ and } c' = \frac{(x + 2y + z)}{2}$$

Now $a = x - a'$, $b = y - b'$, $c = z - c'$

$$a = x - \frac{(x + 2y + z)}{-2}, \; b = y - \frac{(x + 2y + z)}{2}, \; c = z - \frac{(x + 2y + z)}{2}$$

$$a = \frac{+3x + 2y + z}{2}, \; b = \frac{-(x + z)}{2}, \; c = \frac{-(x + 2y - z)}{2}$$

$(x, y, z) = (a, b, c) + (a', b', c')$

$$= \left(\frac{(3x+2y+z)}{2}, \frac{-(x+z)}{2}, \frac{-(x+2y-z)}{2}\right) \in w_1$$

$$+ \left(\frac{-(x+2y+z)}{2}, \frac{(x+2y+z)}{2}, \frac{(x+2y+z}{2}\right) \in w_2$$

Let $(a, b, c) \in w_1 \cap w_2$

$$a + 2b + c = 0 \text{ and } \frac{a}{-1} = \frac{b}{1} = \frac{c}{1}$$

Then $a - 2a - a = 0$

$\Rightarrow$ $a = 0$

$a = -b = -c \Rightarrow b = 0$ and $c = 0$

$w_1 \cap w_2 = \{0\}$

yes sum is the direct sum.

Q.26. Determine the dimension of subspace of W of $\mathbb{R}^4$ generated by S: (1, 1, 2, 0), (1, -1, 2, 0), (0, 1, 2, 1). **(2019)**

Sol.
$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_1$
$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$R_3 \to 2R_3 + R_2$
$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix}$$

Rank of this matrix is 3. So the dim. of space generated by vetor is 3.

Q.27. For the subspace $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0, x_3 + x_4 = 0\}$ of the vector space $\mathbb{R}^4$ over $\mathbb{R}$, give a basis of the quotient space $\mathbb{R}^4/W$. **(2019)**

Sol. $$W = \left\{(x_1, x_2, x_3, x_4) : \begin{matrix} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 \end{matrix}\right\}$$

So $x_1 = -x_2$

$x_3 = -x_4$

Therefore basis of W be $\left\{\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}$

Now this basis of can be extended to besis of $\mathbb{R}^4$.

and $\left\{\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ Will form a basis for $\mathbb{R}^4$.

So, the basis of $\frac{V}{W}$ is given by $\{w + \beta\}$ Where β belongs to basis of $\mathbb{R}^4$ not is basis of W.

$\{W + \{1,0,0,0\}, W + \{0,0,0,1\}\}$ is form basis for $\frac{V}{W}$.

$\{(x_1, x_2, x_3, x_4) + (1,0,0,0), \{x_1, x_2, x_3, x_4\} + (0,0,0,0)(x_1, x_2, x_3, x_4) \in w.$

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Unit II

Q.1. Let M_3 denote the vector space of 3×3 matrices over the field of real numbers. A map T: $M_3 \to M_3$ is defined by $T(A) = A - A^t \; \forall \; A \in M_3$. Then prove that T is linear transformation. Find a basis of Ker T and hence find the nullity of T and rank of T. (2014)

Ans. Let A and B $\in M_3$ and $\alpha, \beta \in F$

$$T(\alpha A + \beta B) = (\alpha A + \beta B) - (\alpha A + \beta B)^t$$
$$= \alpha A + \beta B - \alpha A^t - \beta B^t$$
$$= \alpha(A - A^t) + \beta(B - B^t)$$
$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$$

Hence T is linear transformation.

$$\text{Ker } T = \{A \in M_3 : T(A) = 0\}$$
$$= \{A \in M_3 : A - A^t = 0\}$$
$$= \{A \in M_3 : A^t = A\}$$
$$= \text{Set of all symmetric matrix of } M_3.$$

Nullity of T = dim ker T

$$= \frac{1}{2} \times 3 \times (3+1) = 6$$

Q.2. Prove or disprove -

(i) If $T_1: \mathbb{R}^2 \to \mathbb{R}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}$ are two linear transformations such that Ker T_1 = Ker T_2 and $I_m T_1 = I_m T_2$, then $T_1 = T_2$. (2014)

Sol. If $T_1 : \mathbb{R}^2 \to \mathbb{R}$ and $T_2 : \mathbb{R}^2 \to R$ are two linear transformation such that ker$T1$ = kerT_2 and $I_m T_1 = I_m T_2$.

Let $T_1(x, y) = x - y$ and $T_2(x, y) = y - x$

Then $\text{Ker } T_1 = \left\{(x, y) \in \mathbb{R}^2 \middle|_{x=y \text{ or } x=y=0}^{x-y=0}\right\}$

And $\text{Ker } T_2 = \left\{(x, y) \in \mathbb{R}^2 \middle|_{y=x \text{ o } x=y=0}^{y-x=0}\right\}$

$$\text{Ker } T_1 = \text{Ker} T_2$$

for all a $\in$ R, then there exist (a, ( ) $\in R^2$, such that $T_1(a, 0) = a - 0 = a$

Similarly for all a $\in$ R

$$T_2(0, a) = a - 0 = a$$

Here T & T2 are onto on R.

$$I_m T_1 = I_m T = R$$

So, but $T_1 \neq T_2$

Hence this is disprove.

Q.3. Prove or disapprove -
If $T: V_1 \to V_2$ is linear transformation such that $\rho(T) = 2$ and $\dim V_2 = 5$, then nullity of T^t will be 3. **(2014)**

Sol. Given that $T : V_1 \to V_2$ is a linear transformation such that

$\rho(T) = 2$ and $\dim V_2 = 5$

By rank Nullity theorem. We have

$\text{rank } T + \text{nullity of } (T) = \phi\text{im}V_1$

Let $T^t : V_2 \to V_1$ be linear transformation

By Rank-Nullity Theorem

For T^t we have

$\text{rank}(T^t) + \text{Nullity}(T^t) = \dim V_2$

$2 + \text{Nullity}(T^t) = 5$

$\text{Nullity of } (T^t) = 5 - 2 = 3$

Hence it is true.

Q.4. If W is a subspace of $\mathbb{R}^4(\mathbb{R})$ spanned by subset $S = \{(2, 3, -2, 1), (4, 2, -1, 3), (2, -5, 4, 3), (4, 10, -7, 1)\}$, then dimension of the annihilator of W will be 2. **(2014)**

Sol. $S = \{(2, 3, -2, 1), (4, 2, -1, 3), (2, -5, 4, 3), (4, 10, -7, 1)\}$

$$\begin{bmatrix} 2 & 3 & -2 & 1 \\ 4 & 2 & -1 & 3 \\ 2 & -5 & 4 & 3 \\ 4 & 10 & -7 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -4 & 3 & 1 \\ 0 & -8 & 6 & 2 \\ 0 & -4 & -3 & -1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_3 - 2R_2 \\ R_4 \to R_4 + R_2 \end{matrix} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim W = 2$

We know that $\dim W + \dim W^0 = \dim V$

$2 + \dim W^0 = 4$

$\Rightarrow \quad \dim W^0 = 2$

Hence Proved.

Q.5. Let M_2 denote the vector space of 2×2 matrices over the field of real numbers. A linear transformation $T : M_2 \to M_2$ is defined by $T(A) = A.B - B.A$, $\forall A \in M_2$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, then find $I_m T$, Ker T and trace of T. (2014)

Ans. Standard basis of M_2 is

$S = \{e_{11}, e_{12}, e_{21}, e_{22}\}$

$$T(e_{11}) = e_{11}B - Be_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T(e_{11}) = 0.e_{11} + 1.e_{12} + (-1)e_{21} + 0.e_{22}$$

$$T(e_{12}) = e_{12}B - Be_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= (1).e_{11} + (1).e_{12} + (0).e_{21} + (-1).e_{22}$$

$$T(e_{21}) = e_{21}B - Be_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = (-1)e_{11} + (0)e_{12} + (-1).e_{21} + (1)e_{22}$$

Similarly $T(e_{22}) = e_{22}B - Be_{22}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (0)e_{11} + (-1)e_{12} + (1)e_{21} + (0)e_{22}$$

$$\text{matrix } X(T)_{S,S} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$tr(T) = 0 + 1 + (-1) + 0 = 0$$

$$I_m T = \left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$$

$\because$ $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to$ Reduced in Echlon form

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echlon form

$$\text{Basis of } I_m(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$\rho(T) = 2$

$\dim \ker T = \dim M_2 - \rho(T)$ ($\because$ Using R.N.T) .

$= 4 - 2 = 2$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Ker } T$

$$\Rightarrow \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} B - B \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a+b & a+3b \\ 2c+d & c+3d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad b = c, d = b + a, a = c$$

$$\because \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & b+a \end{bmatrix} = \begin{bmatrix} a+0.b & 0.a+1.b \\ 0.a+1.b & 1.a+1.b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} a + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} b$$

$$\text{Ker } T = \left\{ \begin{bmatrix} a & b \\ c & a+b \end{bmatrix} : a, b \in R \right\}$$

$$\text{Basis of Ker } T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\text{Im}\,T = \left\{ x\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\text{Im}\,T = \left\{ \begin{bmatrix} x & x+y \\ -y & -x \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Q.6. Find dual basis to the ordered basis $X = \{(1,-2,3), (1,-1,1), (2,-4,7)\}$ of $\mathbb{R}^3(\mathbb{R})$. (2014

Sol. Given that $X = \{(1,-2,3), (1,-1,1), (2,-4,7)\}$

Let $X^* = \{f_1, f_2, f_3\}$ be the dual basis of X.

Let $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = \alpha(1,-2,3) + \beta(1,-1,1) + \gamma(2,-4,7) \quad \text{.....(i)}$$

$$\Rightarrow \quad \alpha + \beta + 2\gamma = x \quad \text{.....(ii)}$$

$$-2\alpha - \beta - 4\gamma = y \quad \text{.....(iii)}$$

$$3\alpha - \beta + 7\gamma = z \quad \text{.....(iv)}$$

Solving eq.(ii), (iii) and (iv) we get

$$\alpha = -3x - 5y - 2z$$

$$\beta = 2x + y$$

$$\gamma = x + 2y + z$$

Putting the value α, β, γ in eq.(i) we get

$$f_1(x,y,z) = \alpha \odot f_1(1,-2,3) \oplus \beta \odot f_1(1,-1,1) \oplus \gamma \odot f_1(2,-4,7)$$

$$f_1(x, y, z) = \alpha = -3x - 5y - 2z$$

Similarly $f_2(x, y, z) = \beta = 2x + y$

$$f_3(x, y, z) = \gamma = x + 2y + z$$

Q.7. Prove that a non-zero linear functional on a vector space V is injecti if and only if dim V = 1. (201

Sol. Let $f(\neq f_0) : V \to F$ be a linear function and injective.

$\Leftrightarrow$ $\text{Ker}\, f = \{0\}$

Using rank-nullity theorem for f

$\Leftrightarrow$ $\dim \text{Im}\, f + \dim \text{Ker}\, f = \dim V$

$\Leftrightarrow$ $\dim \text{Im}\, f = \dim V$ .....(i)

Since image f is subspace of 'f'

$\therefore$ $\dim \text{Im}\, f \leq \dim f$ ($\because$ dim of field is always 1)

≤ 1

$\therefore$ $\dim \text{Im}\, f = 0_F$ or 1

If $\dim \text{Im}\, f = 0_F$

$\Rightarrow$ $f = f_0$ which is not possible.

$\therefore$ $\dim \text{Im}\, f = 1$

Q.8. Let V and W be vectort spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Then prove that

rank (T) + nullity (T) = dim V **(2015)**

Sol. We have prove that

rank (T) + nullity (T) = dim V

Proof - Let dim ker T = m

Let $S = \{a_1, a_2, \ldots\ldots a_m\}$ be the basis of ker T.

Since S is ℓ.I. subset of vector space V. Therefore it can be extended to form a basis of V.S. V.

Let $S_1 = \{a_1, a_2, \ldots\ldots, a_m, b_1, b_2, \ldots\ldots b_n\}$ be basis of V.

dim V = m + n

Since S_1 generates V. Therefore $T(S_1)$ generates ImT.

$T(S_1) = \{T(a_1), T(a_2) \ldots\ldots T(a_m), T(b_1), T(b_2) \ldots\ldots, T(b_n)\}$

$T(S_1) = \{O_w, O_w, \ldots\ldots O_w, T(b_1)\, T(b_2) \ldots\ldots T(b_n)\}$

Let $S_2 = \{T(b_1), T(b_2) \ldots\ldots T(b_n)\}$ generates $I_m T$.

Claim - S_2 is basis of $I_m T$ -

For this it is sufficient to prove that S_2 is ℓ.I.

Let $\alpha_1.T(b_1) +_w \alpha_2 T(b_2) +_w \ldots\ldots\ldots\ldots +_w \alpha_n T(b_n) = O_w$

$\Rightarrow$ $T(a_1 b_1 +_v a_2 b_2 +_v \ldots\ldots\ldots +_v \alpha_n b_n) = O_w$ $\{\because$ T is ℓ.T.$\}$

$\Rightarrow$ $\alpha_1 b_1 +_v \alpha_2 b_2 +_v \ldots\ldots\ldots +_v \alpha_n b_n \in \ker T$

$\Rightarrow$ $\alpha_1 b_1 +_v \alpha_2 b_2 +_v \ldots\ldots\ldots +_v \alpha_n b_n = \beta_1 a_1 +_v \beta_2 a_2 + v \ldots\ldots\ldots + v\, \beta_m a_m$

$\Rightarrow$ $(-\beta_1)a_1 + v(-\beta_2)a_2 + v \ldots\ldots\ldots + v\,(-\beta_m)am + \alpha_1 b_1 + v\,\alpha_2 b_2 + v \ldots\ldots + v\alpha_n b_n$

$= 0 = 0v$

$\Rightarrow$ $-\beta_1 = -\beta_2 = \ldots\ldots\ldots\ldots = -\beta_m = \alpha_1 = \alpha_2 = \ldots\ldots\ldots \alpha_n = o_F$

Therefore S_2 is ℓ.I. and S_2 is basis of $I_m T$.

$\therefore$ $\dim I_m T = n = m + n - m = \dim V - \dim \ker T$

$\Rightarrow$ $\dim V = \dim I_m T + \dim \ker T$

Q.9. Prove that any two finite dimensional vector spaces V and W defined over the same field are isomorphic if and only if they have the same dimension. **(2015)**

Sol. We have to show that

$V \approx W \quad \Leftrightarrow \quad \dim V = \dim W$

Proof - Necessary part -

Let $V \approx W$

$\Rightarrow$ $\exists$ a linear transformation

$T : V \to W$ which is one-one and onto

$\Rightarrow$ $\ker T = \{O_v\}$ and $I_m T = W$

Using Rank Nullity Theorem

$\dim \ker T + \dim I_m T = \dim V$

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$0 + \dim I_m T = \dim V$

$\dim W = \dim V$

Sufficient part: -

Let $\dim V = \dim W = n$ (finite)

$\Rightarrow$ $X = \{x_1, x_2, \ldots\ldots x_n\}$

$Y = \{y_1, y_2, \ldots\ldots y_n\}$ are the ordered basis of V and W respectively

define a map $T : V \to W$ as

$$T(\alpha_1 x_1 + v\, \alpha_2 x_2 +_v \ldots\ldots +_v \alpha_n x_n) = \alpha_1 y_1 +_w \alpha_2 y_2 +_w + \ldots\ldots +_w \alpha_n y_n$$

(i) T is linear transformation -

Let $a = \alpha_1 x_1 +_v \alpha_2 x_2 +_v \ldots\ldots +_v \alpha_n x_n \quad \in V$

$b = \beta_1 x_1 +_v \beta_2 x_2 + v \ldots\ldots +_v \beta_n x_n \quad \in V$

And $\lambda, \mu \in F$

$$T(\lambda a +_v \mu\beta) = T(\lambda \odot \alpha_1 \oplus \mu \odot \beta_1) x_1 +_v (\lambda \odot x_2 \oplus \mu \odot \beta_2) x_2 +_v$$

$$\ldots\ldots +_v (\lambda \odot \alpha_n \oplus \mu \odot \beta_n) x_n$$

$$= \{(\lambda \odot \alpha_1) \oplus (\mu \odot \beta_1)\} y_1 +_w \{(\lambda \odot \alpha_2) \oplus (\mu \odot \beta_2)\} y_2 +_w$$

$$\ldots\ldots +_w (\lambda \odot x_n) \oplus (\mu \odot \beta_n) y_n$$

$$= \lambda(\alpha_1 y_1 +_w \alpha_2 y_2 +_w \ldots\ldots + \alpha_n y_n) +_w \mu(\beta_1 y_1 +_w \beta_2 y_2 +_w \beta_n y_n)$$

$$= \lambda T(a) +_w \mu(b)$$

$\Rightarrow$ T is linear transformation.

(ii) T is one-one -

Let $T(a) = T(b)$

$\alpha_1 y_1 +_w \alpha_2 y_2 +_w \alpha_n y_n = \beta_1 y_1 +_w \beta_2 y_2 + w \ldots\ldots +_w \beta_n y_n$.

$\Rightarrow$ $(\alpha_1 - \beta_1) y_1 +_w (\alpha_2 - \beta_2) y_2 +_w \ldots\ldots +_w (\alpha_n - \beta_n) y_n = 0$

$\Rightarrow$ $(\alpha_1 - \beta_1) = O_F, (\alpha_2 - \beta_2) = O_F = \ldots\ldots = (\alpha_n - \beta_n) = O_F$

$\Rightarrow$ $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots\ldots, \alpha_n = \beta_n$

$\Rightarrow$ $a = b$

(iii) T is onto

$\because$ $T : V \to W$ is one-one.

$\Rightarrow$ $\ker T = \{O_V\}$

$\Rightarrow$ $\dim \ker T = 0$

By Rank-Nullity Theorem

$\dim I_m T = \dim V = \dim W \qquad (\because \dim V = \dim W)$

$\Rightarrow$ $I_m T = W$

T is onto

$\therefore$ T is an isomorpism.

Therefore $V \approx W$

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Q.10. Define ranges space, null space of a linear transformation T : V → W. Show that range of T is a subspace of W and null space of T is a subspace of V. **(2015)**

Ans. Range space of a linear transformation: -

Let $T : V \to W$ be a linear transformation. The range of T, denoted by R(T) or T(V) is defined as $T(V) = \{T(x) : x \in V\}$.

Null Space: -

Let $T : V \to W$ be a linear transformation. The kernel of T, denoted by ketT or N(T), is defined as

$\text{Ker } T = \{x \in V : T(x) = 0 \in W\}$

Ker T is also called the null space of T.

Now we prove that range of T is a subspace of W and null space of T is a subspace of V.

By definition, range of T is

$R(T) = \{T(x) : x \in V\}$

Q.11. If A is a real symmetric matrix, then prove that there exists a real symmetric matrix B such that $B^3 = A$. **(2014)**

Sol. If A is symmetric over R.

Then it is diogonalizable

$A = PDP^{-1}$ Such that D is diogonal

let $B = PD_2P^{-1}$ such that D_2 is a diogonal matrix whose values are third root of the matrix so we get .

$B^3 = PD_2^3P^{-1} = PDP^{-1} = A$

Let $u_1\ u_2 \in R(T)$ and $\alpha, \beta, \in F$. Then $u_1 = T(x_1)$, $u_2 = (x_2)$, for some x_1, $x_2 \in V$ we have.

$= \alpha T(x_1) + \beta(x_2)$

$= T(\alpha x_1 + \beta x_2)$ (as T is C.T)

$= T(x)$ where $x = \alpha x_1 + \beta x_2 \in V$.

since T(x) E R(T), so $\alpha u_1 + \beta u_2 \in R(T)$. Hence R (T) is a subspace of W.

By defintion, $\ker T = \{x \in V : T(x) = 0 \in W\}$

We know that T (0) = 0 and so $0 \in$ kev T that is ker T is non - empty.

let $x, y \in \ker T$ and $\alpha, \beta \in F$.

then T (x) = T (y) = 0 . since T is a L.T.

we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$= \alpha . 0 + \beta . 0 = 0$

$\Rightarrow \quad \alpha x + \beta y \in \text{Ker } T$

Hence ker T is a subspace of V.

Q.12. Find the dual basis of the basis
$B = \{(1, -1, 3), (0, 1, -1), (0, 3, -2)\}$ for $\mathbb{R}^3(\mathbb{R})$. **(2015)**

Sol. Let $\alpha_1 = (1, -1, 3)$, $\alpha_2 = (0, 1, -1)$, $\alpha_3 = (0, 3, -2)$

Then $B = \{\alpha_1, \alpha_2, \alpha_3\}$

If $B' = \{f_1, f_2, f_3\}$ is duel basis of B,

then $f_1(\alpha_1) = 1, f_1(\alpha_2) = 0, f_1(\alpha_3) = 0$

$$f_2(\alpha_1) = 0, f_2(\alpha_2) = 0, f_2(\alpha_3) = 0$$

$$f_3(\alpha_1) = 0, f_3(\alpha_2) = 0, f_3(\alpha_3) = 1$$

Now to find explicit expressions for f_1, f_2, f_3

Let $(a, b, c) \in \mathbb{R}^3(\mathbb{R})$

Let $$(a,b,c) = x(1,-1,3) + y(0,1,-1) + z(0,3,-2) \quad \text{.....(1)}$$

$$= x\alpha_1 + y\alpha_2 + z\alpha_3$$

Then $$f_1(a,b,c) = x,\ f_2(a,b,c) = y$$

$$f_3(a,b,c) = z$$

Now to find the values of x, y, z from eq.(1) we have

$$x = a$$

$$-x + y + 3z = b$$

$$3x - y - 2z = 0$$

Solving there equation

$$x = a,\ y = 7a - 2b - 3c,\ z = b + c - 2a$$

Hence $$f_1(a,b,c) = a$$

$$f_2(a,b,c) = 7a - 2b - 3c$$

and $$f_3(a,b,c) = 2a + b - c$$

therefore $B' = \{f_1, f_2, f_3\}$ as a dual basis of B.

Q.13. Let V_1, V_2 and V_3 be finite dimensional vector spaces over a field F. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations. Prove that $T_2 o T_1 : V_1 \to V_3$ is a linear transformation and rank $(T_2 o T_1) \leq$ min {rank(T_1), rank(T_2)}.
(2016, 19)

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Sol. Let v_1, v_2 and v_3 be finite dimensinol vector space over a field F.

let $T_1 : (v_1, +_1, .) \to (v_2, +_2, .)$ and

$T_2 : (v_2, +_2, .) \to (v_3, +_3, .)$ be $\ell.t$

Then $T_2 \circ T_1 : (v_1, +_1, .) \to (v_3, +_3, .)$ defind as

$(T_2 \circ T_1)(x) = T_2(T_1(x)) \quad \forall \ x \in v_1$

Claim :- $T_2 \circ T_1$ is a $\ell.t.$

let $a, b \in v_1$ and $\alpha, \beta \in F$

$$(T_2 \circ T_1)(\alpha a +_1 \beta b) = T_2(T_1(\alpha a +_1 \beta b))$$

$$= T_2(\alpha T_1(a) +_2 \beta T_1(b)) \qquad (\because T_1 \text{ is a } \ell.t)$$

$$= \alpha T_2(T_1(a)) +_3 \beta T_2(T_1(b)) \qquad (\because T_2 \text{ is a } \ell.t)$$

$$= \alpha(T_2 \circ T_1)(a) +_3 \beta\,(T_2 \circ T_1)(b)$$

Now, we have to prove that

$\text{rank}(T_2 \circ T_1) \le \min\{\text{rank}(T_1), \text{rank}(T_2)\}$

Proof :- (I) $\rho(T_2 \circ T_1) \le \rho(T_2)$

$\because \quad T_1 V_1 \subseteq V_2 \qquad (\because T_1 : v_1 \to v_2 \text{ is } \ell.t)$

$\Rightarrow \quad T_2(T_1(v_1)) \le T_2(v_2)$

$\Rightarrow \quad (T_2 \circ T_1)(v_1) \le T_2(v_2)$

$\Rightarrow \quad \text{Im}(T_2 \circ T_1) \le \text{Im}\, T$

Since $\text{Im}(T_1 \circ T_2)$ and $\text{Im}(T_2)$ are subset of v_3 and $\text{Im}(T_2 \circ T_1) \le \text{Im}\, T_2$

Therefore $\text{Im}(T_1 \circ T_2)$ is a subset of $\text{Im}(T_2)$

$\Rightarrow \quad \dim \text{Im}(T_2 \circ T_1) \le \dim \text{Im}\, T_2$

$\text{rank}(T_2 \circ T_1) \le \text{rank}(T_2)$

$\rho(T_2 \circ T_1) \le \rho(T_2) \qquad \ldots\ldots(1)$

(II) Claim :- $\rho(T_2 \circ T_1) \le \rho(T_1)$

Using rank nullity theorem for T_1 and $(T_1 \circ T_2)$

$\rho(T_1) = \dim v_1 - \dim \ker T_1 \qquad \ldots\ldots(2)$

and $\rho(T_1 \circ T_2) = \dim V_1 - \dim\ker(T_2 \circ T_1) \qquad \ldots\ldots(3)$

Claim :- $\ker T_1 \subseteq \ker T_2 \circ T_2$

let $x \in \ker T_1$

$\Rightarrow \quad T_1(x) = 0_2$

$\Rightarrow \quad T_2(T_1(x)) = T_2(o_2)$

$\Rightarrow \quad T_2 \circ T_1(x) = o_3$

$\Rightarrow \quad x \in \ker(T_1 \circ T_2) \qquad \therefore \ \text{Ker}\, T_1 \subseteq \text{Ker}(T_2 \circ T_1)$

$\because \quad$ Ker T_1 and ker $(T_1 \circ T_2)$ are subspaces of vector space v_1 and

$\ker T_1 \le \ker(T_2 \circ T_1)$

$\because$ Ker T_1 is subspace of ker $(T_2 \circ T_1)$

$\Rightarrow$ dim ker $T_1 \leq$ dim ker $(T_2 \circ T_1)$ .....(4)

from eq.(3) and eq.(4) we get

$$\rho(T_2 \circ T_1) < \rho(T_1)$$

$$\rho(T_2 0\, T_1) = \rho(T_1) \text{ if } T_2 \text{ is one - one} \quad \text{.....(5)}$$

ie. multiplication of a ℓ.t. from left side by an injective ℓ.t does not change the rank

from (1) and (5)

$$\text{rank}(T_2 \circ T_1) \leq \min\{\rho(T_1), \rho(T_2)$$

$$\rho(T_2 \circ T_1) \leq \{\rho(T_1), \rho(T_2)\}$$

Q.14. State and prove Fundamental Theorem of Vector space homomorphisms. (2016

Sol. Fundamental theorem of vector space homomorphism:-

Statement - A homomorphic image of a vector space is homomorphic to quotient space of the vector space.

Let $T : v_1 \to v_2$ be v.s homomorphism

$$\text{Im}(T) = T(v_1)$$

Claim - $\dfrac{v_1}{\ker T} \simeq T(v_1)$

Let ker $T = k$ $\Rightarrow$ $\dfrac{v_1}{k} \simeq T(v_1)$

difine a relation

$$f : \left(\frac{v_1}{k}\right) \to T(v_1)$$

$$f(k+x) = T(x), \quad \forall x \in v_1$$

(I) f is a map :-

let $k + x_1$ and $k + x_2 \in \dfrac{v_1}{k}$.

and $k + x_1 = k + x_2$

$\Rightarrow$ $x_1 - x_2 \in k$

$\Rightarrow$ $x_1 - x_2 \in \ker T$ $(\because k = \ker T)$

$\Rightarrow$ $T(x_1 - x_2) = 0_2$

$\Rightarrow$ $T(x_1) - T(x_2) = 0_2$

$\Rightarrow$ $T(x_1) = T(x_2)$

$\Rightarrow$ $f(k + x_1) = f(k + x_2)$

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$\Rightarrow$ f is a map.

(II) F is a linear transformation.

$$\text{let } k+a \text{ and } k+b \in \frac{V_1}{k} \text{ and } \alpha,\beta \in F$$

$$F\{\alpha(k+a)+\beta(k+b)\} = F\{k+(\alpha a +_1 \beta b)\}$$

$$= T(\alpha a_1 +_1 \beta b)\{\because f(k+n) = T(x)\}$$

$$= \alpha.T(a) +_2 \beta.T(b)$$

$$= \alpha.f(k+a) +_2 \beta f(k+b)$$

$\Rightarrow$ f is a linear transformation (ℓ.t)

(III) f is one - one: -

$$\text{let } k+x_1 \text{ and } k+x_2 \in \frac{V_1}{w}$$

$$f(k+x_1) = f(k+x_2)$$

$\Rightarrow$ $T(x_1) = T(x_2)$

$\Rightarrow$ $T(x_1) + (-T(x_2)) = 0_2$

$\Rightarrow$ $T(x_1) + T(-x_2) = 0_2$ $\qquad (T \in a) = -T(a))$

$\Rightarrow$ $T(x_1 - x_2) = 0_2$ $\qquad (\because T \text{ is a } \ell.t.)$

$\Rightarrow$ $x_1 - x_2 \in \ker T$

$\Rightarrow$ $x_1 - x_2 \in k$

$\Rightarrow$ $k + x_1 = k + x_2$

$\Rightarrow$ f is one - one

(IV) f is onto: -

for all $T(x) \in T(v_1)$

$\Rightarrow$ $x \in V_1$

$\Rightarrow$ $k + x \in \frac{V_1}{k}$

$\because$ $f(k+x) = T(x)$

$\Rightarrow$ f is an onto map.

$\because$ f is an isomorphic

$\because$ $\frac{V_1}{k} \simeq T(v_1)$

$\Rightarrow$ $\frac{V_1}{\ker T} \simeq T(v_1)$

Q.15. Let V_1 and V_2 be finite dimensional vector spaces over a field and T :

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$V_1 \to V_2$ be a linear transformation. Define the transpose T^t of T. Prove tha rank (T^t) = rank (T). (2016

Sol. **Transponse T^t of T :-**

Let v_1 and v_2 be finite dimensional vector space over a fiel f and $T: v_1 \to v_2$ be a linear transformation.

Then difine a map $T^t: v_2^* \to v_1^*$ as

$$T^t(f) = fot, \forall \ f \in v_2^*$$

Now, we have to prove that

rank (T^t) = rank (T)

Since dim w + dim w^0 = dim v ($\because$ w is subspace of v_1)

take w = Im T

$\Rightarrow$ dim (Im T) + dim $(\text{Im T})^0$ = dim v_2

dim (Im T) + dim (ker T^t) = dim v_2 ....(1)

using rank - nullity Theorem of T^t

dim $(\text{Im } T^t)$ + dim $(\ker T^t)$ = dim v_2^* = dim v_2 .....(2)

From eq.(1) and (2) we have

rank (T^t) = rank T

Q.16. Let V be a finite dimensional vector space over a field F. Let V* denote the dual of dual space V* of V. Prove that for each $v \in V$, the map fv : V $\to$ F defined by $fv(\phi) = \phi(v)$ ($\phi \in V^*$), is linear transformation and the map $\theta : V \to V^{**}$ defined by $\theta(v) = fv$ is an isomorphism. (201

Sol. If $v \in v$ and $\phi \in v^*$, Then $\phi(v)$ is a unique element of f. Therefore t correspondence fv defined by

$$fv(\phi) = \phi(v) \forall \phi v^* \quad .....(1)$$

is a fuction from v^* into F.

Let $a, b \in F$ and $\phi, \varphi \in v^*$ Then

$fv(a\phi + b\varphi) = (a\phi + b\varphi)(v)$ from eq.(1)

$= (a\phi)(v) + = (b\varphi)(v)$

{by scalar multiplication of linear functionals}

$a[fv(\phi)] + b[fv(\varphi)]$ from eq.(1)

Therefore fv is a linear transformation on v^* and thue $fv \in v^{**}$

Now, let θ be the function from v^* into v^{**} defined by

$$\theta(v) = fv \forall v \in V$$

(I) θ is one - one: -

If $v, w \in v$, Then

$\theta(v) = \theta(w)$

$\Rightarrow$ $fv = fw \Rightarrow fv(\phi) = fw(\phi) \forall \phi \in v^*$

$\Rightarrow$ $\phi(v) = \phi(w) \forall \phi \in v^*$ from eq.(1)

$\Rightarrow$ $\phi(v) - \phi(w) = 0 \forall \phi \in v^*$

$\Rightarrow$ $\phi(v - w) = 0 \forall \phi \in v^*$

$\Rightarrow$ $v - w = 0$

$\Rightarrow$ $v = w$

$\therefore$ θ is one - one

(II) θ is a linear transformation.

let $a,b \in f$ and $v,w \in v$. Then

$$\theta(av + bw) = f_{(av + bw)} \quad \text{(by definition of } \theta)$$

for evry $\phi \in v^*$, we have

$fav + bw(\phi) = \phi(av + bw)$ {from eq.(1)}

$= a\phi(v) + b\phi(w)$

$= af_v(\phi) + bf_w(\phi)$ {from eq.(II)}

$= (afv)^{(\phi)} + (bfw)^{(\phi)}$

$= (afv + bfw)^{(\phi)}$

$\therefore$ $f(av + bw) = af_v + bf_w = a\theta(v) + b\theta(w)$

Thue $\theta(av + bw) = a\theta(v) + b\theta(w)$

$\therefore$ θ is a linear transfarmation $\rightarrow$

From v^* into v^{**} we have dim v^* = dim v^{**}

Therefore θ is one - one implies that q must also be onto

Hence θ is an isomorphism. of v^* into v^{**}.

Q.17. Define the transpose of a linear transformation and annihilator of a non-empty subset of a vector space, and prove that:
(i) The transpose of a linear transformation is a linear transformation.
(ii) The annihilator of a non-empty subset of a vector space is a subspace of the dual space of the vector space.
(iii) kernel of the transpose of a linear transformation is the annihilator of the image of the linear transformation. **(2012)**

Ans. Transpose of a linear transformation: -

Let a linear transformation $T : V_1 \rightarrow V_2$,

Then define a map

$T^t : V_2^* \to V_1^*$ as

$T^t(f) = foT \ \forall f \in V_2^*$ is called a transpose of a linear transformation.

Annihilator of a subset of a vector space: -

Let 'S' be the subset of a V.S. 'V' over field 'F' which is non empty the annihilator of S is denoted by 'S^o' and is defined as

$$S^o = \left\{ f \in V^* : f(s) = O_F \forall s \in S \right\} \subseteq V^*$$

$$\neq \phi, \quad f_o \in S^o$$

Claim - T^t is a linear transformation

Let $f, g \in V_2^*$ and $\alpha, \beta \in F$

$$T^t(\alpha f + \beta g) = (\alpha f + \beta g)\, oT = (\alpha f)oT + (\beta g)oT$$
$$= \alpha(foT) + \beta(goT) = \alpha T^t(f) + \beta T^t(g)$$

$\Rightarrow$ T^t is a linear transformation.

(ii) Claim S^o is a subspace of V^* -

Let $f, g \in S^o$ and $\alpha, \beta \in F$

$\Rightarrow$ $f(s) = O_F$ and $g(s) = O_F \ \forall s \in S$ and $\alpha, \beta \in F$

$\Rightarrow$ $\alpha \odot f(s) = O_F$ and $\beta \odot g(s) = O_F$

$\Rightarrow$ $(\alpha - f)(s) = O_F$ and $(\beta . g).(s) = O_F$

$\Rightarrow$ $(\alpha f + \beta g)(s) = O_F \quad \forall s \in S$

$\Rightarrow$ $(\alpha f + \beta g)\, S^o$

$\therefore$ S^o is subspace of V^*.

(iii) We have to prove that

$Ker T^t = (Im T)^o$

$T : V_1 \to V_2$ and $T^t : V_2^* \to V_1^*$

let $f \in ker T^t$

$\Leftrightarrow$ $T^t(f) = fo$

$\Leftrightarrow$ $foT = fo$

$\Leftrightarrow$ $(foT)(x) = fo(x) = o_F$

$\Leftrightarrow$ $f[T(x)] = o_F, \quad \forall T(x) \in I_m T$

$\Leftrightarrow$ $f \in [I_m T]^o$

Therefore $Ker T^t = (Im T)^o$

Q.18. (i) Define linear transformation. If $T : V_1 \to V_2$ is a linear transformation and W is a subspace of V_2, then prove that $T^{-1}(W)$ will be subspace of V_1.
(ii) Let $T : V_1 \to V_2$ be a surjective, linear transformation. Suppose that $\dim V_1 = 12$ and $\dim V_2 = 8$ and W be a subspace of V_2 such that $\dim W = 3$. Then

find the dimension of $T^{-1}(W)$. **(2012)**

Sol. **Linear transformation: -**

Let $(V_1, +_1, .)$ and $(V_2, +_2, .)$ be two vector space over a filed F.

Then a map $T : V_1 \to V_2$ is called a linear transformation iff

(i) $T(a +_2 b) = T(a) +_2 T(b)$ $\quad \forall a, b \in V_1$

(ii) $T(\alpha a) = \alpha T(a)$ $\quad \forall \alpha \in F$ and $\forall a \in V_1$

Let W is a subspace of V_2. Then we have to prove that $T^{-1}(W)$ is a subspace of V_1.

Let $C_1, d_1 \in T^{-1}(W)$ and $\lambda, \mu \in F$

$\Rightarrow$ $T(c_1), T(d_1) \in W$ $\quad \left(\because x \in f^{-1}(V) \Leftrightarrow f(x) = V\right)$

$\Rightarrow$ $\lambda T(c_1) +_2 \mu T(d_1) \in W$ $\quad (\because$ W is subspace)

$\Rightarrow$ $T(\lambda c_1 +_1 \mu d_1) \in W$

$\Rightarrow$ $\lambda c_1 +_1 \mu d_1 \in T^{-1}(W)$

$\Rightarrow$ $T^{-1}(W)$ is a subspace of V_1.

(ii) Since $T' : T^{-1}(W) \to V_2$ is restriction of T over $T^{-1}(W)$ and T is onto map.

$$T'\left(T^{-1}(W)\right) = T\left[T^{-1}(W)\right] = W = I_m T' \quad \text{.....(1)}$$

Also $\ker T' = \ker T \cap T^{-1}(W) = \ker T$ .....(2) $\left(\because \ker T \subseteq T^{-1}(W)\right)$

Using Rank-Nullity theorem, we have

$\dim \ker T' + \dim \text{Im}T' = \dim(T^{-1}(W))$

$\Rightarrow$ $\dim \ker T + \dim W = \dim T^{-1}(W)$ $\quad (\because$ using eq.(1) and (2))

$\Rightarrow$ $\dim V_1 - \dim \text{Im}T + \dim W = \dim T^{-1}(W)$ $\quad$ (Using R.N.T)

$\Rightarrow$ $\dim V_1 - \dim V_2 + \dim W = \dim T^{-1}(W)$ $\quad (\because T^{-1}$ is onto $\Leftrightarrow \text{Im}T = V_2)$

$\Rightarrow$ $12 - 8 + 3 = \dim T^{-1}(W)$

$\Rightarrow$ $\dim T^{-1}(W) = 7$

Q.19. Define dual space of a vector space and prove that the dimension of th dual space of a finite dimensional vector space is equal to the dimension of the vector space. **(2012)**

Ans. **Dual space of vector space: -**

A linear transformation $f : V \to F$ is called linear function. If

$\text{Hom}(V, F) = \{\text{all linear functional} : V \to F\}$

$= V^*$

and V^* is called dual space of vectgor space V.

We have to prove that

$\dim V = \dim V^*$

Proof - Construction of a basis of a dual space from the basis of vector space -

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Let dim V = n and
let $X = \{x_1, x_2, \ldots\ldots x_n\}$ be basis of V
and $X^* = \{f_1, f_2, \ldots\ldots f_n\} \subseteq V^*$
Where $f_i : V \to F$ defined as -

$$f_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ o_F & \text{if } i \neq j \end{cases}$$

Claim - X^* is a basis of V^*.
(I) X^* is a linear independent

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots\ldots + \alpha_n f_n = f_0.$$
$$(\alpha_1 f_1 + \alpha_2 f_2 + \ldots\ldots + \alpha_n f_n)(x_1) = f_0(x_1)$$
$$\Rightarrow \quad \alpha_1 \odot f_1(x_1) \oplus \alpha_2 \odot f_2(x_2) \oplus \ldots\ldots \oplus \alpha_n \odot f_n(x_n) = O_F$$
$$\Rightarrow \quad \alpha_1 . 1 \oplus \alpha . 0 \oplus \ldots\ldots \oplus \alpha_n . 0 = O_F \qquad \left(\because f_i(x_j)\right) = 1 \text{ if } i = j, \; = 0 \text{ if } i \neq j$$
$$\Rightarrow \quad \alpha_1 = O_F$$

Similarly $\alpha_2 = \alpha_3 = \ldots\ldots = \alpha_n = O_F$
Therefore X^* is linearly independent.
(ii) $V^* = [X^*]$

Let $g \in V^* \Rightarrow g(x_i) = \beta_i$ where $i = 1, 2, 3, \ldots\ldots n$

Claim - $g = \beta_1 f_1 + \beta_2 f_2 + \ldots\ldots + \beta_n f_n$
let $x \in V$

$$\Rightarrow \quad x = \gamma_1 x_1 + \gamma_2 x_2 + \ldots\ldots + \gamma_n x_n \qquad \ldots\ldots(1) \cdot \; V_i \in F$$
$$f_1(x) = f_1(\gamma_1 x_1 + \gamma_2 x_2 + \ldots\ldots + \gamma_n x_n)$$
$$\Rightarrow \quad f_1(x) = \gamma_1 \odot f_1(x_1) \oplus f_2 \odot f_2(x_2) \oplus \ldots\ldots \oplus \gamma_n \odot f_n(x_n)$$
$$\Rightarrow \quad f_1(x) = \gamma_1 \odot 1 \oplus \gamma_2 \odot O \oplus \ldots\ldots \oplus \gamma_n \odot O$$
$$\Rightarrow \quad f_1(x) = \gamma_1$$

Similarly $f_2(x) = \gamma_2 = \ldots\ldots = f_n(x_n) = \gamma_n$
Consider R.H.S.

$$(\beta_1 f_1 + \beta_2 f_2 + \ldots\ldots + \beta_n f_n$$
$$= \beta_1 \odot f_1(x) \epsilon \; \beta_2 \odot f_2(x) \oplus \ldots\ldots \oplus \beta_n \odot f_n(x)$$
$$= \beta_1 \odot \gamma_1 \oplus \beta_2 \odot \gamma_2 \oplus \ldots\ldots\ldots\ldots \oplus \beta_n \odot \gamma_n \qquad \ldots\ldots(2)$$

Consider L.H.S.

$$g(x) = g(\gamma_1 x_1 + \gamma_1 x_1 + \ldots\ldots + \gamma_n x_n)$$
$$= \gamma_1 \odot g(x_1) \oplus \gamma_2 \odot g(x_2) \oplus \ldots\ldots \oplus \gamma_n \odot g(x_n)$$
$$= \gamma_1 \odot \beta_1 \oplus \gamma_2 \odot \beta_2 \oplus \ldots\ldots \oplus \gamma_n \beta_n \qquad \ldots\ldots(3)$$

From eq.(2) and (3)

$$g = \beta_1 f_1 + \beta_2 f_2 + \ldots\ldots\ldots + \beta_n f_n$$

Hence $V^* = [X^*]$

$\therefore$ X^* is a basis of V^* and X^* is called dual basis of X.

Hence $\dim V = \dim V^*$

Note - $\dim (\text{Hom}(V_1, V_2)) = \dim V_1 . \dim V_2$

$\dim(\text{Hom}(V, F)) = \dim V . \dim F$

$\dim V^* = \dim V$

Q.20. Let M_2 be the set of all 2×2 real matricces. If $T : M_2 \to M_2$ is linear transformation defined by $T(A) = A.B - B.A$, $\forall A \in M_2$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ then find (i) ker T (ii) Im T and (iii) Trace of T. (2012)

Sol. Standard basis of M_2 is -

$S = \{e_{11}, e_{12}, e_{21}, e_{22}\}$

we have to find matrix $(T)_{s,s} = ?$

$$T(e_{11}) = e_{11}B - B.e_{11}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$$

$$= 0.e_{11} + 4.e_{12} + (-1)e_{21} + 0.e_{22}$$

$$T(e_{12}) = e_{12}B - Be_{12}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= 1.e_{11} + 0.e_{12} + 0.e_{21} + (-1)e_{22}$$

$$T(e_{21}) = e_{21}B - Be_{21}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= -4e_{11} + 0e_{12} + 0e_{21} + 4e_{22}$$

$$T(e_{22}) = e_{22}B - Be_{22}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

$$= 0.e_{11} + 4.e_{12} + 1.e_{12} + 0.e_{22}$$

$$\text{mat}(T)_{SS} = \begin{bmatrix} 0 & 1 & -4 & 0 \\ 4 & 0 & 0 & 4 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & 0 \end{bmatrix}$$

trace of matrix $= 0$

$$\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} = \text{basis of Im}(T)$$

$$\rho(T) = 2$$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 2a+b & 4a+2b \\ 2c+d & 4c+2d \end{bmatrix} - \begin{bmatrix} 2a+4c & 2b+4d \\ a+2c & b+2d \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} b-4c & 4a-4d \\ d-a & 4c-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $b - 4c = 0$, $d - a = 0$, $4(a - d) = 0$, $4c - b = 0$

Solving these equations $a = d, b = 4c$

So let $a = \lambda, c = \mu$

$a = \lambda, d = \lambda, b = 4\mu, c = \mu$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda & 4\mu \\ \mu & \lambda \end{bmatrix}$

$$\text{basis of ker } T = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right\}$$

$\dim(\ker T) = 2$

Q.21. Define kernel of a linear transformation. Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that ker T and $I_M T$ are respectively spanned by {(1, 1, 1, 1), (1, 0, 0, 1)} and {(1, 1, 0), (1, 0, 1)}. **(2013)**

Ans. Kernel of a linear transformation: -

Let U(F) and V(F) be two vector spaces and let T be a linear transformations from U into V. Then the kernel of T or null space of T written as N(T) is the set of all vectors α in U such that $T(\alpha) = 0$.

Thus $N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\}$

Let $S = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$ extending to {(1, 1, 1, 1) (1, 0, 0, 1)} in basis of $\mathbb{R}^4$.

$\{(1, 1, 1, 1), (1, 0, 0, 1), e_3, e_4\}$ are linear independent.

$T(1, 1, 1, 1) = (0, 0, 0)$ and $T(e_3) = (1, 1, 0)$

$T(1, 0, 0, 1) = (0, 0, 0)$ and $T(e_4) = (1, 0, 1)$

Let $(x, y, z, 4) \in \mathbb{R}^4$

$\therefore \quad (x, y, z, u) = \alpha(1, 1, 1, 1) + \beta(1, 0, 0, 1) + \gamma e_3 + \delta e_4$

$T(x, y, z, u) = \alpha T(1, 1, 1, 1) + \beta(1, 0, 0, 1) + \gamma T(e_3) + \delta T(e_4)$

$= \alpha(0, 0, 0) + \beta(0, 0, 0) + \gamma(1, 1, 0) + \delta(1, 0, 1)$

$T(x, y, z, u) = (\gamma + \delta, \gamma, \delta)$

Q.22. Define transpose of a linear transformation and prove that the rank of the transpose of a linear transformation is equal to the rank of the linear transformation. **(2013)**

Ans. Transpose of a linear transformation: -

Let a linear transformation $T : V_1 \to V_2$. Then define a map $T^t : V_2^* \to V_1^*$ as

$T^t(f) = foT \quad \forall f \in V_2^*$ is called transpose of linear transformation.

Now we have to show that

Rank (T^t) = Rank (T), i.e. $\rho(T^t) = \rho(T)$

Since $\dim W + \dim W^\circ = \dim V$

Take $W = ImT$

$\Rightarrow \quad \dim(ImT) + \dim(ImT)^\circ = \dim V_2$

$\dim(ImT) + \dim(\ker T^t) = \dim V_2$ .....(i)

$(\because (ImT)^\circ = (\ker T^t))$

Using Rank Nullity Theorem of T^t.

$\dim(ImT)^t + \dim(\ker T^t) = \dim V_2^* = \dim V_2$ .....(ii)

from eq.(i) and (ii) we get

Rank (T^t) = Rank (T)

Q.23. If a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $T[(x, y, z, u)] = (x - y + z + u, x + 2z - u, x + y + 3z - 3u) \ \forall \ (x, y, z, 4) \in \mathbb{R}^4$, then find (i) a basis of ker T and (ii) a basis of $I_m T$. **(2013)**

Sol. $T(x, y, z, u) = (x - y + z + u, x + 2z - u, x + y + 3z - 3u)$ the matrix form can

be written as

$$[T]=\begin{bmatrix}1 & -1 & 1 & 1\\ 1 & 0 & 2 & -1\\ 1 & 1 & 3 & -3\end{bmatrix}$$

Changing echalon form

$$\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1}\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 1 & -2\\ 0 & 2 & 2 & -4\end{bmatrix}\xrightarrow{R_3-R_3-2R_2}\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\text{basis of ImT} = \left\{\begin{pmatrix}1\\-1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}\right\}$$

Let $(x, y, z) \in \ker T$, then

$$\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\\u\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x - y + z + u = 0$$
$$y + z - 2u = 0$$

Let $u = k, z = \lambda.$

$$y = 2k - \lambda$$
$$x = y - z - u = 2k - \lambda - \lambda - k = k - 2\lambda$$

so the basis, ker T

$$\left\{\begin{pmatrix}1\\2\\0\\1\end{pmatrix},\begin{pmatrix}-2\\2\\1\\0\end{pmatrix}\right\}$$

Q.24. Define the annihilator of a non-empty subset of a vector space a prove that the sum of the dimensions of a sub-space of a finite dimensio vector space and its annihilator, is equal to the dimension of the vector spa (20

Sol. Annihilator of a subset of a vector space: -

Let 'S' be the subset of a vector space 'V' over field

which is non empty, then annihilator of S is denoted by 'S^o' and is defined as -

$$S^o = \{f \in V^* : f(s) = O_F \quad \forall S \in S\} \subseteq V^*$$

Now we prove that

$\dim W + \dim W^o = \dim V$

let $\dim W = r$ and

$X_1 = \{x_1, x_2, \ldots\ldots x_r\}$ be the basis of W. Since X_1 is linearly independent subset of vector space V therefore it can be extended to form a basis of vector space 'V' (By Extension Theorem). Let

$$X_2^* = \{f_1, f_2, \ldots\ldots, f_r, f_{r+1}, \ldots\ldots, f_n\} \subseteq V^*$$

be the dual basis of X_2 and basis of V*.

Consider $S = \{f_{r+1}, f_{r+2}, \ldots\ldots f_n\}$

Claim - $S \subseteq W^o$

Let $x \in W$

$\Rightarrow \quad x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots\ldots + \alpha_r x_r + \quad$ where $\alpha_i \in F$

$\Rightarrow \quad f_{r+1}(x) = \alpha_1 \odot f_{r+1}(x_1) \oplus \alpha_2 \odot f_{r+1}(x_2) \oplus \ldots\ldots\ldots \oplus x_r \odot f_{r+1}(x_r)$

$\Rightarrow \quad f_{r+1}(x) = O_F$

$\Rightarrow \quad f_{r+1} \in W^o$

Similarly $f_{r+2}, \ldots\ldots\ldots, f_n \in W^o$

$\Rightarrow \quad S \subseteq W^o$

(I) S is linearly independent -

$S \subseteq X_2^*$ and X_2^* is linearly independent.

Therefore S is also linearly independent.

(II) $W^o = [S]$ -

Let $g \in W^o \quad \Rightarrow \quad g(x_i) = O_F, \quad 1 \leq i \leq r$

$\therefore \quad g \in V^*$

$$\Rightarrow \quad g = \sum_{i=1}^{n} g(x_i) f_i$$

$$g = g(x_1)f_1 + g(x_2)f_2 + \ldots\ldots\ldots + g(x_r)f_r + g(x_{r+1})f_{r+1} + \ldots\ldots\ldots + g(x_n)f_n$$

$$\Rightarrow \quad g = g(x_{r+1})f_{r+1} + \ldots\ldots\ldots\ldots + g(x_n)f_n$$

g is a linear combination of $f_{r+1}, \ldots\ldots\ldots, f_n$.

$\therefore \quad W^o = [S]$

$\therefore$ S is a basis of W^o.

$\dim W^o = n - r$

$\Rightarrow \quad \dim W^o = \dim V - \dim W$

$\Rightarrow \quad \dim W^o + \dim W = \dim V$

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Q.25. Define an isomorphism between two vector spaces over the same field. Prove that two finite dimensional vector spaces over a field F are isomorphic if and only if they have same dimension. (2017)

Ans. Let V_1 and V_2 be two V.S. over the same field then we say that V_1 isomorphic to V_2 if $\exists$ a l.t. $T : V_1 \to V_2$ which is one-one onto.

Q.26. Define the annihilator of a non-empty subset of a vector space over field. Determine the annihilator of the subset $S = \{(x, y, z) \in \mathbb{R}^r : xy = 1\}$ of the real vector space $\mathbb{R}^3$. (2017)

Sol. Let V be a V.S. over field F and $f : V \to F$ be linear functional. Let S be a non-empty subset of vector space V. Then the annihilator of S is denoted by S° and defined as -

$$S^\circ = \{f \in V^* : f(s) = 0, \ \forall\, s \in S\}$$

where, V^* is dual basis of V.

given that $S = \{(x, y, z) \in \mathbb{R}^3 : xy = 1\}$

Let $x = \dfrac{1}{y}$

Take, $x = 1, y = 1, z = 0$

$x = -1, y = -1, z = 0$

Let $S = \{(1, 1, 0), (-1, -1, 0)\}$

then S is ℓ.d.

we take $S = \{(1, 1, 0), (0, 1, 0), (0, 0, 1)\}$

Let $a_1 = (1, 1, 0), a_2 = (0, 1, 0), a_3 = (0, 0, 1)$

define $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$.

$f_1(a_1) = 1, f_1(a_2) = 0, f_1(a_3) = 0$

$f_2 : \mathbb{R}^3 \to \mathbb{R}^3$ and $f_3 : \mathbb{R}^3 \to \mathbb{R}^3$ as

$f_2(a_1) = 0, f_2(a_2) = 1, f_2(a_2) = 0, f_3(a_3) = 0, f_3(a_2) = 0, f_3(a_3) = 1.$

Let $(x, y, z) \in \mathbb{R}^3$ and S is basis of $\mathbb{R}^3$.

$$(x, y, z) = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = \beta_1(1, 1, 0) + \beta_2(0, 1, 0) + \beta_3(0, 0, 1)$$

$$\Rightarrow \quad (x, y, z) = (\beta_1, \beta_1 + \beta_2, \beta_3)$$

$$\Rightarrow \quad \beta_1 = x, \beta_1 + \beta_2 = y \Rightarrow \beta_2 = y - x$$

$$\beta_3 = z$$

then $f_1(x, y, z) = x, f_2(x, y, z) = y - x, f_3(x, y, z) = z$

Then $\{f_1, f_2, f_3\}$ is dual basis of S and this is annihilator of S.

Q.27. State and prove the fundamental theorem of vector space homomorphisms. (2018)

Sol. Fundamental theorem of a vector space homomorphism: -

A homomorphic image of a vector space is isomorphic to quatient space of the vector space.

Proof: -

Let $T : V_1 \to V_2$ be a vector space homomorphism.

$\text{Im}(T) = T(V_1)$ (homomorphic image of V_1)

Claim: $\frac{V_1}{\text{Ker}T} \approx T(V_1)$:-

define a realation -

$f : \left(\frac{V_1}{K}\right) \to T(V_1)$ $\qquad \left[\frac{V}{w} \Rightarrow w + x\right]$

$f(k+x) \to T(x)$ $\qquad \forall\ x \in V_1$

(i) "f" is a map -

Let $k + x_1$ and $k + x_2 \in V_1 / K$

and $k + x_1 = k + x_2$

$\Rightarrow$ $x_1 - x_2 \in K$ $\qquad (w + a = w + b \Leftrightarrow a - b \in w)$

$\Rightarrow$ $x_1 - x_2 \in \text{Ker}\,T$ $\Rightarrow$ $T(x_1 - x_2) = O_2$

$\Rightarrow$ $T(x_1) - T(x_2) = O_2$ $\Rightarrow$ $T(x_1) = T(x_2)$

$\Rightarrow$ $f(k + x_1) = f(k + x_2)$

therefore "f" is a map.

(ii) "f" is a linear transformation -

Let $k + a$ and $k + b \in V_1/K$ and $\alpha, \beta \in F$

$f\{\alpha(k+a) +_1 \beta(k+b)\}$

$f\{k + (\alpha_1 +_1 \beta b)\}$ $\qquad (\because\ w + \alpha a = a(w + \alpha))$

$T(\alpha a +_1 \beta b)$ $\qquad (\because\ f(k+x) \to T(x))$

$= \alpha T(a) +_2 \beta . T(b) = \alpha f(k+a) +_2 \beta f(k+b)$

"f" is a linear transformation.

(iii) "f" is one-one -

Let $k + x_1$ and $k + x_2 \in V_1/k$ and $f(k + x_1) = f(k + x_2)$

$\Rightarrow$ $T(x_1) = T(x_2)$

$\Rightarrow$ $T(x_1) + (-T(x_2)) = O_2$ $\Rightarrow$ $T(x_1) + t(-x_2) = O_2$

$\Rightarrow$ $T(x_1 - x_2) = O_2$ (T is a linear transformation)

$\Rightarrow$ $x_1 - x_2 = \text{Ker}\,T$ $\Rightarrow$ $x_1 - x_2 \in K$

$\Rightarrow$ $K + x_1 = K + x_2$ $\qquad (\because\ a - b \in W \Leftrightarrow w + a = w + b)$

(iv) "f" is onto -

$\forall\ T(x) \in T(V_1)$

$\Rightarrow$ $x \in V_1$ $\Rightarrow$ $K + x \in V_1$

$f(K + x) = T(x)$

$\Rightarrow$ f is an onto map.

$\therefore$ f is an isomorphim.

$\therefore$ $\frac{V_1}{K} \approx T(V_1)$ $\Rightarrow$ $\frac{V_1}{\text{Ker}\,T} \approx T(V_1)$

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Q.28. Define rank and nullity of linear transformation. Let T be a linea operator on a finite dimensional vector space V such that rank(T^2) = rank(T Prove that range (T) ∩ Kernel (T) = {0} (201

Sol. Let $T : V \to V$ be a linear transformation of finite dimensional vect space over field F such that

$$\rho(T^2) = \rho(T)$$

Then we have to prove that

$$\text{Ker } T \cap \text{Im } T = \{0\}$$

Let, $T : V \to V$ be a linear transformation

$T^2 : V \to V$ be also a linear transformation using Rank - Nullity theorem we hav

$$\dim \ker T + \dim \text{Im} T = \dim V \quad \text{.....(i)}$$

And $\dim \ker T^2 + \dim \text{Im } T^2 = \dim V$ .....(ii)

respectively then from eq.(i) and (ii)

$$\dim \ker T + \dim \text{Im } T = \dim \ker T^2 + \dim \text{Im } T^2$$

$\Rightarrow$ $\dim \ker T = \dim \ker T^2$ $\{\because \rho(T^2) = \rho(T)\}$

$\Rightarrow$ $\ker T = \ker T^2 \Rightarrow \ker T^2 = \ker T$ $\{\because \ker T$ is a subspace of $\ker T$

$\Rightarrow$ $x \in \ker T^2 \Rightarrow x \in \ker T$

$\Rightarrow$ $T^2(x) = 0 \Rightarrow T(x) = 0$ .....(iii)

Let $a \in \ker T \cap \text{Im } T \Rightarrow a \in \ker T$ and $a \in \text{Im } T$

$\Rightarrow$ $T(a) = 0$ and $\exists\ b \in V$ such that $T(b) = a$

$\Rightarrow$ $T(a) = 0$ and $\exists\ b \in V$ such that $T(T(b)) = T(a)$

$\Rightarrow$ $T(a) = 0$ and $\exists\ b \in V$ such that $T^2(b) = T(a)$

$\Rightarrow$ $T^2(b) = T(a) = 0 \Rightarrow T^2(b) = 0$

$\Rightarrow$ $T(b) = 0$ $\{\because T^2(x) = 0 \Rightarrow T(x) = 0\}$

$\Rightarrow$ $a = 0$ $(\because$ Using (ii))

Hence $\text{Ker} T \cap \text{Im } T = \{0\}$

Q.29. Define the transpose of a linear transformation. Prove that the ran of the transpose of a linear transformation T between finite dimensional ve tor space over the same field is equal to the rank of T. (201

Sol. Transpose of a linear transformation: -

Let $V_1(F)$ and $V_2(F)$ be two vetor space and $T : V_1 \to V_2$ be a linear transform tion. And V_1^* and V_2^* is dual space of V_1 and V_2 respectively.

Let $f : V_2 \to F$ be linear functional

$\because$ Composition of two linear map is also a linear map.

$\therefore$ $foT : V_1 \to F$ be a linear functional.

Theorem: -

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If $T : V_1 \to V_2$ is linear transformation.

then $\rho(T^t) = \rho(T)$

Proof : -

$\because$ $T : V_1 \to V_2$ is a linear transformation then by (R - N) Rank - Nullity theorem.

$$\dim \ker T + \dim \operatorname{Im} T = \dim V$$

$$\Rightarrow \quad \dim \ker T + \rho(T) = \dim V_1 \qquad \text{.....(i)}$$

and $T^t : V_2^* \to V_1^*$ is also a linear transformation, then by R-N theorem

$$\dim \ker T^t + \dim \operatorname{Im} T^t = \dim V_1^*$$

$$\Rightarrow \quad \dim \ker T^t + \rho(T^t) = \dim V_2^*$$

$$\because \quad \dim V_2^* = \dim V_2 \quad \Rightarrow \quad \dim \ker T^t + \rho(T^t) = \dim V_2 \qquad \text{.....(ii)}$$

Since, we know that ImT is a subspace of V_2.

$$\therefore \quad \dim \operatorname{Im} T + \dim[\dim T]^0 = \dim V_2 \qquad \text{.....(iii)}$$

from eq.(ii) and (iii)

$$\dim \ker T^t + \rho(T^t) = \dim \operatorname{Im} T + \dim [\operatorname{Im} T]^0$$

$$\Rightarrow \quad \dim \ker T^t + \rho(T^t) = \rho(T) + \dim[\operatorname{Im} T]^0 \qquad \text{.....(iv)}$$

Cliam - $\ker T^t = \{\operatorname{Im} T]^0$

Let $f \in \ker T^t$

$\Leftrightarrow$ $T^t(f) = f_0$ (zero linear function and by definition of Ker T)

$\Leftrightarrow$ $foT = f_0$ (from definition of T^t)

$\Leftrightarrow$ $(foT)(x) = fo(x)$ (by definition of equal map $x \in V_1$)

$\Leftrightarrow$ $f[T(x)] = 0$ $\{\forall x \in V_1\}$

$\Leftrightarrow$ $f[T(x)] = 0$ $\{\forall T(x) \in \operatorname{Im} T = T(V_1)\}$

$\Leftrightarrow$ $f \in [\operatorname{Im} T]0$ {by definition of Annihilator}

Hence, $\ker T^t = [\operatorname{Im} T]^0$

$$\Rightarrow \quad \dim \ker T^t = \dim[\operatorname{Im} T]^0 \qquad \text{.....(v)}$$

from eq.(iv) and (v)

$$\dim \ker T^t + \rho(T^t) = \rho(T) + \dim(\operatorname{Im} T)^0$$

$$\Rightarrow \quad \rho(T^t) = \rho(T)$$

Q.30. Let V_1 and V_2 be vector spaces over the same field F and $T : V_1 \to V_2$ be a surjective linear transformation. Suppose that W is a subspace of V_2 of dimension 2. If $\dim V_1 = 7$ and $\dim V_2 = 5$, then find $\dim T^{-1}(W)$. (2018)

Sol. Let us define a map-

$$T^1 : T^{-1}(W) \to V_2$$

$$\therefore \quad T[T^{-1}(W)] = W \quad \{\because T \text{ is a surjective map}\}$$

Since T^1 is restriction of linear transformation T over subspace $T^{-1}(W)$

$$\therefore \quad T^1[T^{-1}(W)] = W$$

$\Rightarrow$ $\text{Im}T^1 = W$ $\Rightarrow$ $\dim \text{Im}\, T^1 = \dim W$

$\Rightarrow$ $\dim \text{Im}\, T^1 = 2$ $(\because \dim W = 2)$

Since $\ker T^1 = \ker T \cap T^{-1}(W)$

$\because$ we know that,

$\ker T \subseteq T^{-1}(W)$, $\forall$ W subspace of V_2

$\therefore$ $\ker T^1 = \ker T$

$\therefore$ From Rank-nullity theorem,

$\dim \ker T + \dim \text{Im}\, T = \dim V_1$

$\Rightarrow$ $\dim \ker T = \dim V_1 - \dim \text{Im}\, T$

$\dim \ker T = \dim V_1 - \dim \text{Im}_m T = 7 - 5 = 2$

Again by Rank Nullity theorem-

$\dim \ker T^1 + \dim I_m T^1 = \dim [T^{-1}(W)]$

$\Rightarrow$ $2 + 2 = \dim [T^{-1}(W)]$

$\Rightarrow$ $\dim [T^{-1}(W)] = 4$

Q.31. Define the dual space of vector space. Find the dual basis of the dua space of $\mathbb{R}^3$ relative to the ordered basis $S = \{(1, 1, 0), (1, 2, 0), (1, 1, 1)\}$ of $\mathbb{R}$

Sol. Dual space: -

A linear trousformation $f : V \to F$ is called linear function if.

Hom $(V, f) = (\text{all linear functional} : V \to F) = V^*$

and V^* is called dual space of vector space V.

Here $S = \{(1,1,0)(1,2,0),(1,1,1)\}$

Let $S^* = \{f_1, f_2, f_3\}$ be the dual basis of S.

Let $(x, y, z) \in \mathbb{R}^3$.

$(x, y, z) = \alpha(1,1,0) + \beta(1,2,0) + \gamma(1,1,1)$ ..... (1)

$\Rightarrow$ $\alpha + \beta + \gamma = x$ ..... (2)

$\alpha + 2\beta + \gamma = y$ ..... (3)

$\gamma = z$ ..... (4)

From eq.(2) and (3) we get

$\beta = y - x$

Putting the value of γ and β in eq.(2) we get .

$\alpha + y - x + z = x$

$\Rightarrow$ $\alpha = 2x - y - z$

Now . Putting the value of α , β , γ in eq. (1)

$f_1(x, y, z) = \alpha \odot f_1(1, 1, 0) \oplus \beta \odot f_1(1, 2, 0) \oplus \gamma \odot f_1(1, 1, 1)$

$f_1(x, y, z) = \alpha$

$f_1(x, y, z) = 2x - y - z$

Similarly $f_2(x, y, z) = \beta = y - x$

$f_3(x, y, z) = \gamma = z$

Q.32. Define the annihilator of non-empty subset of a vector space. If V is a finite dimensional vector space over a field F and W is a subspace of V, then prove that:

$$\dim W + \dim W^0 = \dim V,$$

Where W^0 is the annihilator of W. **(2019)**

Sol. Annihilator of a Non - empty subset of a vectar space b: -

Let 'S' be the subset a vector space 'V' over field F which is non-empty then annihilator of S is denoted by 'S^0' and is defined as -

$$S^0 = \{f \in V^* : f(s) = O_F \ \forall s \in S\} \subseteq V^*$$

$$\neq \phi \quad f_0 \in S^0$$

Claim : S^0 is sub space of V^*.

Let $f, g \in S^0$ and $\alpha, \beta \in F$

$\Rightarrow$ $f(s) = O_F$ and $g(s) = O_F$ $\quad \forall s \in S$ and $\alpha, \beta \in F$

$\Rightarrow$ $\alpha \odot f(s) = O_F$ and $\beta \odot g(s) = O_F$

$\Rightarrow$ $(\alpha.f)(s) = O_F$ and $(\beta.g)(s) = O_F$

$\Rightarrow$ $(\alpha.f + \beta.g)(s) = O_F$ $\quad \forall s \in S$

$\Rightarrow$ $\alpha.f + \beta.g \in S^0$

$\therefore$ S^0 is subsace of V*.

Now we prove that

$\dim W + \dim W^0 = \dim V$

Let $\dim W = r$ and $X_1 = \{x_1, x_2, ...x_r\}$ be the basis of W.

Since X_1 is linearly independent subset of vetor space V therefore it can be extended to form a basis of V.S.V (By extension theorem).

Let $X_2^* = \{f_1, f_2, ...f_r, f_{r+1}, ...f_n\} \subseteq V^*$ be the dual basis of X_2 and basis of V^*.

Consider $S = \{f_{r+1}, f_{r+2}, ..., f_n\}$

Claim: -

$$S \subseteq W^*$$

Let $x \in W$

$\Rightarrow$ $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_r x_r$ $\quad$ where $\alpha_i \in F$

$\Rightarrow$ $f_{r+1}(x) = \alpha_1 \odot f_{r+1}(x_1) \oplus \alpha_2 \odot f_{r+1}(x_2) \oplus ..... + \alpha_r \odot f_{r+1}(x_r)$

$\Rightarrow$ $f_{r+1}(x) = O_F$ $\Rightarrow$ $f_{r+1} \in W^0$

Similarly $f_{r+2}, ..., f_n \in W^0$

$\Rightarrow$ $S \subseteq W^0$

Now we prove that S is a basis of W^0.

(I) S is linearly independent

$S \subseteq X_2^*$ and X_2^* is linear independent. Therefore S is also linear independent. ($\because$ A subset of L.I. set in L.I.)

(II) $W^0 = [S]$

Let $g \in W^0 \Rightarrow g(x_i) = O_F, \quad 1 \le i \le r$

$\therefore \quad g \in V^* \Rightarrow g = \sum_{i=1}^{n} g(x_i) f_i$.

$g = (x_1)f_1 + g(x_2)f_2 + \ldots\ldots\ldots\ldots + g(x_r)f_r + g(x_{r+1})f_{r+1} + \ldots\ldots + g(x_n)f_n$

$g = g(x_{r+1})f_{r+1} + \ldots\ldots\ldots + g(x_n)f_n$

$\Rightarrow$ g is a linear combination of $f_{r+1}, \ldots\ldots, f_n$.

$\therefore \quad W^0 = [S]$

$\therefore$ S is a basis of W^0

$\dim W^0 = n - r$

$\dim W^0 = \dim V - \dim W \Rightarrow \dim W^0 + \dim W = \dim V$

Q.33. Define an elementary matrix. Write

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{R})$$

(201

as a product of elementary matrices.

Sol. Elementary Matrix: -

A matrix obtained from a unit matrix by a single elementa transformation is called an elementary matrix (or E-matrix).

Given that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{R})$$

Now $A = I_3 A$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$R_3 \to R_3/2$

$$\begin{pmatrix}1 & 1 & 1\\ 0 & -4 & -1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$R_1 \to R_1 - R_3$

$$\begin{pmatrix}1 & 1 & 0\\ 0 & -4 & -1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$C_2 \to C_2 - C_1$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -4 & -1\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$R_2 \to R_2/4$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1/4\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$C_3 \to C_3 - \frac{1}{4}C_2$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{pmatrix}}_{E_1}\underbrace{\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{E_2}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{pmatrix}}_{E_3}\underbrace{\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{E_4}$$

$$\underbrace{\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{E_5}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1/4\\ 0 & 0 & 1\end{pmatrix}}_{E_6}A$$

So $A = E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}E_6^{-1}E_5^{-1}$

Or $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$

Unit III

Q.1 If a linear transformation associated with a square matrix is nonsingular, then prove that the linear transformations associated with the matrix will be non singular. **(2014, 16)**

Sol. Let $A = [\alpha_{ij}]$ be n×n matrix over a field and $T_1 : V \to V$ be a linear transformation associated with 'A' related to ordered basis -

$$X = \{x_1, x_2, \ldots\ldots, x_n\} \text{ of } V$$

$$\therefore \quad T_1(x_j) = \sum_{i=1}^{n} \alpha_{ij} x_i \qquad \forall j \qquad \ldots\ldots\ldots(1)$$

Let $T_2 : V \to V$ be another linear transformation associated with matrix A relative to basis.

$$X' = \{x'_1, x'_2, \ldots\ldots, x'_n\} \text{ of } V$$

$$\therefore \quad T_2(x_j) = \sum_{i=1}^{n} \alpha_{ij} x'_i \qquad \forall j \qquad \ldots\ldots\ldots(2)$$

Let T_1 be non-singular.

Claim - T_2 is non-singular

$$\Rightarrow \quad X = \{x_j : J = 1, 2, \ldots\ldots, n\} \qquad X = \{x_i; i = 1, 2, \ldots, n\}$$

$$L_1(x_j) = x'_j \qquad \begin{array}{ccc} V & \xrightarrow{T_1} & V \\ \downarrow & & \uparrow \\ V & \xrightarrow[T_2]{} & V \end{array} \qquad L_2(x'_i) = x_i$$

L_1 and L_2 are transition map which is one-one and onto that is non singular.

Now we will show that

$$L_2 0 T_2 0 L_1 = T_1$$

$$L_2 0 T_2 0 L_1(x_j) = L_2 0 T_2(L_1(x_j)) = (L_2 0 T_2)(x_j')$$

$$= (L_2 T_2(x_j')) = L_2\left(\sum_{i=1}^{n} \alpha_{ij} x'_i\right)$$

$$= \sum_{i=1}^{n} \alpha_{ij} L_2(x'_i) = \sum_{i=1}^{n} \alpha_{ij} x_i = T_1(x_j)$$

$\therefore$ from equality of map

$$L_2 0 T_2 0 L_1 = T_1$$

Therefore, T_1 is non-singular.

$\Rightarrow$ $L_2 0 T_2 0 L_1$ is non-singular.

$\Rightarrow$ $L_2 0 T_2$ is non-singular. ($\because$ L_1 is non-singular)

$\Rightarrow$ T_2 is non-singular. ($\because$ L_3 is non-singular)

Q.2. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

and hence write A^{-1} as the product of elementary matrices. **(2014)**

Sol. Given that

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to (R_1 + R_2) - R_3$

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & -3 & -3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & -6 & -5 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_3 \to (R_1 + R_2) + R_3$ $\qquad$ $R_2 \to R_2 + 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$ $\qquad$ $R_1 \to R_1 + 3R_3$ $\qquad$ $R_1 \to R_1 + R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ $\qquad$ $R_1 \to R_1 - 3R_3$ $\qquad$ $R_2 \to R_2 + 2R_3$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to (R_1 + R_2) - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

Q.3. Define row-rank and determinantial rank of matrix and prove that row-rank of a matrix is equal to the determinantial rank of matrix. (2014)

Ans. Row rank: -

Let $A = [\alpha_{ij}]$ be nxn matrix over field F.

$$\text{Therefore } A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots\ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots\ldots\ldots & \alpha_{2n} \\ . & . & & . \\ . & . & & . \\ \alpha_{m1} & \alpha_{m2} & \ldots\ldots\ldots & \alpha_{mn} \end{bmatrix}$$

Rows of A -

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$R_1 = (\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}) \in F^n$

$R_2 = (\alpha_{21}, \alpha_{22}, \ldots\ldots, \alpha_{2n}) \in F^n$

.

.

$R_m = (\alpha_{m1}, \alpha_{m2}, \ldots\ldots, \alpha_{mn}) \in F^n$

$S_1 = \{R_1, R_2, \ldots\ldots R_m\} \in F^n$

$L(S_1) = [S_1]$ = {all possible linear combination of element of S_1}

$[S_1]$ is a subspace of F^n and is called as row-space of 'A' and dimension of row space of 'A' is called row rank of A.

Determinantial-Rank of a matrix: -

An mxn matrix over a field F is said to be of determinantial rank 'd' iff there exist a dxd submatrix D of A such that

$|D| \neq O_F$ and determinants of all submatrices $(d + l) \times (d + 1)$ is zero.

Let 'A' be mxn matrix over a field 'F' and 'd' be the determinantial rank of A and 'r' be rank of matrix.

Claim: - $d = r$

Since 'd' is determinantial rank of A.

$\Rightarrow$ there exists d×d submatrix (D) of A i.e. $D \neq O_F$

$\Rightarrow$ D is non singular.

$\Rightarrow$ $\rho(D) = d$ (Because the rank of non-singular matrix is equal to order of matrix.)

$\Rightarrow$ all d-rows of D are linear independent.

$\Rightarrow$ Those d-rows of matrix 'A' which are contributing the elements of D will be linear independent.

$\Rightarrow$ $d \leq$ .....(i)

Let rank (A) = r

$\Rightarrow$ row rank of A = r

$\Rightarrow$ There are 'r' rows in 'A' which are linear independent. Consider a submatrix 'B' having r-rows of 'A' which are L.I.

$\Rightarrow$ row rank (B) = r

$\Rightarrow$ Coloum rank of B = r

$\Rightarrow$ $\exists$ B_1 of order rxr submatrices of B (hence submatrix of A) by those r-colums of B formed by those coloums B which are L.I.

$\Rightarrow$ $\rho(B_1) = r$

$\Rightarrow$ B_1 is non-singular.

$\Rightarrow$ $|B_1| \neq O_F$

$\Rightarrow$ $r \leq d$ .....(ii)

From eq.(i) and (ii)

$r = d$

Q.4. Define adjoint of a square matrix. If A is nxn $(n \geq 2)$ matrix, th

prove that adj adj $(|A|)^{n-2}$ A. (201

Sol. Case I - When n = 2 -

Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be 2×2

$$\text{adj}A = \begin{bmatrix} \alpha & -\beta \\ -\gamma & \delta \end{bmatrix}$$

$$\text{adj adj}A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = A = (|A|)^{n-2} A$$

Case II - If n > 2 or n ≥ 3

(i) If A is singular: -

$\rho(A) \leq n-1$ and $|A| = 0_F$

$\Rightarrow$ $\rho(\text{adj}A) \leq 1$ and $|A| = 0_F$

$\Rightarrow$ $\rho(\text{adj } A) < n\text{-}1$ $\Rightarrow$ $\rho(\text{adjadj } A) = 0$

$\Rightarrow$ adj adj $A = 0 = 0_F . A = (|0_F|)^{n-2} . A$

$\Rightarrow$ adj adj $A = (|A|)^{n-2} A$

(ii) If A is Non-singular: -

Using adjoint theorem for adj A

adj A . adjadjA = |adjA|In

$\Rightarrow$ A(adj A adjadj A) = A(|adjA|In)

$\Rightarrow$ |A|In(adjadjA) = |adjA|(AIn)

$\Rightarrow$ |A|(adjadjA) = |adjA|.A

$\Rightarrow$ |A|(adjadj A) = $(|A|)^{n-1}$.A

$|A^{-1}||A|(\text{adjadj } A) = |A^{-1}|(|A|)^{n-1} A$

$\Rightarrow$ adjadjA = $(|A|)^{n-2}$.A

Q.5. Reduce the matrix

$$\mathbf{a} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix}$$

(201

Sol. $$a = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix} \qquad R_3 \to R_3 - (R_1 + R_2)$$

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$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \to \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \to C_4 - 2C_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_3 \to C_3 - 3C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \to C_4 + C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \to C_4 - C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is normal form of matrix a.

Hence $\rho(a) = 3$

Q.6. Let $V = \mathbb{R}^3$ and $T : V \to V$ be a linar transformation defined by T(a, b, c) = (a + c, -2a + b, -a + 2b + c).

Find the matrix of T with respect to the basis.

$\{(1,0,1),(-1,1,1),(0,1,1)\}$.

Sol. Given that $v = \mathbb{R}^3$

and $T: V \to V$ be a linear frausformat on defined by

$$T(a,b,c) = (a+c, -2a+b, -a+2b+c) \quad \text{.....(1)}$$

Let $S = \{\alpha_1 = (i,0,1), \alpha_2 = (-1,1,1), \alpha_3 = (0,1,1)\}$

from eq.(1) we have

$$T(\alpha_1) = T(1,0,1) = (2,-2,0)$$

$$T(\alpha_2) = T(-1,1,1) = (0,3,4)$$

$$T(\alpha_3) = T(0,1,1) = (1,1,3)$$

let $(2,-2,0) = x\alpha_1 + y\alpha_2 + z\alpha_3$

$$(2,-2,0) = x-y, y+z, x+y+z)$$

implies $x-y=2, y+z=-2, x+y+z=0$

After solving we get

$$x=2, y=o \text{ and } z=-2$$

$$\therefore T(1,0,1) = (2,-2,0) = 2\alpha_1 - 2\alpha_3$$

Similarly $(0,3,4) = (x\text{-}y, y+z, x+y+z)$

implies $x\text{-}y=0, y+z=3, x+y+z=4$

Solving we get

$$x=1, \quad y=1 \text{ and } z=2$$

$T(-1, 1, 1) = (0, 3, 4) = \alpha_1 + \alpha_2 + 2\alpha_3$

Similarly $(1,1,3) = (x\text{-}y, y+z, x+y+z)$

$$x\text{-}y=1, \quad y+z=1 \quad x+y+z=3$$

Solving we get, $y=1, \quad x=2 \quad z=0$

$$T(0,1,1) = (1,1,3) = 2\alpha_1 + \alpha$$

Hence the matix of T w.r.t

basic $s = \{\alpha_1 = (1,0,1), \alpha_2 = (-1,1,1), \alpha_3 = (0,1,1)\}$

is $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

Q.7. Show that $x+y+z=6, x+2y+3z=14, x+4y+7z=30$ are consiste and solve them. (201

Solve.

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+7z=30$$

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$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(A^*) = 2$

Hence it is consitant

$x + y + z = 6$ .....(1)

$y + 2z = 8$

let $x = k$

then $y + z = 6 - k$

$y + 2z = 8$

$-z = 6 - k - 8$

$-z = -k - 2 \quad \Rightarrow \quad z = k + 2$

Putting the value of x ∫ z in ...(1)

$k + y + k + 2 = 6 \quad \Rightarrow \quad y = 4 - 2k$

Hence $x = k, \quad y = 4 - 2k, \quad z = k + z$

Hence there is infinite no. of Solution.

Q.8. Define the adjoint of a square matrix and prove that adj (AB) = (adj B)(adj A). **(2015)**

Sol. Adjoint of a Square matrix -

Let $A = (a_{ij})$ be an n - square matrix, let C_{ij} be the co-factor of a_{ij} then the transpose of the matrix $[C_{ij}]$ that $[C_{ij}]^t$ is called the adjoint of the matrix A and is written as Adj A. Let A and B are non singular. Therefore A.B is also non-singular

By adjoint theorem for A.B

$(A.B)\text{adj}(A.B) = |A.B|I_n$ .....(1)

Consider

(A.B)(adj B. adj A)

$= A(B \text{ adj } B).\text{ adj } A$

$= A(|B| I_n).\text{ adj } A$ (by adjoint theorem)

$= |B|(A.I_n) \text{ adj } A$

$= |B|(A.\text{adj } A)$

$= |B|(|A|I_n)$ (by adjoint theorem)

$= (|B| \odot |A|) I_n$

$= (|A| \odot |B|) I_n$ (commututive property of field)

$= |AB| I_n$ .....(2)

From eq.(1) and (2)

$(A.B) \text{ adj } (A.B) = (A.B)(\text{adj } B.\text{adj } A)$

$(A.B)^{-1} (AB) \text{ adj } (A.B) = (A.B)^{-1} (A.B)(\text{adj } B.\text{adj } A)$

$\text{adj}(A.B) = \text{adj } B.\text{ adj } A$

Q.9. **Let** $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$. **Find the basis of ker T.** (2012)

Ans. Given that

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$$

since $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the liner transformation ossociated to A with respect to the standard bassis of $\mathbb{R}^3$ and $\mathbb{R}^2$

Therefore

$T(1,0,0) = 2(1,0) + 1(0,1) = (2,1)$

$T(0,1,0) = (1,0) + 1(0,1) = (0,1)$

$T(0,0,1) = -1(1,0) + (-1)(0,1) = (-1,-1)$

$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$T(x,y,z) = x(2,1) + y(0,1) + z(-1,-1)$

$T(x,y,z) = (2x-2, x+y-z)$

Now $\ker T = \{(x,y,z) \in \mathbb{R}^3 . T(x,y,z) = 0\}$

ie:- $2x - z = 0$

$x + y - z = 0$

Let $z = k$ then $x = +\frac{k}{2}$

$$\frac{k}{2} + y - k = 0$$

$$y = \frac{k}{2}$$

So $\ker T = \{(k/2, k/2, k) : k \in \mathbb{Z}\}$

Basis of $\ker T = \left\{\left(\frac{1}{2}, \frac{1}{2}, 1\right)\right\}$

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or basis of ker T = {(1, 1, 2)}

Q.10. Define rows-space of a matrix. Find a basis of the row-space of the matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & -3 & 1 \\ -1 & -2 & -2 & 2 \\ 4 & 6 & 4 & -6 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Hence obtain the rank of A. **(2012)**

Sol. Row space of a matrix: -

Let $A = [a_{ij}]_{m\times n}$ be an m×n matrix over the filed F. The row vectors of A are the vectors $\alpha_1, \ldots\ldots \alpha_m \in V_n(F)$ defined by

$$\alpha_i = (a_{i1}, a_{i2}, \ldots\ldots, a_{in}) \qquad 1 \le i \le m$$

The row space of A is the subspace of $V_n(F)$ spanned by these vectors.

$$A = \begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & -3 & 1 \\ -1 & -2 & -2 & 2 \\ 4 & 6 & 4 & -6 \\ 1 & 3 & 4 & -3 \end{bmatrix} \xrightarrow[\substack{R_3\to R_3+R_1 \\ R_4\to 2R_4+R_1 \\ R_5\to R_5-2R_1 \\ R_6\to 2R_6-R_1}]{\substack{R_2\to 2R_2-R_1}} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & -2 & -6 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -8 & -6 \\ 0 & 2 & 2 & -6 \end{bmatrix}$$

$$\xrightarrow[R_6\to R_6+R_5]{R_2\to 2R_3+R_2} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -8 & -6 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{R_5\to R_4+R_5} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -6 & -2 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 2R_3 + R_5} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow[R_6 \to R_6/6]{R_3 \to R_3/2} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[\text{and interchanging rows}]{R_3 \to R_3 + R_6} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank A = 3

Q.11. Define adjoint of a square matrix and prove that

(i) If A is an n-square non singular matrix, then

adj adj A = $(|A|)^{n-2}$.A

(ii) If A and B are non-singular matrices of the same order, then

adj(A.B) = adj B . adj A **(2012)**

Sol. Adjoint of a square matrix: -

Let $A = [\alpha_{ij}]$ be n×n matrix over a filed F and C_{ij} be the matrix of co-factors of elements of 'A' then the transpose of matrix of the co-factor of

elements of A is called as adjoints of 'A' and it is denoted as A adj A or adj A.

(i) We have to prove that

$$\text{adj adj } A = (|A|)^{n-2}A$$

If A is an n-square non singular matrix then using adjoint theorem, we get

$$\text{adj } A \text{ adj adj } A = |\text{adj } A|\, I_n$$

$$A.(\text{adj } A.\text{adj adj } A) = A(|\text{adj } A|.I_n)$$

$$\Rightarrow |A|\, I_n (\text{adj}.\text{adj}A) = |\text{adj } A|(A.I_n)$$

$$\Rightarrow |A|(\text{adj adj } A) = |\text{adj } A|A$$

$$\Rightarrow |A|(\text{adj adj } A) = (|A|)^{n-1} A$$

$$\Rightarrow \text{adj adj } A = (|A|)^{n-2}.A$$

(ii) We have to prove that

$$\text{adj } (A.B) = \text{adj } B \,.\, \text{adj } A$$

Proof -

Since A and B are non-singular matrix therefore A.B is also non-singular

By adjoint theorem for A.B

$$(A.B)\,\text{adj}\,(A.B) = |A.B|\, I_n \qquad \text{.....(1)}$$

Conside $(A.B)(\text{adj } B \,.\, \text{adj } A) = A \,.\, (B \text{ adj } B) \text{ adj } A$

$$= A\,(|B|\, I_n) \text{ adj } A = |B|(A.I_n) \text{ adj } A$$

$$= |B|(A \,.\, \text{adj } A) = |B|\,(|A|.I_n)$$

$$= \left(|B|\odot A\right) I_n = \left(|A|\odot|B|\right) I_n$$

$$= |\, A.B\,|\, I_n \qquad \text{.....(2)}$$

From eq.(1) and (2) we get

$$(A.B)\,\text{adj}\,(A.B) = (A.B)\,(\text{adj } B \,.\, \text{adj } A)$$

$$\Rightarrow \text{adj}(A.B) = \text{adj } B \,.\, \text{adj } A$$

Q.12. Define trace of a square matrix. If A and B are m×n and n×m matrices, respectively, over a filed, then prove that A.B and B.A have the same trace. Hence prove that similar matrices have the same trace. (2012)

Sol. Trace of a square matrix: -

Let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots\ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots\ldots & \alpha_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \alpha_{n1} & \alpha_{n2} & \ldots\ldots & \alpha_{nn} \end{bmatrix} \quad \text{be } n \times n$$

square matrix over filed $(F, \oplus, \odot)$. Then element $\alpha_{11} \oplus \alpha_{22} \oplus \ldots\ldots \oplus \alpha_{nn}$ is

called trace of matrix and denoted by 'tr(A)'.

Now we have to prove that tr (A.B) = tr (B.A)

Let $A = [\alpha_{ij}]$ be m×n matrix

and $B = [\beta_{jk}]$ be n×m matrix.

where $A.B = [\gamma_{ik}]_{m\times m}$

where $\gamma_{ik} = \sum_{i=1}^{n} \alpha_{ij} \odot \beta_{jk}$

$\therefore$ $\text{trace}(A.B) = \gamma_{11} \oplus \gamma_{22} \oplus ..... \oplus \gamma_{mm}$

$$= \sum_{i=1}^{m} \gamma_{ii} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \odot \beta_{ji} \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \odot \beta_{ji} \qquad .....(1)$$

$B = [\beta_{jk}]_{n\times m}$

$A = [\alpha_{k\ell}]_{m\times n}$

$B.A = [\theta_{j\ell}]_{n\times m}$

where $\theta_{j\ell} = \sum_{k=1}^{m} \beta_{jk} \odot \alpha_{k\ell}$

$$t_r(B.A) = \sum_{j=1}^{n} \theta_{jj} = \sum_{k=1}^{m} (\beta_{jk} \odot \alpha_{kj})$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} \beta_{ji} \odot \alpha_{ij} \qquad .....(2)$$

from eq.(i) and (ii)

$\text{tr}(A.B) = \text{tr}(B.A)$

we have to show that

Similarly matrices has same trace. Let A and B are similar matrices then

$B = PAP^{-1}$

$\text{tr } B = \text{tr}(PAP^{-1}) = \text{tr}(PP^{-1}A)$

$= \text{tr}(In.A)$

$\text{tr } B = \text{tr } A$

Q.13. Find the basis of the solution space of the following system of linear equations over the field of real numbers:

$2x + z - u + v = 0, x + z - u - v = 0, 6x + y + 4z + v = 0.$ **(2012)**

Sol. We have given that

$2x + z - u + v = 0$

$$x+z-u-v=0$$
$$6x+y+4z+v=0$$

$$A=\begin{bmatrix}2 & 0 & 1 & -1 & 1\\ 1 & 0 & 1 & -1 & 1\\ 6 & 1 & 4 & 1 & 0\end{bmatrix}$$

Reduce in echlon form we get

$$\begin{bmatrix}1 & 0 & 1 & -1 & 1\\ 0 & 1 & -2 & 6 & 5\\ 0 & 0 & -1 & 1 & -1\end{bmatrix}$$

Basis of Ker T
$= \{(1,0,1,-1),(0,1,-2,6,5),(0,0,-1,1,-1)\}$

Q.14. Define trace of square matrix. If A is a real matrix such that trace $(A.A^t)=0$, then prove that $A=0$. **(2013)**

Ans. Trace of a square matrix: -

Let A be a square matrix of order n over a field F. The sum of elements of A lying along the principal diagonal is called the trace of A, i.e. if $A=[a_{ij}]_{n\times m}$ then trace $A=\sum_{i=1}^{n} a_{ij}$.

Let $A=\left[a_{ij}\right]_{m\times n}$,

Then $A^t=[b_{ij}]_{n\times m}$, where $b_{ij}=a_{ji}$.
Now $A^t.A$ is a matrix of the type $n\times n$.
Let $A^tA=[C_{ij}]_{n\times n}$. Then
C_{ii} = the sum of the products of A^t and the i^{th} column of A.

$$= b_{i1}a_{1i}+b_{i2}a_{2i}+\ldots\ldots\ldots+b_{im}a_{mi}$$
$$= a_{1i}a_{1i}+a_{2i}a_{2i}+\ldots\ldots\ldots+a_{mi}a_{mi} \qquad [\because b_{ij}=a_{ji}]$$
$$= a_{1i}^2+a_{2i}^2+\ldots\ldots\ldots+a_{mi}^2$$

Now $\text{trace}\left(A^tA\right)=\sum_{i=1}^{n} C_{ii}$

$$=\sum_{i=1}^{n}\left\{a_{1i}^2+a_{2i}^2+\ldots\ldots+a_{mi}^2\right\}$$

= The sum of the squares of all the element of A.

Now the elements of A are real numbers.
Therefore $\text{trace}(A^t.A)=0$
$\Rightarrow$ The sum of the squares of all the elements of A is zero.

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$\Rightarrow$ Each element of A is zero.

$\Rightarrow$ A is null matrix i.e. $A = 0$.

Q.15. Define determinantal rank of a matrix. If A is an n-square matrix of rank n - 1, then prove that the rank of adjoint of A is 1. (2013)

Ans. Determinantal rank of matrix: -

An $m \times n$ matrix over a field F is said to be of determinantal rank 'd' iff there exists a $d \times d$ submatrix D of A such that $|D| \neq O_F$ and determinants of all submatrix $(d+1)\times(d+1)$ is zero.

Let $\rho(A) = n - 1 < n$

$\Rightarrow$ A is singular matrix and $\exists$ atleast one co-factor of element of A different from zero.

$\Rightarrow$ $|A| = O_F$ and $\text{Adj}\, A \neq 0$

$\Rightarrow$ $A.\text{adj}\, A = 0$ and $\text{adj}\, A \neq 0$ $\quad (\because A.\text{adj}\, A = |A| I_n = 0.I_n = 0)$

$\Rightarrow$ $\rho(A.\text{adj}\, A) = 0$ and $\rho(\text{adj}\, A) \neq 0$

$\because$ $\rho(A.\text{adj}A) \geq \rho(A) + \rho(\text{adj}A) - n$

$\Rightarrow$ $0 \geq \rho(A) + \rho(\text{adj}A) - n$ and $\rho(\text{adj}A) \neq 0$

$\Rightarrow$ $0 \geq n - 1 + \rho(\text{adj}A) - n$ and $\rho(\text{adj}A) \neq 0$

$\Rightarrow$ $\rho(\text{adj}A) \leq 1$ and $\rho(\text{adj}A) \neq 0$

$\Rightarrow$ $\rho(\text{adj}A) = 0 \text{ or } 1$ and $\rho(\text{adj}A) \neq 0$

Q.16. Can we express $(1, 4, 5) \in \mathbb{R}^3(\mathbb{R})$ as a linear combination of (1, 1, 2), (2, 3, 5) and (3, 5, 8) in $\mathbb{R}^3$? If yes, in how many ways? (2013)

Sol. Let $(1, 4, 5) = x(1, 1, 2) + y(2, 3, 5) + z(3, 5, 8)$

Then $x + 2y + 3z = 1$

$x + 3y + 5z = 4$

$2x + 5y + 8z = 5$

$$A^* = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 4 \\ 2 & 5 & 8 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A^*) = 2$ (infinite solution)

Therefore $(1, 4, 5) \in \mathbb{R}^3(\mathbb{R})$ can be expressed as a linear combination of {(1, 1, 2), (2, 3, 5) and (3, 5, 8)} in infinite number of ways.

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Q.17. Find the value of k for which the system of linear equation $kx+y+z=1, x+ky+z=k, x+y+kz=k^3$ over the field of real numbers does not have a solution. **(2013)**

Sol. $AX=B$

Where $A^*=\begin{bmatrix}k & 1 & 1\\ 1 & k & 1\\ 1 & 1 & k\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}1\\ k\\ k^2\end{bmatrix}.$

Then $A^*=\begin{bmatrix}k & 1 & 1 & . & 1\\ 1 & k & 1 & . & k\\ 1 & 1 & k & . & k^3\end{bmatrix}$

$$R_1 \leftrightarrow R_3\begin{bmatrix}1 & 1 & k & . & k^3\\ 1 & k & 1 & . & k\\ k & 1 & 1 & . & 1\end{bmatrix}$$

$$\begin{matrix}R_2 \to R_2-R_1\\ R_3 \to R_3-kR_1\end{bmatrix}\begin{bmatrix}1 & 1 & k & . & k^3\\ 0 & k-1 & 1-k & . & k-k^3\\ 0 & 1-k & 1-k^2 & . & 1-k^4\end{bmatrix}$$

$$R_2 \to \frac{1}{k-1}R_2\begin{bmatrix}1 & 1 & k & . & k^3\\ 0 & 1 & -1 & . & -k(k+1)\\ 0 & 1-k & 1-k^2 & . & 1-k^4\end{bmatrix}$$

$$R_3 \to R_3-(1-k)R_2\begin{bmatrix}1 & 1 & k & . & k^3\\ 0 & 1 & -1 & . & -k(k+1)\\ 0 & 0 & (k+2)(1-k) & . & (1+k)(1-k^3)\end{bmatrix}$$

$$R_3 \to \frac{1}{1-k}R_3\begin{bmatrix}1 & 1 & k & k^3\\ 0 & 1 & -1 & -k(k+1)\\ 0 & 0 & (k+2) & (1+k)(1+k+k^2)\end{bmatrix}$$

for $k=-2$

$\rho(A^*)=3$ and $\rho(A)=2$

Since $\rho(A^*)\neq\rho(A)$

Therefore in this case solution does not exist.

Q.18. Define non-singular matrix. Prove that an n-square matrix is non-singular if and only if the rank of the matrix is n. (2013)

Sol. Non-singular Matrix: -

A square matrix over a filed is said to be non-singular iff a linear transformation associated with matrix is non-singular.

Let 'A' be n×n matrix over filed F and $T_A : F^n \to F^n$ be a linear transformation associated with matrix A.

A is a non-singular $\Leftrightarrow$ T_A is non-singular

$\Leftrightarrow$ T_A is one-one

$\Leftrightarrow$ $\ker T_A = \{O_v\}$

$\Leftrightarrow$ $\dim \ker T_A = 0$

$\Leftrightarrow$ $\dim \text{Im}T_A + \dim \ker T_A = \dim F^n$

$\Leftrightarrow$ $\rho(T_A) + 0 = n$

$\Leftrightarrow$ $\rho(A) = n$

Q.19. Define an elementary matrix. Write

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \in M_3(\mathbb{R}).$$ (2017)

Sol. The matrix associated with elementry linear transformation is known as elementry matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \in M_3(\mathbb{R})$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \frac{R_3}{3}$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$R_2 \to \left(-\frac{1}{2}\right)R_2$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$$R_1 \to R_1 - R_2 \quad \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$R_2 \to R_2 + \frac{3}{2}R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -5/6 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{5}{2}R_3$$

t^3 $\qquad$ A^{-1}

A as the product of elementary matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.20. Find the values of a for which the following system of linear equations over $\mathbb{R}$ is consistent:

$x + 3y - z = 1, 2x + 7y + az = 3, x + ay - 7z = 0$ **(2017)**

Sol. Given system

$x + 3y - z = 1,$

$2x + 7y + az = 3,$

$x + ay - 7z = 0$

$AX = B$

Where $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & a \\ 1 & a & -7 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Agumented matrix

$$A^* = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 2 & 7 & a & 3 \\ 1 & a & -7 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -1 & . & 1 \\ 0 & 1 & a+2 & . & 3 \\ 0 & a-3 & -6 & . & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (a-3)R_2} \begin{bmatrix} 1 & 3 & -1 & . & 1 \\ 0 & 1 & a+2 & . & 1 \\ 0 & 0 & 6-(a+2)(a-3) & . & 0 \end{bmatrix}$$

$$(a+2)(a-3)=6 \quad \Rightarrow \quad a^2-a-6=6$$

$$\Rightarrow \quad a^2-a-12=0 \quad \Rightarrow \quad a^2-4a-+3a-12=0$$

$$\Rightarrow \quad a(a-4)+3(a-4)=0 \quad \Rightarrow \quad a=4,-3$$

If $a = 4, -3$ then system is consistant with infinite solution and if $a \neq 4, -3$ the system is consistent with unique solution.

Q.21. Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

and write A^{-1} as the product of elementary matrices. **(2018)**

Sol. $$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ w get-

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{bmatrix} \text{ and } \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$ we get-

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} ; \qquad I_3 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_3 \to (-1) R_3$

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} ; \qquad I_3 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$R_1 \to R_1 - 5R_3$ and $R_2 \to R_2 + 2R_3$ we get-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \qquad I_3 = \begin{bmatrix} -2 & -6 & 5 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

$= I_3$ $\qquad\qquad$ $= A^{-1}$

$$\therefore \quad A^{-1} = \begin{bmatrix} -2 & -6 & 5 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

A and A^{-1} as product of elementary matrices i.e.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

and $$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.22. Reduce the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

to normal form and find its rank. **(2018)**

Sol. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$C_2 \to C_2 + C_1, C_3 \to C_3 + 2C_1, C_4 \to C_4 + 4C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

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$R_3 \to R_3 - 4R_2, R_4 \to R_4 - 9R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$C_3 \to C_3 + 6C_2, C_4 \to C_4 + 3C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_3 \to \frac{1}{33} R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 \to R_4 - 66R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - \frac{2}{3} C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} \end{bmatrix}$$

$\therefore$ $\rho(A) =$ order of I_3

$\therefore$ $\rho(A) = 3$

Q.23. Define the adjoint of a square matrix. If A and B are non-singular

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matrices of the same order over the same field F, then prove that adj(AB) = adj(B) adj(A). **(2018)**

Sol. **Adjoint of a square matrix:-**

Let, $A = [\alpha_{ij}]$ be $n \times n$ matrix over field F and $[c_{ij}]$ be the matrix of co-factor of element A then the transpose of matrix of co-factor of element A will be called as adjoint of the matrix A and it is denoted by adj(A) or adj A i.e.,

adj a **definition** Transpose of matrix of co-factor of elements of A.

If A and B are non-singular matrix of same order over it then

$$\textbf{adj (A.B) = adj B . adj A}$$

Proof:-

Since, A and B are non-singular.

$\therefore$ A.B is also non-singular.

$$\therefore \quad (A.B)\,(\text{adj}\, A.B) = |A.B|\, I_n \qquad \text{.....(i)}$$

Now, $(A.B)\,(\text{adj}\, B.\, \text{adj}\, A) = A\,(B\, \text{adj}\, B)\, \text{adj}\, A$

$$= A\left(|B|.I_n\right)\text{adj}\, A = |B|\left(A.\text{adj}\, A\right)$$

$$= |B| \odot \left(|A|.I_n\right) = \left(|A| \odot |B|\right) I_n$$

$$= |A.B|\, I_n \qquad \text{.....(ii)}$$

From eq.(i) and eq.(ii)

$$(A.B)\,(\text{adj}\, A.B) = (A.B)\,(\text{adj}\, B.\, \text{adj}\, A)$$

$$\Rightarrow \quad (A.B)^{-1}(A.B)(\text{adj}\, A.B) = (A.B)^{-1}(A.B)(\text{adj}\, B.\, \text{adj}\, A)$$

$$\Rightarrow \quad \text{adj}\, A.B = \text{adj}\, B.\, \text{adj}\, A$$

Q.24. Let the rank of the coefficient matrix of a non-homogeneous system of m linear equations in n variables be p. If the system is consistent, the show that the number of linearly independent solutions of the system is n - p + 1. **(2018)**

Sol. Let $AX = B$ .....(i)

Let Y be a solution of $AX_0 = 0$

$$AY = 0 \qquad \text{.....(ii)}$$

Let X_0 be a perticular solution of eq.(i)

$$AX_0 = B \qquad \text{.....(iii)}$$

Claim: -

$(X_0 + Y)$ will be a solution of eq.(i)

For $A(X_0 + Y) = AX_0 + AY = B + 0 = B$

So $(X_0 + Y)$ will be a solution of eq.(i)

Let $Y_1, Y_2, \ldots\ldots Y_{n-p}$ be linearly independent solution of homogeneous system $AX = 0$

Claim -. $X_0, X_0 + Y_1, X_0 + Y_2, \ldots\ldots, X_0 + Y_{n-p}$, are solutions of $A.X = B$.

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$\{X_0 + Y_0 + Y_1, \ldots\ldots, X_0 + Y_{n-p}\}$ is linearly independent subset of F^n -

For this we have to show that X_0 is not a linear combination of $Y_1, Y_2, \ldots, Y_{n-p}$. For if possible let X_0 be a linear combination of $Y_1, Y_2, \ldots\ldots Y_{n-p}$.

$\Rightarrow \quad X_0 = \alpha_1 Y_1 + \ldots\ldots\ldots\ldots + \alpha_{n-p} Y_{n-p}$

$\Rightarrow \quad AX_0 = \alpha_1(AY_1) + \ldots\ldots\ldots + \alpha_{n-p}(AY_{n-p})$

$\Rightarrow \quad AX_0 = 0$

Which is wrong as X_0 is particular solution of $A.X = B$

Therefore, our assumption that X_0 is linear combination of $Y_1, Y_2, \ldots\ldots, Y_{n-p}$ is wrong. So X_0 is not linear combination of $Y_1, Y_2, \ldots\ldots, Y_{n-p}$.

$\therefore \quad \{Y_1, Y_2, \ldots\ldots, Y_{n-p}\}$ is linearly independent subset of F^n and X_0 is not linear combination of $\{Y_1, Y_2, \ldots\ldots, Y_{n-p}\}$.

So, $\{X_0, Y_1, Y_2, \ldots\ldots, Y_{n-p}\}$ is also linearly independent.

$\Rightarrow \quad \{X_0, X_0 + Y_1, \ldots\ldots, X_0 + Y_{n-p}\}$ is also linearly independent.

Q.25. Define the row reduced Echelon form of a matrix. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} \in M_{3\times 4}(\mathbb{R})$$

Find a non-singular matrix $P \in M_3(\mathbb{R})$ such that PA is in row reduced Echelon form. **(2019)**

Sol. Row-Reduced Echelon form of a Matrix: -

An $m \times n$ matrix R is called a row reduced echelon matrix if:

(i) Every row of R which has all its entries O occurs below every row which has a non-zero entry.

(ii) The first non-zero entry in each non-zero row is equal to 1.

(iii) If the first non-zero entry in row i appears in column k_i are zero.

(iv) If r is the number of non zero rows, then $k_1 < k_2 < \ldots\ldots < k_r$.

Given that $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} \in M_{3\times 4}(R)$

$A = IA$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 1 \\ 0 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} A$$

$$R_3 \to 5R_3 - R_2 \quad \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 1 \\ 0 & 0 & 15 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -1 & 5 \end{pmatrix} A$$

So $$P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -1 & 5 \end{pmatrix}$$

Q.27. Define adj(A), the adjoint of a matrix, $A \in M_n(F)$, where F is a field and prove that $A.adj(A) = det(A)I_n = adj(A).A$. **(2019)**

Ans. Adjoint (A): -

Let $A = [\alpha_{ij}]$ be $n \times n$ matrix over a field F and $[C_{ij}]$ be the matrix of cofactor of elements of A. The the transpose of the matrix of cofactor of elements of A is called a adjoint of 'A' and is denoted by 'adjA'.

Now, we prove that

$$A.adjA = adjA.A = |A|.I_n$$

Proof: Let $A = [\alpha_{ij}]$ be $n \times n$ matrix over a field F therefore

$adj[C_{JK}]$ where $C_{JK} = C_{KJ}$ .....(i)

$A.adjA = [\gamma_{iK}]$

Where $\gamma_{iK} = \sum_{J=1}^{n} \alpha_{iJ} \odot C_{JK} = \sum_{J=1}^{n} \alpha_{iJ} \odot C_{KJ} = \delta_{iK} \odot |A|$ using eq.(i)

Therefore

$$A.adjA = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \odot |A| & O_F \odot |A| & \cdots & O_F \odot |A| \\ O_F \odot |A| & 1 \odot |A| & \cdots & O_F \odot |A| \\ O_F \odot |A| & O_F \odot |A| & \cdots & 1 \odot |A| \end{bmatrix}$$

$$= |A| I_n \quad \text{.....(ii)}$$

Similarly $adjA.A = |A|.In$ .....(iii)

From eq.(ii) and (iii)

$$A.adjA = adjA.A = |A|.I_n$$

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Q.28. Define the row-rank and the column-rank of a matrix. Prove that for any field F and $A \in M_{m\times n}(F)$, (row-rank)(A) = (column-rank)(A). (2019)

Sol. Row-rank and Column-rank of matrix: -

Let $A = [\alpha_{ij}]$ be n×n matrix over field F

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \cdot & & & \\ \cdot & & & \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix}$$

Rows of A -

$R_1 = (\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}) \in F^n$

$R_2 = (\alpha_{21}, \alpha_{22}, \ldots\ldots, \alpha_{2n}) \in F^n$

.

.

$R_m = (\alpha_{m1}, \alpha_{m2}, \ldots\ldots, \alpha_{mn}) \in F^n$

$S_1 = \{R_1, R_2, \ldots\ldots, R_m\} \in F^n$

$L(S_1) = [S_1]$ = {all possible linear combination of element of S_1}

$[S_1]$ is a subspace of F^n and is called as row space of A and dimension of row-space of A is called row-rank of A.

Column of A is

$C_1 = (\alpha_{11}, \alpha_{21}, \ldots\ldots, \alpha_{m1}) \in F^m$

$C_2 = (\alpha_{21}, \alpha_{22}, \ldots\ldots, \alpha_{m2}) \in F^m$

.

.

$C_m = (\alpha_{1n}, \alpha_{2n}, \ldots\ldots, \alpha_{mn}) \in F^m$

$S_1 = \{C_1, C_2, \ldots\ldots, C_n\} \in F^m$

$L(S_2)$ = {all possible linear combination of elements of S_2}

is a subspace of F^m and is called column space of A and dimension of column space of A is called column rank of A.

Now - We prove that

(Row-rank) (A) = (Column-rank)(A)

Let $$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1j} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2j} & \ldots & \alpha_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mj} & \ldots & \alpha_{mn} \end{bmatrix}$$

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be m×n matrix over the field $(F, \oplus, \odot)$.

$$R_1 = (\alpha_{11}, \alpha_{12}, \ldots \alpha_{1j} \ldots, \alpha_{1n}) \in F^n$$
$$R_2 = (\alpha_{21}, \alpha_{22}, \ldots \alpha_{2j} \ldots, \alpha_{2n}) \in F^n$$
$$\vdots$$
$$R_m = (\alpha_{m1}, \alpha_{m2}, \ldots \alpha_{mj} \ldots, \alpha_{mn}) \in F^n$$
$$C_j = \{\alpha_{1j}, \alpha_{2j}, \ldots\ldots, \alpha_{mj}\}$$

Claim - Column-rank of A ≤ Row-rank of A

Let dimension of row-rank A = r

$\Rightarrow$ Number of row rank A = r

$\Rightarrow$ Number of elements in a basis of row-space = r

Let $S = \{X_1, X_2, \ldots\ldots, X_r\}$ be the basis of row-space.

$\therefore$
$$X_1 = (\beta_{11}, \beta_{12}, \ldots\ldots \beta_{1j}, \ldots\ldots \beta_{1n})$$
$$X_2 = (\beta_{21}, \beta_{22}, \ldots\ldots \beta_{2j}, \ldots\ldots \beta_{2n})$$
$$\vdots$$
$$X_r = (\beta_{r1}, \beta_{r2}, \ldots\ldots \beta_{rj}, \ldots\ldots \beta_{rn})$$

$\because$ $R_1, R_2, \ldots\ldots, R_n$ belongs to row-space of A.

$\therefore$ It can be uniquelly expressed as a linear combination of elements of S.

$$R_1 = (\alpha_{11}, \alpha_{12}, \ldots \alpha_{1j} \ldots, \alpha_{1n}) = (\lambda_{11}x_1 + \lambda_{12}x_2 + \ldots\ldots + \lambda_{1r}x_r)$$
$$R_2 = (\alpha_{21}, \alpha_{22}, \ldots \alpha_{2j} \ldots, \alpha_{2n}) = (\lambda_{21}x_1 + \lambda_{22}x_2 + \ldots\ldots + \lambda_{2r}x_r)$$
$$\vdots$$
$$R_m = (\alpha_{m1}, \alpha_{m2}, \ldots \alpha_{mj} \ldots, \alpha_{mn}) = (\lambda_{m1}x_1 + \lambda_{m2}x_2 + \ldots\ldots + \lambda_{mr}x_r)$$

Now J^{th} column of A

$$C_j = \begin{bmatrix} \alpha_{1J} \\ \alpha_{2J} \\ . \\ . \\ \alpha_{mJ} \end{bmatrix} = \begin{bmatrix} \lambda_{11} \odot \beta_{1j} \oplus \lambda_{12} \odot \beta_{2j} \oplus \ldots\ldots\ldots \oplus \lambda_{1r} \odot \beta_{rj} \\ \lambda_{21} \odot \beta_{2j} \oplus \lambda_{22} \odot \beta_{2j} \oplus \ldots\ldots\ldots \oplus \lambda_{2r} \odot \beta_{rj} \\ . \\ . \\ \lambda_{m1} \odot \beta_{1j} \oplus \lambda_{m2} \odot \beta_{2j} \oplus \ldots\ldots\ldots \oplus \lambda_{mr} \odot \beta_{mr} \end{bmatrix}$$

$$= \beta_{1j} \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ . \\ . \\ \lambda_{m1} \end{bmatrix} + \beta_{2j} \begin{bmatrix} \lambda_{12} \\ \lambda_{22} \\ . \\ . \\ \lambda_{m2} \end{bmatrix} + \ldots\ldots + \beta_{rj} \begin{bmatrix} \lambda_{1r} \\ \lambda_{2r} \\ . \\ . \\ \lambda_{mr} \end{bmatrix}$$

Y_1 $\quad$ Y_2 $\quad$ Y_r

$= \beta_{1j} Y_1 + \beta_{2j} Y_2 + \ldots\ldots + \beta_{rj} Y_r \quad \forall j$

which are show that all column of A is linear combination of elements $Y_1, Y_2, \ldots Y_r$ which are elements of F^m.

$\therefore$ $\{Y_1, Y_2, \ldots Y_r\}$ generates column space of A.

$\Rightarrow$ dimension of column space of A is always less or equal to r.

i.e. dimension of column space of $A \leq r \leq$ dim row space of A .....(i)

$\Rightarrow$ dim of column space of $A' \leq$ dim of row space of A'

$\Rightarrow$ dim of row space of $A \leq$ dim of column of A .....(ii)

from eq.(i) and (ii)

Column rank of A = Row rank of A.

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Unit IV

Q.1. Using Gram-Schmidt orthonormalization process orthonormalize the ordered basis {(0, 1, 1), (1, 0, 1), (1,1,0)} of $\mathbb{R}^3(\mathbb{R})$. (2014)

Sol. Let $S = \{(0,1,1),(1,0,1),(1,1,0)\}$

$\quad\quad\quad\quad b_1 \quad\quad b_2 \quad\quad b_3$

$$a_1 = \frac{b_1}{\| b_1 \|} = \frac{(0,1,1)}{\sqrt{2}} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$c_2 = b_2 - \langle b_2 a_1 \rangle a_1 = (1,0,1) - \frac{1}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,0,1) - \left(0, \frac{1}{2}, \frac{1}{2}\right) = \left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$a_2 = \frac{c_2}{\|c_2\|} = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1 + \frac{1}{4} + \frac{1}{4}}} = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{6}{4}}}$$

$$= \left(\frac{1}{\sqrt{3/2}}, \frac{-1/2}{\sqrt{3/2}}, \frac{1/2}{\sqrt{3/2}}\right)$$

$$c_3 = b_3 - \langle b_3 . a_1 \rangle a_1 - \langle b_3, a_2 \rangle a_2$$

$$c_3 = (1,1,0) - \frac{1}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{3/2}} - \frac{1}{2\sqrt{3/2}} + 0\right)\left(\frac{1}{\sqrt{3/2}}, \frac{-1/2}{\sqrt{3/2}}, \frac{1/2}{\sqrt{3/2}}\right)$$

$$= (1,1,0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{2\sqrt{3/2}}\right)\left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right)$$

$$= (1,1,0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{2\sqrt{3/2}}\right)\left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right)$$

$$= (1,1,0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right) = \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$$

$$a_3 = \frac{a_3}{\|a_3\|} = \frac{\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)}{\sqrt{\frac{4}{9} + \frac{16}{9} + \frac{4}{9}}} = \left(\frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}}, \frac{2}{\sqrt{24}}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

therefore normalization basis is $= \{a_1, a_2, a_3\}$

$$= \left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$$

Q.2. Define adjoint T* of a linear transformation T on an inner product space (V, <>) and prove that
Ker (T*oT) = Ker T and hence prove that ρ(T*oT) = ρ(T) = ρ (ToT*). (2014)

Sol. Adjoint of a linear trasformation: -

Let T: (V, <>) → (V, <>) be a linear transformation. Then we defined a map.

$T^* : (V, <>) \to (V, <>)$ as

$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$. $\forall x, y \in V$

Then the map T* is called adjoint of T.

Claim : Ker (T*oT) = Ker T

Let $x \in$ Ker (T*oT)

$\Rightarrow$ $(T^*oT)(x) = 0$ $\Rightarrow$ $T^*(T(x)) = 0$

$\Rightarrow$ $\langle x, (T^*oT)(x)\rangle = 0$ $\Rightarrow$ $\langle x, T^*(T(x))\rangle = 0$

$\Rightarrow$ $\|T(x)\|^2 = 0$ $\Rightarrow$ $T(x) = 0$

$\Rightarrow$ $x \in$ kerT

$\text{Ker}(T^*oT) \subseteq \text{Ker } T$ .....(i)

let $x \in$ Ker T

$\Rightarrow$ $T(x) = 0$ $\Rightarrow$ $T^*(T(x)) = 0$

$\Rightarrow$ $(T^*oT)(x) = 0$ $\Rightarrow$ $x \in$ Ker (T*oT)

$\therefore$ $\text{Ker } T \subseteq \text{Ker}(T^*oT)$

From eq.(i) and (ii)

Ker (T*oT) = Ker T .....(iii)

Using Rank-Nullity Theorem for T and (T*oT)

dim Ker T + dim ImT = dim V .....(iv)

dim Ker(T*oT) + dim Im(T*oT) = dim V .....(v)

from eq,(iii), (iv) and (v)

we get ρ(A) = ρ(T*oT)

Q.3. Prove that eign values of a Hermitian Matrix are all real. (2014)

Sol. Suppose A is a Hermitian matrix, λ an eigen value of A and X a corresponding eigen vector. Then

$AX = \lambda X$ .....(i)

Pre multiplying both side by X^θ, we get

$X^\theta AX = \lambda X^\theta X$ .....(ii)

Taking conjugate transpose of both sides of eq.(ii), we get

$(X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$

or $X^{\theta}A^{\theta}(X^{\theta})^{\theta} = \bar{\lambda}\, X^{\theta}(X^{\theta})^{\theta}$

or $X^{\theta}AX = \bar{\lambda}\, X^{\theta}X$ .....(iii)

[$\because (X^{\theta})^{\theta} = X$ and $A^{\theta} = A$, A being Hermitian]

From eq.(ii) and (iii) we get

$$\lambda X^{\theta}X = \bar{\lambda}\, X^{\theta}X \quad \Rightarrow \quad (\lambda - \bar{\lambda})X^{\theta}X = 0$$

But X is not a zero vector,

Therefore $X^{\theta}X \neq 0$

Hence $\lambda - \bar{\lambda} = 0$, so that $\lambda = \bar{\lambda}$ and cosequently λ is real.

Q.4. Prove that eigen vectors of a Hermitian matrix corresponding to distinct eigen values of the matrix are orthogonal. **(2014)**

Sol. Given that A is Hermitian.

i.e. $A^{\theta} = A$

Also, we have

$$Ax_1 = \lambda_1 x_1 \text{ and } Ax_2 = \lambda_2 x_2$$

or $\lambda_1 x_1^{\theta}\left(x_2^{\theta}\right)^{\theta} = \lambda_1 x_1^{\theta} x_2$ $\left[\bar{\lambda}_1 \text{ is real } \Rightarrow \lambda_1^{\theta} = \lambda\right]$

or $\lambda_1 x_1^{\theta} x_2 = \lambda_2 x_1^{\theta} x_2 \quad \Rightarrow \quad (\lambda_1 - \lambda_2)x_1^{\theta}x_2 = 0$

Since $\lambda_1 - \lambda_2 \neq 0$,

we have $x_1^{\theta}x_2 = 0$

$\Rightarrow$ x_1 and x_2 are orthogonal with respect to each other.

Q.5. Prove that the determinant of a skew-symmetric matrix of odd order is zero. **(2014)**

Sol. Suppose that n is an odd integer and let A be an n×n skew symmetric matrix.

Thus we have $A^T = -A$ .....(i)

Then $\det(A) = \det(A^T) = \det(-A)$ from eq.(i)

$= (-1)^n \det(A) = -\det(A)$ Since n is odd

Therefore $2\det(A) = 0$ and

Hence $\det(A) = 0$

Q.6. If A is real skew-symmetric matrix such that $A^2 + I_n = 0$, then prove that A is of even order. **(2014)**

Ans. Given that A is real skew symmetric matrix such that

$$A^2 + I_n = 0 \quad \Rightarrow \quad A^2 = -I_n$$

$$|A|^2 = (-1)^n$$

So, $|A|^2 = -1$ Only if $|A|$ is pure imaginary.

In case of skew symmetric matrices. Since every skew-symmetric matrix of odd

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order is always singular.

$$|A|^2 = \pm 1$$

$$|A| \neq 0$$

Hence A is of even order.

Q.7. Define inner product on a vector space V(F). Let V(C) be the vector space of all continuous complex-valued functions on the unit interval $0 \leq t \leq 1$ If $f(t), g(t) \in V$, Let us define

$$f(t).g(t) = \int_0^1 f(t)\overline{g(t)}dt$$

Show that (f(t),g(t)) is an inner product on V(C). **(2015)**

Sol. Innerproduct space: -

Let V be a vector space over field of complex number (or real no.) Then an inner product in V is a map i.e. $<>: V \times V \to C$ which assigns each ordered pair (a,b) of $V \times V$ to a unique scalar (complex no.)

$<a.b.>$ or a.b satisfying the fallowing conditions

(1) $<a+b,c> = <a.c> + <b,c> \ \forall \ a,b,c \in V$

(2) $<\lambda a, b> = \lambda <a.b> \ \forall \ \lambda \in \subset, \ \forall \ a,b, \in V$

(3) $< a, b> = \overline{\langle b,a \rangle}, \ \forall \ a,b, \in V$

(4) $< a, a > \geq 0$ and $< a,a > = 0$ iff $a = 0 \ \forall \ A \in V$ and $(v,<>)$ is called an inner product space

Given that $(f(t).g(t)) = \int_0^1 f(t).\overline{g(t)}dt \quad 0 \leq t \leq 1$ .....(1)

We have to show that all the properties of an inner product hold in eq.(i)

(1) Conjugate symmetry: -

We have

$$(g(t),f(t)) = \int_0^1 g(t)\overline{f(t)}dt \qquad \text{from eq.(1)}$$

$$\therefore \quad \overline{(g(t).f(t))} = \left[\overline{\int_0^1 g(t)\overline{f(t)}dt}\right] = \int_0^1 \left[\overline{g(t)\overline{f(t)}}\right]dt$$

$$= \int_0^1 \overline{g(t)}f(t)dt = \int_0^1 f(t)\overline{g(t)} = [f(t),g(t)]$$

(2) Linearity: -

Let $a, b \in c$ and $h(t) \in V$.

Then $[af(t) + bg(t), h(t)] = \int_0^1 (af(t) + bg(t))\overline{h(t)}dt$

$$= a\int_0^1 f(t)\overline{h(t)}dt + b\int_0^1 g(t)\overline{h}(t)dt$$

$$= a(f(t), h(t)) + b(g(t), h(t))$$

Non - negativity:-

We have

$$(f(t), f(t)) = \int_0^1 f(t)\overline{f(t)}dt = \int_0^1 |f(t)|^2 \, dt \qquad \text{.....(2)}$$

Since $[f(t)]^2 \geq 0$ for every t lying in the closed interval $[0,1]$. Therefore eq.(2) ≥ 0. Thus $(f(t), f(t)) \geq 0$

Also $(f(t), f(t) = 0$

$$\Rightarrow \quad \int_0^1 [f(t)]^2 = 0$$

$$\Rightarrow \quad [f(t)]^2 = 0 \text{ for every t lying in } [0,1]$$

$$\Rightarrow \quad f(t) = 0 \quad \text{for every t lying in } [0,1]$$

Hence the product defined eq. (1) is an inner product on V (c).

Q.8. If a, b are vectors in an inner product space V, prove that

(i) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

(ii) $\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\left(\|\alpha\|^2 + \|\beta\|^2\right)$. **(2015)**

Sol. (i). $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ **(triangle inequality)**

Proof:- We have

$$\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta)$$

$$= (\alpha, \alpha + \beta) + (\beta, \alpha + \beta)$$

$$= (\alpha, \alpha) + (\alpha, \beta) + (\beta, \alpha) + (\beta, \beta)$$

$$= \|\alpha\|^2 + (\alpha, \beta) + (\alpha, \beta) + \|\beta\|^2$$

$$= \|\alpha\|^2 + (\gamma + \bar{\gamma}) + \|\beta\|^2 \qquad \text{Where } \gamma = (\alpha, \beta) \in \mathbb{C}$$

$$= \|\alpha\|^2 + 2\,\text{Re}\,\gamma + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2|\gamma| + \|\beta\|^2 \qquad \therefore (\text{Re}\,\gamma \leq |\gamma|)$$

$$= \|\alpha\|^2 + 2|(\alpha, \beta)| + \|\beta\|^2$$

$$= \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2 \text{ (by Cauchy - Schwarz inequality)}$$

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$$= \|\alpha\| + \|\beta\|^2$$

$\therefore$ $\|\alpha+\beta\|^2 \le (\|\alpha\| + \|\beta\|)^2$

Hence $\|\alpha+\beta\| \le \|\alpha\| + \|\beta\|$

(ii) $\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2(\|\alpha\| + \|\beta\|^2)$ **(Parallelogram law)**

Proof:- We have

$$\|\alpha+\beta\|^2 = (\alpha+\beta, \alpha+\beta)$$

$$= (\alpha, \alpha+\beta) + (\beta, \alpha+\beta)$$

$$= (\alpha,\alpha) + (\alpha,\beta) + (\beta,\alpha) + (\beta,\beta) \quad \text{.....(1)}$$

And $\|\alpha-\beta\|^2 = (\alpha-\beta, \alpha-\beta)$

$$= (\alpha, \alpha-\beta) - (\beta, \alpha-\beta)$$

$$= (\alpha,\alpha) - (\alpha,\beta) - [(\beta,\alpha) - (\beta,\beta)]$$

$$= (\alpha,\alpha) - (\alpha,\beta) - (\beta,\alpha) + (\beta,\beta) \quad \text{.....(2)}$$

Adding eq.(1) and (2) we get

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2(\alpha,\alpha) + 2(\beta,\beta)$$

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\left(\|\alpha\|^2 + \|\beta\|^2\right)$$

Q.9. State and prove Bessel's inequality for an inner product space. (2015)

Ans. Bassel is inequality:-

If V is an inner product space and if $\{w_1 w_2 .. w_n\}$ is an orthonormal set in V. Then

$$\sum_{i=1}^{n} |(w_i, V)|^2 \le \|(V)\|^2, \text{ for all } v \in V$$

Proof:- Let $v \in V$ be arbitrary consider the vector

$$x = V - \sum_{i=1}^{n} \alpha_i w_i \qquad \text{where } \alpha_i = (v, w_i), \qquad \text{.....(1)}$$

Then $(x,x) = \left(v - \sum_{i=1}^{n} \alpha_i w_i, v - \sum_{j=1}^{n} \alpha_j w_j\right)$

$$= (v,v) - \left(v, \sum_{j=1}^{n} \alpha_j w_j\right) - \left(\sum_{i=1}^{n} \alpha_i w_i, v\right) + \left(\sum_{i=1}^{n} \alpha_i w_i, \sum_{j=1}^{n} \alpha_j w_j\right)$$

$$= \|v\|^2 - \sum_{j=1}^{n} \alpha_j (v, w_j) - \sum_{i=1}^{n} \alpha_i (w_i, v) + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \bar{\alpha}_j (w_i, w_j)$$

$$= \|v\|^2 - \sum_{j=1}^{n} \overline{(v, w_j)}(v, w_i) - \sum_{i=1}^{n} (v, w_i)\overline{(v, w_i)} + \sum \alpha_i \bar{\alpha}_i .1$$

$$\left[\because (w_i w_j) = 0 \quad \text{for } i \neq j \text{ and } (w_i w_i) = 1\right]$$

$$= \|v\|^2 - \sum_{i=1}^{n} |(v, w_i)|^2 - \sum_{i=1}^{n} |(v, w_i)|^2 + \sum_{i=1}^{n} |(v, w_i)^2|$$

$$= \|v\|^2 - \sum_{i=1}^{n} \left|\overline{(v, w_i)}\right|^2 = \|v\|^2 - \sum_{i=1}^{n} |(v, w_i)|^2 \qquad (\because |z| = |\bar{z}|)$$

$$\therefore \quad \|x\|^2 = \|v\|^2 - \sum_{i=1}^{n} |(w_i, v)|^2 \qquad \text{.....(2)}$$

Since $\|x\|^2 \geq 0$,

so by eq.(2)

$$\|v\|^2 - \sum_{i=1}^{n} |(w_i, v)|^2 \geq 0 \quad \Rightarrow \quad \|v\|^2 \geq \sum_{i=1}^{n} |(w_i, v)|^2$$

Hence $$\sum_{i=1}^{n} |(w_i, v)|^2 \leq \|v\|^2, \quad \forall v \in V$$

Q.10. Define eigen values and eigen vectors of a square matrix A. Prove that the eigen vectors corresponding to distinct eigen values of a matrix are linearly independent. **(2015)**

Ans. Eigen valuves and eigen vectors of a square matrix matrix:-

To show eigen vector corresponding to distinet eigen valeues of a matrix are linealy independen

Let v_1, v_2 be eigen vectors corresp nding to eigen valies $\lambda_1 \lambda_2$ respectively.

To show that the vectors v_1, v_2 are ℓ.I. consider linear combination

$$c_1 v_1 + c_2 v_2 = 0 \qquad \text{....(1)}$$

Where c_1 and c_2 are complex numbers,

We have to show that $c_1 = c_2 = 0$

By definition of eigen values and eigen vectors

$$Av_1 = \lambda_1 v_1 \text{ and } Av_2 = \lambda_2 v_2$$

$$0 = A.0 = A(c_1 v_1 + c_2 v_2)$$

$$= c_1 Av_1 + c_2 Av_2$$

$$0 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \quad \ldots(2)$$

Multiplying by λ_2 in eq. (1) we get

$$0 = c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 \quad \ldots(3)$$

Subtracting (1) from (3) we get

$$0 = (c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2) - (c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2)$$

$$0 = c_1(\lambda_2 - \lambda_1) v_1$$

$$\Rightarrow \quad c_1(\lambda_2 - \lambda_1) = 0 \qquad (\text{Since } v_1 \neq 0)$$

Since λ_1 and λ_2 are distinct we have

$$c_1 = 0$$

Putting c_1 in eq.(1) we get

$$c_2 = 0 \qquad (\text{since } v_2 \neq 0)$$

Therefore, the values of c_1 and c_2 are both zero, and hence the eigen vector v_1, v_2 are lirearly independent.

Q.11. Prove that two eigen vectors of Skew-Hermitian matrix corresponding to two distinct eigen values of the matrix are orthogonal. **(2012)**

Sol. Let A is skew Hermitian matrix. Then iA is Hermitian.

Let u and v be eigen values and λ and μ be eigenvectors respectively.

$$Au = \lambda u, \; Av = \mu v$$

$$(iA)u = (i\lambda)u, \qquad (iA)v = (i\mu)v$$

Take $(i\lambda)u^*v = (i\lambda u)^*v = ((iA)u)^*v$

$$= u^*(iA)^*v = u^*(-i)A^*v = (-i)u^*(-A)v \qquad \{\because A^* = -A\}$$

$$= i\,u^*Av = iu^*\mu v \qquad \{\because Av = \mu v\}$$

$$\Rightarrow \quad (i\lambda)u^*v = i\mu u^*v$$

$$\Rightarrow \quad i(\lambda - \mu)u^*v = 0 \quad \Rightarrow \quad u^*v = 0$$

$$\Rightarrow \quad u \text{ and } v \text{ are orthogonal to each other.}$$

Q.12. If $S = \{x_1, x_2, \ldots\ldots x_n\}$ is an orthogonal subset of an inner product space $(V, <>)$, then for $x \in V$ prove that

$$\sum_{i=1}^{n} |\langle x, x_i \rangle|^2 = \|x\|^2$$

(2012)

if and only if S is a basis of V.

Sol. **Sufficient Part: -**

Let $X = \{x_1, x_2, \ldots\ldots, x_n\}$ be an orthogonal basis of $(V, <>)$.

$$\therefore \quad x \in V \quad \Rightarrow \quad x = \alpha_1 x_1 + \alpha_1 x_1 + \ldots\ldots\ldots + \alpha_n x_n$$

$$\therefore \quad <x, x> = <\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots\ldots + \alpha_n x_n, \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots\ldots + \alpha_n x_n>$$

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$$= \alpha_1\bar{\alpha}_1 + \alpha_2\bar{\alpha}_2 + \ldots\ldots\ldots + \alpha_n\bar{\alpha}_n$$

$$= |\alpha_1|^2 + |\alpha_2|^2 + \ldots\ldots\ldots + |\alpha_n|^2$$

$$\|x\|^2 = |\langle x, x_1\rangle|^2 + |\langle x, x_2\rangle|^2 + \ldots\ldots\ldots + |\langle x, x_n\rangle|^2$$

Necessary Part: -

Let $\|x\|^2 = \sum_{i=1}^{n} |\langle x, x_i\rangle|^2$ .....(i)

Then we will show that x will be basis of IPS (V, <>).

$\because$ X is an orthonormal set which is linearly independent.

Let $x \in V$ and

$y = x - \langle x, x_1\rangle x_1 \ldots\ldots\ldots\ldots \langle x, x_n\rangle x_n$ .....(ii)

$$\|y\|^2 = \|x\|^2 - \sum_{i=1}^{n} |\langle x, x_i\rangle|^2 = 0 \quad \text{(using eq.(i))}$$

$\Rightarrow \quad y = 0$

$\Rightarrow \quad x = \langle x, x_1\rangle x_1 + \ldots\ldots\ldots\ldots + \langle x, x_n\rangle x_n$

$\Rightarrow \quad x = \alpha_1 x_1 + \ldots\ldots\ldots\ldots + \alpha_n x_n$

$\therefore \quad V = [X]$

Therefore X is basis.

Q.13. Define eigen value of a square matrix. Let $A = [a_{ij}]$ be an n-square real matrix such that $|a_{ij}| \le m, \forall_i = 1, 2 \ldots\ldots n$ and λ be an eigen value of A. Then prove that

$$|\lambda| \le mn$$ **(2012)**

Ans. Eigen Value of a square matrix: -

Let $A = [a_{ij}]_{n\times n}$ be any n-rowed square matrix and λ an indeterminate. Then the equation $|A - \lambda I| = 0$ is called the characteristic equation of A and the roots of this equation are called the eigen values of the matrix A.

Given that $A = [a_{ij}]$ be an n-square real matrix such that $|a_{ij}| \le m$

$\forall i = 1, 2, \ldots\ldots n$ we know that

Trace of matrix = Sum of eigen values

Then $\sum a_{ii} \le \dfrac{m + m + \ldots\ldots + m}{n - \text{time}} \le |mn|$

or trace of matrix $\le |mn|$

or sum of eigen value $\le |mn|$

or $|\lambda| \leq mn$.

(Hence Proved)

Q.14. Define the orthogonal components $S^{\perp}$ of a non-empty subset S of an inner product space. If $\{(1, 1, 2, 0), (1, 1, 0, 3)\}$ generates the subspace W of Euclidean space $\mathbb{R}^4$ then find an orthogonal basis of $W^{\perp}$. (2013)

Sol. Orthogonal complement of a non empty set of an inner product space: -

Let $S \neq \phi$ be a subset of an IPS $(V, < >)$. Then orthogonal complement of "S" in "V" is collection of all those elements of "V" which are orthogonal elements of "V" which are orthogonal to every element of "S" and it is denoted by "$S^{\perp}$".

i.e. $S^{\perp} \underline{\underline{def}} \{x \in V :< X, s >= 0, \forall s \in S\}$

Since $S = \{(1, 1, 2, 0), (1, 1, 0, 3)\}$ is linear independent.

$\therefore$ S is a basis of W.

$$\dim W = 2$$

Since $$\dim V = \dim W + \dim W^{\perp}$$

$$\dim R^4 = 2 + \dim W^{\perp}$$

$$4 = 2 + \dim W^{\perp}$$

$\Rightarrow$ $$\dim W^{\perp} = 2$$

Let $(x, y, z, u) \in W^{\perp}$

$$<(x, y, z, u), (1, 1, 2, 0)> = 0$$

and $$<(x, y, z, u), (1, 1, 0, 3)> = 0$$

$\Rightarrow$ $$x + y + 2z = 0$$

$$x + y + 3u = 0$$

$$AX = 0$$

Where $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$

$$R_2 \to R_2 - R_1 \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

The above equation are equivalent to

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + 2z = 0$$

$$-2z+3u=0 \quad \Rightarrow \quad z=\frac{3}{2}u$$

$$x=-y-3u$$

$$X=\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}=\begin{bmatrix} -1.y-3.u \\ 1.y+0.u \\ 0.y+\frac{3}{2}.u \\ 0.y+1.u \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}y+\begin{bmatrix} -3 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}u$$

Therefore linear independent solution is (-1, 1, 0, 0) and (-3, 0, 3/2, 1) or (-1, 1, 0, 0), (-6, 0, 3, 2).

Therefore $S_1 = \{(-1, 1, 0, 0), (-6, 0, 3, 2)\}$ is basis of $W^{\perp}$.

Now orthonormalize above basis by Gram-Schmidt orthonormalization process we get.

$$S_2=\left\{\frac{1}{\sqrt{2}}(-1,1,0,0),\frac{1}{\sqrt{49}}(-6,0,3,2)\right\}$$

Q.15. Define eigen values and eigen vectors of a linear operator on a vector space. Find eigen values and corresponding eigen-vectors of the linear operator T on a vector space $\mathbb{R}_2[x]$ of all real poynomial of degree at most 2, given by $T(a_0+a_1x+a_2x^2)=(3a_0-2a_1)-(2a_0-3a_1)x+4a_2x^2 \forall (a_0+a_1x+a_2x_2) \in \mathbb{R}_2[x]$.

Sol. Eigen value and eigen vectors of a linear operator: -

For a given linear operator $T: V \to V$, a non zero vector X and a constant scalar λ are called an eigen vector and its eigen values respectively when $T(X)=\lambda X$.

Let $\{1, x, x^2\}$ be basis of $R_2[x]$.

Then $T(1)=3-2x=3.1-2x+0.x^2$

$T(x)=-2+3x=-2+3x+0.x^2=-2.1+3.x+0.x^2$

$T(x^2)=4x^2=0.1+0.x+4.x^2$

matrix of T

$$=\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}=A(\text{say})$$

Then characteristics equation is given by

$|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}=0$$

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$$(4-\lambda)[(3-\lambda)^2-4]=0$$

$$(4-\lambda)(9+\lambda^2-6\lambda-u)=0 \quad \Rightarrow \quad (4-\lambda)(\lambda^2-6\lambda+5)=0$$

$$(4-\lambda)(\lambda-5)(\lambda-1)=0$$

$$\lambda=1,4,5$$

Eigen vector corresponding to eigen values $\lambda = 1$.

$$(A-I)X=0 \qquad \text{where } X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{bmatrix} 3-1 & -2 & 0 \\ -2 & 3-1 & 0 \\ 0 & 0 & 4-1 \end{bmatrix} X=0 \quad \text{i.e.} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} X=0$$

$$2x_1-2x_2=0$$

$$-2x_1+2x_2=0$$

$$3x_3=0 \quad \Rightarrow \quad x_3=0$$

$$2x_1=2x_2 \quad \Rightarrow \quad x_1=x_2$$

Then $X=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Similarly we find eigen vectors corresponding to eigen values $\lambda = 4$ and $\lambda = 5$.

$$X=\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } X=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ respectively.}$$

Q.16. Define adjoint of a linear operator. Find the adjoint of the linear operator T_A on $M_n(F)$ (Set of all n×n matrices over the field F) given by $T_A(X) = A.X \; \forall \; X \in M_n(F)$, where A is a fixed matrix in $M_n(F)$. (2013)

Sol. Adjoint of a linear operator: -

Let $T : (V, <>) \to (V, <>)$ be a linear transformation. Then we defined a map $T^* : (V, <>) \to (V, <>)$ as

$$<T(x), y> = <x, T^*(y)> \quad \forall \; x, y \in V$$

Then the map T* is called adjoint of T.

Given that

$$T(X) = A.X, \qquad \forall x \in M_n$$

We have

$$<T(x), y> = <x, T^*(y)> \qquad \forall x, y \in M_n(F) \qquad \text{.....(i)}$$

Also $<T(x), y> = <Ax, y> = tr(AXY^*)$

$= tr(XY^*A) = tr(X(A^*Y)^*) = \langle x, A^*Y \rangle$ .....(ii)

From eq.(i) and (ii) we get

$T^*(y) = A^*Y$ or $T^*(X) = A^*X$

Q.17. If T* is the adjoint of a linear operator T on the inner product space (V, <>) over the field of complex numbers, then prove that rank of T* = rank of T. **(2013)**

Sol. First we prove that

$Im T^* = (Ker\ T)^\perp$ .....(i)

$T : V \to V; \quad T^* : V \to V$

$\dim W + \dim W^\perp = \dim V$ (W is subspace of V)

$\dim \ker T + \dim (Ker\ T)^\perp = \dim V$

$\Rightarrow \quad \dim \ker T + \dim (Im T^*) = \dim V$ (using (1))

$\Rightarrow \quad \dim \ker T + \rho(T^*) = \dim V$ .....(ii)

Using Rank-Nullity theorem for T we have

$\Rightarrow \quad \dim \ker T + \rho(T) = \dim V$ .....(iii)

Form eq.(ii) and (iii)

$\rho(T^*) = \rho(T)$

Q.18. If S is a real skew-symmetric matrix such that $S^2 + I = 0$, then Prove that S is an orthogonal matrix of even order. **(2013)**

Sol. S is real skew symmetric matrix then $-S = S^t$

or $SS^t = -S.S \quad \Rightarrow \quad SS^t = -S^2$

$\Rightarrow \quad SS^t = -S^2 = I \quad \{\text{given } S^2 + I = 0 \Rightarrow S^2 = -I\}$

$\Rightarrow \quad SS^t = I \quad \Rightarrow$ S is orthogonal.

Since $S^2 + I_n = 0 \quad \Rightarrow \quad S^2 = -I_n$

$(\det(S))^2 = (-1)^n$

L.H.S. is square quantity and must be positive because of S have real entries. Then n must be even. (Proved)

Q.19. State and prove the Bessel's inequality for an inner product space. **(2018)**

Sol. Bessel's Inequlity: -

If $\{x_1, x_2, ..., x_n\}$ is an orthonormal subset of an inner product space (V, <>), then any $x \in V$, we have -

$$|\langle x, x_1 \rangle|^2 + |\langle x, x_2 \rangle|^2 + ... + |\langle x, x_n \rangle|^2 \leq \|x\|^2$$

Or $$\sum_{i=1}^{n} |x, x_i|^2 \leq \|x\|^2$$

Proof: -

Let us consider

$$y = x - \langle x, x_1 \rangle x_1 - \langle x, x_2 \rangle x_2, ... - \langle x, x_n \rangle x_n$$

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$$= x - \sum_{i=1}^{n} < x, x_i > x_i$$

$$\Rightarrow \quad y = x - \sum_{i=1}^{n} \alpha_i x_i \qquad \text{.....(i)} \qquad \text{where } \alpha_i = <x, x_i>$$

Now $$\|y\|^2 = <y, y> = \left\langle x - \sum_{i=1}^{n} \alpha_i x_i, x - \sum_{i=1}^{n} \alpha_i x_i \right\rangle$$

$$= \langle x, x \rangle - \left\langle x, \sum_{i=1}^{n} \alpha_i x_i \right\rangle - \left\langle \sum_{i=1}^{n} \alpha_i x_i, x \right\rangle + \left\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i x_i \right\rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} \bar{\alpha}_i \langle x, x_i \rangle - \sum_{i=1}^{n} \alpha_i \langle x, x_i \rangle + \sum_{i=1}^{n} \alpha_i \bar{\alpha}_i \langle x_i, x_i \rangle$$

$$\Rightarrow \quad \|y\|^2 = \|x\|^2 - \sum_{i=1}^{n} \overline{\langle x, x_i \rangle} \langle x, x_i \rangle - \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle x, x_i \rangle} + \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle x, x_i \rangle}$$

$$\left\{ \begin{array}{ll} \because & \alpha_i = \langle x, x_i \rangle \\ \text{And} & \langle x, x_i \rangle = \| x_i \|^2 = 1 \end{array} \right\}$$

$$\Rightarrow \quad \|y\|^2 = \|x\|^2 - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2$$

Since $\|y\|^2 \geq O_F$

$$\Rightarrow \quad \|x\|^2 - \sum_{i=1}^{n} |\langle x, x_i \rangle|^2 \geq O_F \quad \Rightarrow \quad \sum_{i=1}^{n} |\langle x, x_i \rangle|^2 \leq \|x\|^2$$

Q.20. Let W be a subspace of a finite dimensional inner product space V, Prove that $V = W \oplus W^{\perp}$. **(2018)**

Sol. Proof: -

Since, W is a subspace of a finite dimensional I.P.S. $(V, <>)$.

$\therefore$ W must be a finite dimensional and posses an orthonormal basis.

Let $S_1 = \{a_1, a_2, ..., a_m\}$ be an orthonormal basis of W.

$\therefore$ $\dim W = m$ .....(i)

Since, S_1 is a linearly independent subset of V. So,by extension theorem it can be extendend to form an orthonormal basis of V.

Let $S_1 = \{a_1, a_2, ..., a_m, b_1, b_2, ..., b_n\}$ is an orthogonal basis of V. .....(ii)

i.e. $\dim V = m + n$

Claim - $b_1, b_2, ... b_n \in W^{\perp}$

Since S_2 is an orthonormal basis of V.

$b_1, b_2, \ldots\ldots b_n$ are orthonormal to $a_1, a_2, \ldots\ldots, a_m$.

$\Rightarrow$ $b_1, b_2, \ldots\ldots b_n$ are orthogonal to every linar combination of $a_1, a_2, \ldots\ldots, a_m$.

Hence $b_1, b_2, \ldots\ldots b_n$ are orthogonal to every element of W(or S_1).

Hence $b_1, b_2, \ldots\ldots b_n \in W^{\perp}$

Now for $V = W \oplus W^{\perp}$

We have to show that

(i) $V = W + W^{\perp}$ (ii) $W \cap W^{\perp} = \{0\}$

(i) $V = W + W^{\perp}$ -

Let $x \in V$

$\Rightarrow$ $x = \langle x, a_1\rangle a_1 + \langle x, a_2\rangle a_2 + \ldots\ldots + \langle x, a_m\rangle a_m$

$+ \langle x, b_1\rangle b_1 + \langle x, b_2\rangle b_2 + \ldots\ldots + \langle x, b_n\rangle b_n$

$\Rightarrow$ $x = \underbrace{\alpha_1 a_1 + \alpha_2 a_2 + \ldots\ldots + \alpha_m a_m}_{\in W} + \underbrace{\beta_1 b_1 + \beta_2 b_2 + \ldots\ldots + \beta_n b_n}_{\in W^{\perp}}$

$\Rightarrow$ In every $x \in V$ can be written as one element of W plus one element of $W^{\perp}$.

$\Rightarrow$ $V = W + W^{\perp}$

(ii) $W \cap W^{\perp} = \{0\}$

Let $x \in W \cap W^{\perp}$ $\Rightarrow$ $x \in W$ and $x \in W^{\perp}$

$\Rightarrow$ $x \perp x$ $\Rightarrow$ $\langle x, x\rangle = O_F$

$\Rightarrow$ $x = 0$ $\therefore$ $W \cap W^{\perp} = \{0\}$

(Proved)

thus $V = W \oplus W^{\perp}$

Q.21. Let $S = \{(1, 1, 0), (1, 3, 0), (1, 1, 1)\} \subseteq \mathbb{R}^3$. Use Gram-Schmidt process to orthonormalize S. **(2019)**

Sol. $S = \{(1,1,0)(1,3,0),(1,1,1)\} \subseteq \mathbb{R}^3$

Let $V_1 = (1, 1, 0), V_2 = (1, 3, 0), V_3 = (1, 1, 1)$

Let $w_1 = V_1 = (1, 1, 0)$ $\Rightarrow$ $\|w_1\|^2 = (w_1, w_2)$

$= 1.1 + 1.1 + 0.0 = 1 + 1 = 2$

$\therefore$ $\dfrac{w_1}{\|w_1\|} = \dfrac{1}{\sqrt{2}}(1,1,0) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right)$

Let $w_2 = V_2 - \dfrac{(V_2, w_1)w_1}{\|w_1\|^2}$ .....(i)

$(V_2, \omega_1) = (V_2, V_1) = 1.1 + 3.1 + 0.0 = 1 + 3 = 4$

from eq.(i)

$$w_2 = (1,3,0) - \frac{1}{2}(1,1,0)4 = (1,3,0) - (2,2,0) = (-1,1,0)$$

$$\Rightarrow \quad \|w_2\|^2 = (w_2, w_2) = (-1) + (-1) + 1.1 + 0.0 = 1 + 1 = 2$$

$$\therefore \quad \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(1,3,0) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\right)$$

Let $$w_3 = V_3 - \frac{(V_3, w_1)w_1}{\|w_1\|^2} - \frac{(V_3, w_2)w_2}{\|w_2\|^2} \qquad \text{.....(2)}$$

We have $(V_3, w_1) = (V_3, V_1) = 1.1 + 1.1 + 1.0 = 2$

$(V_3, w_2) = 1.(-1) + 1.1 + 1.0 = -1 + 1 = 0$

Putting in eq.(ii) we get

$$w_3 = (1,1,1) - \frac{2}{2}(1,1,0) - \frac{0.(1,3,0)}{2} = (1,1,1) - (1,1,0) = (0,0,1)$$

$$\Rightarrow \quad \|w_3\|^2 = (w_3, w_3) = 0.0 + 0.0 + 1.1 = 1$$

$$\frac{w_3}{\|w_3\|} = \frac{(0,0,1)}{1} = (0,0,1)$$

Hence an orthonormal basis for $\mathbb{R}^3$ is

$$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\right), (0,0,1)\right\}$$

Q.22. Let V be a finite dimensional inner product space and $T \in V^*$. Prove that there exists a unique $y \in V$ such that $T(x) = \langle x, y \rangle \ \forall\ x \in V$. (2019)

Sol. Since T is a linear functional on V. So $T : V \to F$ is a homomorphism. We know that every finite dimensional inner product space has an orthogonal basis. Let $\{x_1, x_2, ...x_n\}$ be an orthogonal basis of V, then

$\langle x_i, x_j \rangle = 0$ for $i \neq j$ and $\langle x_i, x_j \rangle = 1$ for each i .....(i)

Any $x \in V$ can be written as

$x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n; \qquad \alpha_i \in F$ .....(ii)

Let $T(x_i) = \beta_i \in F, \qquad$ for $i = 1, 2, ...n$ .....(iii)

Let $y = \bar{\beta}_1 x_1 + \bar{\beta}_2 x_2 + ... + \bar{\beta}_n x_n \in V$.

We have

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \bar{\beta}_j x_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j \langle x_i, x_j \rangle = \sum_{i=1}^{n} \alpha_i \beta_i \quad \text{by eq.(i)}$$

$$\therefore \quad \langle x, y \rangle = \sum_{i=1}^{n} \alpha_i \beta_i \qquad \text{.....(iv)}$$

From eq.(ii)

$$T(x) = T(\alpha_1 x_1 + ... + \alpha_n x_n) = \alpha_1 T(x_1) + ...+ \alpha_n T(x_n)$$

(since T is a homomorphism)

$$= \alpha_1\beta_1 + ... + \alpha_n\beta_n = \sum_{i=1}^{n} \alpha_i\beta_i \qquad .....(v)$$

From eq.(iv) and (v)

$$T(x) = \langle x, y\rangle \qquad \forall x \in V \qquad .....(vi)$$

Now, we show that

$y \in V$ satisfying eq.(vi) is unique.

Let $y_1 \in V$ be such that $T(x) = (x, y_1)\ \forall\ x \in V$ .....(vii)

from eq.(vi) and (vii)

$$(x, y) = (x, y_1) \qquad \forall x \in V \qquad .....(viii)$$

We have $(x, y - y_1) = (x, y) - (x, y_1) = 0$ by eq.(viii)

$\therefore \quad (x, y - y_1) = 0 \qquad \forall x \in V$

In particulars, $(y - y_1, y - y_1) = 0$ (Take $x = y - y_1 \in V$)

$\Rightarrow \quad y - y_1 = 0 \quad \Rightarrow \quad y = y_1$

This proves the uniqueness of y.

Q.23. Define eigen values and eigen vectors of a square matrix. Prove that eigen values of a Hermitian matrix are real. **(2018)**

Sol. Eigen value and eigen vectors of a square matrix: -

An eigen value of square matrix is defined as eigen value of a transformation associated with matrix.

Let A be a square matrix of order n×n over field F.

A map $T_A : F^n \to F^n$ defined as

$$T_A(X) = AX, \qquad \forall AX \in F^n$$

written as column vector i.e. $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$

eigen value of A $\underline{\underline{\text{def}}}$ eigen value of T_A.

Let $\lambda \in F$ be an eigen value value of A.

$\Leftrightarrow$ λ is an eigen value of T_A.

$\Leftrightarrow \exists X(\neq 0) \in F^n$ such that $T_A(x) = \lambda x$ or $Ax = \lambda x$

To Prove - All the eigen value of a Hermitian matrix are real number.

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roof -

Let 'A' be Hermitian matrix over ε then $A^* = A$ .....(i)

et λ be an eigen value of A (Hermitoin matrix)

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $Ax = \lambda x$

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $(Ax)^* = (\lambda x)^*$

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $x^*A^* = \bar{\lambda}x^*$ $\begin{cases} \because (AB)^* = B^*A^* \\ \& (\lambda A)^* = \bar{\lambda}A^* \end{cases}$

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $x^*A = \bar{\lambda}x^*$ $\{\because A^* = A\}$

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $x^*Ax = \bar{\lambda}x^*x$ {multiply by x}

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $x^*\lambda x = \bar{\lambda}x^*x$ {by eq.(ii)}

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $(\lambda - \bar{\lambda})(x^*x) = 0$ $\because \langle x, y\rangle = y^*x$

$\Rightarrow$ $\exists x(\neq 0) \in \mathbb{C}^n$ such that $(\lambda - \bar{\lambda})\langle x, x\rangle = 0$ $(\langle x, x\rangle = x^*x)$

$\Rightarrow$ $\lambda - \bar{\lambda} = 0$ $\{\because \langle x, x\rangle = \|x\|^2 \neq 0 \text{ as } x \neq 0\}$

$\Rightarrow$ $\lambda = \bar{\lambda}$ $\Rightarrow$ λ is purely real number.

Q.24. Let F be a field and $A \in M_n(F)$. Define the trace of A. If $B \in M_n(F)$ is similar to A, then prove that A and B have same trace. Is the converse true? Justify your answer. **(2019)**

Sol. Trace of a Matrix: -

Let A be a square matrix of order n over a field F. The sum of the elements of A lying along the principal diagonal is called the trace of A. i.e. if $A = [a_{ij}]_{n\times n}$. Then

$$trA = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots\ldots + a_{nn}$$

if $A, B \in M_n(F)$

Now we prove that A and B have the same trace.

Suppose that A and B are two similar matrices. Then there exists an invertible matrix C such that

$B = C^{-1}AC$

Let $C^{-1}A = D$

Then $tr\,B = tr(DC) = tr(CD)$ $(\because tr(AB) = tr(BA))$

$= tr(CC^{-1}A) = tr(IA) = trA$

Hence similar matrices have the same trace.

Unit V

Q.1. Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. (2014)

Sol. The characteristic equation of A is

$|A - xI| = 0$

$$\begin{vmatrix} 1-x & 2 \\ 0 & 1-x \end{vmatrix} = 0 \quad \Rightarrow \quad (1-x)^2 - 0 \times 2 = 0$$

$\Rightarrow \quad (1-x)^2 = 0$

$\Rightarrow \quad x = 1, 1$

Since roots are equal. Hence A is not diagonalizable.

Q.2. Prove that real quadratic from on a vector space V detremines a unique symetric bilinear from fs on V given by

$$f_s(x,y) = \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right], \quad \forall x, y \in V. \qquad (2014)$$

Sol. Let f be a bilinear from associated with quadratic from q on V.

$\therefore \quad q(x) = f(x, x) \ \forall x \in V$ .....(i)

$\therefore \quad f = f_s + f_{ss}$ .....(ii)

From (i) and (ii) we have

$q(x) = (f_s + f_{ss})(x, x) \quad \forall x \in V$

$= f_s(x, x) + f_{ss}(x, x) \quad \forall x \in V$

$\therefore \quad q(x) = f_s(x, x) \quad \forall x \in V$

$\because \quad q(x+y) = f_s(x+y, x+y)$

$q(x+y) = f_s(x, x) + f_s(x, y) + f_s(y, x) + f_s(y, y)$

$\Rightarrow \quad q(x+y) = q(x) + 2f_s(x, y) + q(y)$

$$\therefore \quad f_s(x,y) = \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right]$$

Q.3. Reduced the equation $2y^2 + 2zx + 4y + 4 = 0$ to standard form by suitable orthogonal change of variables an identify the conicoid represented by it. (2014, 19)

Sol. Matrix of quadratic form $2y^2 + 2zx$ relative to standard basis of $\mathbb{R}^3$ is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

characteristic equation is $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & \lambda \end{vmatrix} = 0$

$\lambda = 1, 2, -1$

(i) eigen vectors corresponding to $\lambda = 1$. Let (ℓ, m, n) be an eigen vector corresponding to 1.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$\left.\begin{aligned} -\ell + n &= 0 \\ m &= 0 \end{aligned}\right| \quad \Rightarrow \quad m = 0, \ell = n$$

$(1, 0, 1) \to$ Unit vector $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(ii) Eigen vector corresponding to $\lambda = 2$. Let (ℓ, m, n) be an eigen vector corresponding to $\lambda = 2$.

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$\Rightarrow \quad \ell - 2n = 0$

$-3n = 0 \quad \Rightarrow \quad n = 0$

then $\ell = 0$

$(\ell, m, n) = (0, 1, 0)$

(iii) eigen vectors corresponding to $\lambda = -1$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$R_3 \to R_3 - R_1 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$\Rightarrow \quad \ell + n = 0, \ell = -n$

$3m = 0 \quad \Rightarrow \quad m = 0$

(-1, 0, 0) or (1, 0, -1)

Let orthogonal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotating the axis without changing the origin: -

Now axis are x′, y′, z′ where x, y, z and x′, y′, z′ are related by transformation -

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z', \; y = y', \; z = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z'$$

from these transformation the given eq.(i) reduced to

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$$

$\therefore$ eq.(i) becomes

$$x'^2 + 2y'^2 - z'^2 + 4y' = -4$$

$$x'^2 + 2y'^2 + 4y' - z'^2 = -4$$

$$(x')^2 + 2\left(y'^2 + 2y'\right) - z'^2 = -4$$

$$(x')^2 + 2(y' + 1)^2 - (z')^2 = -4 + 2$$

$$-1\,(x')^2 - 2y^2 + (z')^2 = 2$$

Nature - This is hyperboloid of two sheets.

Here $y = y' + 1$

New transformation x′, y, z′.

Final transformtion are -

$$x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z'$$

$$y = y - 1$$

$$z = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z'$$

Q.4. Reduce the quadratic form

$q(x, y, z) = x^2 - 2yz + xz$ on $\mathbb{R}^3(\mathbb{R})$ to conical form $\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$ by congruent reduction and find the transformation relation between (x', y', z') and (x, y, z). Also find the index and signature of the quadratic form. (2014)

Sol. q on $\mathbb{R}^3$, $q(x, y, z) = x^2 - 2yz + xz$ matrix of q relative to standard basis of $\mathbb{R}^3$ is

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 1/2 & -1 & 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 1/2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_1 \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 0 & -1 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 - \frac{1}{2}C_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1/4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \leftrightarrow C_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$R_3 \to R_3 - 4R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$C_3 \to C_3 - 4R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$\therefore$ Normal form of q is

$$x'^2 - \frac{1}{4}y'^2 + 4z'^2$$

when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

Rank = 3, Index = 2 Singneture = 1

Q.5. Find the matrix of the quadratic form - $q(x_1, x_2, x_3) = 2x_1x_2 + 4x_1x_3 + 2x_2x_3$ on $\mathbb{R}^3(\mathbb{R})$ relative to the ordered basis [(1, 1, 0), (1, 0, 1), (0, 0, 1) of $\mathbb{R}^3(\mathbb{R})$]. (2012)

Sol. Given that

$$q(x_1, x_2, x_3) = 2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

The polar form of the smmetric bilinear form f associated with q is given by

$$f(x, y) = \frac{1}{2}[q(x+y) - q(x) - q(y)]$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$

$$(x+y) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$f((x_1, x_2, x_3)(y_1, y_2, y_3)) = \frac{1}{2}[2(x_1 + y_1)(x_2 + y_2) + 4(x_1 + y_1)(x_3 + y_3)$$

$$+2(x_2 + y_2)(x_3 + y_3) - (2x_1x_2 + 4x_1x_3 + 2x_2x_3) - (2y_1y_2 + 4y_1y_3 + 2y_2y_3)]$$

$$= x_1y_2 + y_1x_2 + 2(x_1y_3 + y_1x_3) + x_2y_3 + y_2x_3$$

Basis $S = \{e_1 = (1, 1, 0), e_2 = (1, 0\ 1), e_3 = (0, 0, 1)\}$

Now

$f(e_1, e_1) = f((1, 1, 0), (1, 1, 0)) = 2$ $f(e_2, e_1) = 4$, $f(e_3, e_1) = 3$

$f(e_1, e_2) = f((1, 1, 0), (1, 0, 1)) = 4$, $(e_2, e_2) = 4$, $f(e_3, e_3) = 2$

$f(e_1, e_3) = f((1, 1, 0), (0, 0, 1)) = 3$, $(e_2, e_3) = 2$, $f(e_3, e_3) = 0$

Hence matrix of f relative to the basis S is

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

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Q.6. Reduce the following equation in standard form by suitable orthogonal change of variables

$$4y^2+4z^2+4yz-2x-14y-22z+33=0$$

and tell the nature of the surface represented by it. **(2012)**

Sol. Given that

$$4y^2+4z^2+4yz-2x-14y-22z+33=0 \quad \text{.....(i)}$$

$$A=\begin{bmatrix}0&0&0\\0&4&2\\0&2&4\end{bmatrix}$$

Characteristic Equation of A is

$$\begin{vmatrix}-\lambda&0&0\\0&4-\lambda&2\\0&2&4-\lambda\end{vmatrix}=0 \quad \Rightarrow \quad \lambda=2,6,0$$

Eigen vectors corresponding to $\lambda=2$ is $(0,-1,1)$
Eigen vectors corresponding to $\lambda=6$ is $(0,1,1)$
Eigen vec tors corresponding to $\lambda=0$ is $(1,0,0)$
orthogonal matrix

$$P=\begin{bmatrix}0&0&1\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\end{bmatrix}$$

Let us rotate the co-ordinate axis to new axis (x, y, z) and (x', y', z') are related by transformation

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=P\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$$

$$x=z'$$

$$y=-\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$

$$z=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$

Using these transformation eq.(i) becomes

$$2x'^2+6y'^2+0z'^2+\left(\frac{14}{\sqrt{2}}-\frac{22}{\sqrt{2}}\right)x'+\left(-\frac{14}{\sqrt{2}}-\frac{22}{\sqrt{2}}\right)y'+(-2)z'=-33$$

or $$\left(2x'^2-\frac{8}{\sqrt{2}}x'\right)+\left(6y'^2-\frac{36}{\sqrt{2}}y'\right)-2z'=-33$$

or $$2\left(x'^2-\frac{4}{\sqrt{2}}x'\right)+6\left(y'^2-\frac{6}{\sqrt{2}}y'\right)-2z'=-33$$

or $$2\left(x'-\frac{2}{\sqrt{2}}\right)^2+6\left(y'-\frac{3}{\sqrt{2}}\right)^2-2z'=-33+4+27=-2$$

or $2X^2+6Y^2=2Z'^2-2$

or $2X^2+6Y^2=2(Z'-1)$

or $2X^2+6Y^2=2Z$

or $X^2+3Y^2=Z$ Elliptical Paraboloid

$$X=x'-\sqrt{2}$$

$$Y=y'-\frac{3}{\sqrt{2}}$$

$$Z=z'-1$$

$$x=z+1$$

$$y=-\frac{1}{\sqrt{2}}\left(x+\sqrt{2}\right)+\frac{1}{\sqrt{2}}\left(y+\frac{3}{\sqrt{2}}\right)$$

$$z=\frac{1}{\sqrt{2}}\left(x+\sqrt{2}\right)+\frac{1}{\sqrt{2}}\left(y+\frac{3}{\sqrt{2}}\right)$$

Q.7. Is matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ **diagonalizable? If yes, find a matrix P such that $P^{-1}.A.P$ is a diagonal matrix.** **(2013)**

Sol. Given that $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

Characteristics equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=0$$

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or $\lambda^3+\lambda^2-21\lambda-45=0$

$\therefore \quad \lambda=-3,-3,5$

(i) Eigen vectors corresponding to $\lambda=-3$.

Let $X=(x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda=-3$.

$\therefore \quad [\lambda+3I]X=0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

On solving using Echlon form we get homogeneous equation having solution (-2, 1, 0) and (3, 0, 1).

(ii) Eigen vectors corresponding to $\lambda=5$.

Let $X=(x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda=5$. Then

$$(A-5I)X=0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$R_1 \leftrightarrow R_3 \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{matrix} R_3 \to R_3-7R_1 \\ R_2 \to R_2+2R_1 \end{matrix}\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{matrix} R_1 \to -R_1 \\ R_3 \to R_3+2R_2 \end{matrix}\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$R_2 \to -\frac{R_2}{8}\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

On further solving Echlon form whose equation is equivalent to

$$x_1+2x_2+5x_3=0$$
$$x_2+2x_3=0$$

We get one linear independent solution (1, 2, -1).

Thus there are three linear independent eigen vectors (-2, 1, 0), (3, 0, 1) and (1, 2, -1). Hence given matrix A is diagonalizable.

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Let $P = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Then $P^{-1} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$

Therefore $P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Q.8. Reduce the quadratic form $q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4zx$ on $\mathbb{R}^3(\mathbb{R})$ to canonical form by $\alpha x'^2 + \beta y'^2 + \gamma z'^2$ by congruent reduction and find the transformation relation between (x, y, z) and (x', y', z') and tell the index and signature of the quadratic form. (2013)

Sol. Given that

$q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4zx$

$q(x, y, z) = x^2 + xy - 2xz + xy + 5y^2 + 0yz - 2zx + 0zy - 4z^2$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 5 & 0 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 5 & 0 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 4 & 2 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 2 \\ -2 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 2C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2.2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Q^t $\qquad$ Q

Normal form of q is

$$X'^2 + 2Y'^2 - 6Z'^2$$

Where $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

$x = x' - y' + 4z'$

$y = y' - 2z'$

$z = z'$

Rank = 3, index = 2, signature = 1.

Q.9. Prove that every bilinear form on a real vector space can be uniquely expressed as the sum of a symmetric and skewsymmetric bilinear forms. (2013, 19)

Sol. Let f be a bilinear form on V($\mathbb{R}$) consider the following bilinear form

on V(ℝ).

$$f_s(x,y)=\frac{1}{2}\left[f(x,y)+f(y,x)\right] \quad \text{.....(i)} \quad \forall x,y\in\mathbb{R}$$

and $$f_{ss}(x,y)=\frac{1}{2}\left[f(x,y)-f(y,x)\right] \quad \text{.....(ii)}$$

Claim: - f_s is symmetric and f_{ss} is skew symmetric.

Since we have $f_s(y,x)=\frac{1}{2}\left[f(y,x)+f(x,y)\right]=\frac{1}{2}\left[f(x,y)+f(y,x)\right]$

$$=f_s(x,y)$$

Therefore $f_s(x,y)=f_s(y,x) \quad \forall x,y\in V$

$\Rightarrow$ f_s is symmetric.

Since $f_{ss}(x,y)=\frac{1}{2}\left[f(y,x)-f(x,y)\right]$

$$=-\frac{1}{2}\left[f(x,y)-f(y,x)\right]=-f_{ss}(x,y) \quad \forall x,y\in V$$

$\therefore$ $f_{ss}(x,y)=-f_{ss}(x,y) \quad \forall x,y\in V$

Adding eq.(i) and (ii) we get

$$f(x,y)=f_s(x,y)+f_{ss}(x,y) \quad \text{.....(iii)}$$

or $f(x,y)=(f_s+f_{ss})(x,y) \quad \forall x,y\in V$

Uniqueness: -

Let if possible f have another representation viz

$$f=g_s+g_{ss} \quad \text{.....(iv)}$$

where g_s is symmetric and gss is skew-symmetric.

$\because$ $f(x,y)+f(y,x)=(g_s+g_{ss})(x,y)+(g_s+g_{ss})(y,x)$

$$=g_s(x,y)+g_{ss}(x,y)+g_s(y,x)+g_{ss}(y,x)$$
$$=g_s(x,y)+g_{ss}(x,y)+g_s(x,y)-g_{ss}(x,y)$$
$$=2g_s(x,y)$$

$\therefore$ $g_s(x,y)=\frac{1}{2}\left[f(x,y)+f(y,x)\right] \quad \forall x,y\in V$

$$=f_s(x,y) \quad \forall x,y\in V \quad \text{from eq.(i)}$$

$\Rightarrow$ $g_s=f_s$ .....(v)

from eq.(iv) and (v) we get

$$f_s+f_{ss}=g_s+g_{ss}$$
$$f_{ss}=g_{ss}$$

Hence representation is unique.

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Q.10. Reduce the equation

$$22x^2 - 12xy + 17y^2 - 112x + 92y + 178 = 0$$

into standard form by suitable orthogonal change of variables and tell the nature of the conic section represented by it. **(2013)**

Ans. Given that

$$22x^2 - 12xy + 17y^2 - 112x + 92y + 178 = 0$$

Hence $A = \begin{bmatrix} 22 & -6 \\ -6 & 17 \end{bmatrix}$

Its characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 22-\lambda & -6 \\ -6 & 17-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 39\lambda + 338\lambda = 0$$

Solving we get $\lambda = 13, 26$

Eigen vector corresponding to $\lambda = 13$

$$(A - 13I)X = 0$$

$$\begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$9x - 6y = 0 \quad \Rightarrow \quad -6x + 4y = 0$$

Solving we get

Choose $y = k$ then $n = 2k/3$

then $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k/3 \\ k \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

or $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Similarly eigen vector corresponding to $\lambda = 26$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Let $P = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

Then $x = \frac{2}{\sqrt{13}}x' - \frac{3}{\sqrt{13}}y'$

$$y = \frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'$$

Therefore eq.(i) becomes

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$$22x'^2 - 12\left[\frac{2}{\sqrt{13}}x' - \frac{3}{\sqrt{13}}y'\right]\left[\frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'\right] + 17y'^2$$

$$-112\left[\frac{2}{\sqrt{13}}x' - \frac{3}{\sqrt{13}}y'\right] + 92\left[\frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'\right] + 178 = 0$$

Solving we get

$$\frac{X^2}{2} + \frac{Y^2}{1} = 1$$

Where $X = x' + \frac{2}{\sqrt{13}}$, $Y = y' - \frac{14}{\sqrt{13}}$

Nature is ellipse.

Q.11. Prove that an n×n matrix is diagonalizable if and only if it possesse n linearly independent eigen vectors. **(2013)**

Ans. Suppose A is diagonalizable. Then A is similar to a diagonal matrix D $= dia[\lambda_1, \lambda_2, ... \lambda_n]$. Therefore there exists an invertible matrix

$P = [x_1, x_2, ... x_n]$ such that $P^{-1}AP = D$

i.e. $AP = PD$

$A[x_1, x_2, ... x_n] = [x_1, x_2, ... x_n] dia[\lambda_1, \lambda_2, ... \lambda_n]$

i.e. $[Ax_1, Ax_2, ... Ax_n] = [\lambda_1 x_1, \lambda_2 x_2, ...\lambda_n x_n]$

i.e. $Ax_1 = \lambda_1 x_1, Ax_2 = \lambda_2 x_2, ... Ax_n = \lambda_n x_n$

Therefore $x_1, x_2, ... x_n$ are eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2, ... \lambda_n$ respectively. Since the matrix P is non-singular, therefore its column vectors $x_1, x_2, ... x_n$ are linearly independent. Hence A posses n linearly independent eigen vectors.

Conversely, suppose that A possesses n linearly independent eigen vectors $x_1, x_2, ... x_n$ and let $\lambda_1, \lambda_2, ... \lambda_n$ be the corresponding eigen values. Then

$Ax_1 = \lambda_1 x_1, Ax_1 = \lambda_1 x_1, ... Ax_n = \lambda_n x_n$

Let $AP = [x_1, x_2, ...x_n] = [Ax_1, Ax_2, ... Ax_n]$

$= [\lambda_1 x_1, \lambda_2 x_2, ... \lambda_n x_n] = [x_1, x_2, ...x_n] dia[\lambda_1, \lambda_2, ... \lambda_n]$

$= PD$

Since the column vectors $x_1, x_2, ...x_n$ of the matrix p are linearly independent, therefore X is invertible and x^{-1} exists.

Therefore $AP = PD \Rightarrow P^{-1}AP = P^{-1}PD$

$\Rightarrow P^{-1}AP = D$

$\Rightarrow$ A is similar to a diagonal matrix D.

$\Rightarrow$ A is diagonalizable.

Q.12. Let f be the bilinear form on $\mathbb{R}^3$ defined by

$f((x_1, x_2, x_3), (y_1, y_2, y_3)) = 2x_1y_1 - 3x_2y_2 + 4x_3y_3 + x_1y_2 + 2x_2y_1$

Find

(i) Symmetric and skew symmetric parts of f:

(ii) The matrix of f with respect to ordered basic

$\{(1,1,0),(1,-1,0),(1,1,1)\}$ of $\mathbb{R}^3$. **(2017)**

Sol. Given that

$$f\left(\underbrace{(x_1,x_2,x_3)}_{x},\overbrace{(y_1,y_2,y_3)}^{y}\right)=2x_1-3x_2y_2+4x_3y_3+x_1y_2+2x_2y_1$$

(i) Symmetric

$$f_S=\frac{1}{2}\left[(y,x)+f(x,y)\right]$$

$$=\frac{1}{2}\left[2y_1x_1-3y_2x_2+4y_3x_3+y_1x_2+2y_2x_1\right.$$

$$\left.+2y_1x_1-3y_2x_2+4x_3y_3+y_2x_1+2x_2y_1\right]$$

$$=\frac{1}{2}\left[4y_1x_1-6y_2x_2+8x_3y_3+3y_2x_2+3x_2y_1\right]$$

Skew symmetric

$$f_{ss}=\frac{1}{2}\left[f(y,x)-f(x,y)\right]$$

$$=\frac{1}{2}\left[2y_1x_1-3y_2x_2+4x_3y_3+y_1x_2+2y_2x_1\right.$$

$$\left.-2y_1x_1+3y_2x_2-4x_3y_3-y_2x_1-2x_2y_1\right]$$

$$f_{ss}=\frac{1}{2}\left[x_1y_2-x_2y_1\right]$$

(II)

$a_{11}: f[(1,1,0),(2,2,0)]=2-3+1+2=2$

$a_{12}: f((1,1,0),(1,-1,0))=2+3-1+2=6$

$a_{13}: f(1,1,0),(1,1,1)]=2-3+1+2=2$

$a_{21}: f((1,1,0),(1,1,0))=2+3+1-2=4$

$a_{22}: f(2,-1,0),(1,-1,0))=2-3-1-2=-4$

$a_{23}: f((1,-1,0),(1,1,1))=2+3+1-2=4$

$a_{31}: f((1,-1,0),(1,1,0))=2-3+1+2=2$

$a_{32}: f((1,1,1),(1,-1,0))=2+3-1+2=6$

$a_{33}: f(1,1,1),(1,2,1)=2-3+4+1+2=6$

The matrix of f with respect to the ordered basis $\{(1,1,0),(1,-1,0),(1,1,1)\}$ is

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$$A=\begin{bmatrix}2 & 6 & 2\\ 4 & 4 & 4\\ 2 & 6 & 2\end{bmatrix}$$

Q.13. Prove that every quadratic form q on a real vector space V determines a unique symmetric bilinear form f_s on V given by:

$$f_s(x,y)=\frac{1}{2}\left[q(x+y)-q(x)-q(y)\right],\quad \forall x,y\in V \qquad (2018)$$

Sol. Let f be bilinear from on V such that

$$q(x)=f(x,x), \qquad \forall x\in V \qquad .....(i)$$

Also f can be expressed uniquely as

$$f=f_s+f_{ss} \qquad .....(ii)$$

where $f_s(x,y)=\frac{1}{2}\left(f(x,y)+f(y,x)\right)$

is a symmetrical bilinear form and

$$f_{ss}(x,y)=\frac{1}{2}\left(f(x,y)-f(y,x)\right)$$

is a skew symmetrical bilinear form.

Hence $q(x)=f(x,x)$ from eq.(i)

$=f_s(x,x)+f_{ss}(x,x)$ from eq.(ii)

$=f_s(x,x)$ $\left(\because f_{ss}(x,y)=0\right)$

Then q determines the unique symmetrical bilinear form f_s.

Now $q(x+y)=f_s(x+y,x+y)$

$=f_s(x,x)+f_s(x,y)+f_s(y,x)+f_s(x,x)$

$=f_s(x,x)+2f_s(x,y)+f_s(y,y)$ $\left(\because f_s(x,y)=f_s(y,x)\right)$

$=q(x)+2f_s(x,y)+q(y)$

Hence $2f_s(x,y)=q(x+y)-q(x)-q(y)$

$$f_s(x,y)=\frac{1}{2}\left(q(x+y)-q(x)-q(y)\right)$$

Q.14. Let V be a vector space of dimension $n\in\mathbb{N}$ over a field F. Let $T: V\to V$ be a linear transformation. If T has n distinct eigen values, then prove that T is diagonalizable. Is the converse true? Justify your answer. (2019)

Sol. Suppose that T has m distinct eigen values say, $C_1, C_2, \ldots C_m$. Let $V_1, V_2, \ldots V_m$ be the corresponding eigen vectors.

Claim - $S=\{V_1, V_2, \ldots V_m\}$ are linearly independent over F.

By induction on m.

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if m = 1 Then S = $\{V_1\}$ is linearly independent. (Since $V_1 \neq 0$)

Suppose that the set $\{V_1, V_2, ..., V_K\}$ is linearly independent where K < m.
We shall prove that the set $\{V_1, V_2, ..., V_K, V_{K+1}\}$ is linearly independent.

Let $\alpha_1 V_1 + ... \alpha_{K+1} V_{k+1} = 0$ where $\alpha_i \in F$ .....(i)

$\Rightarrow$ $T(\alpha_1 V_1 + ... \alpha_{K+1} V_{k+1}) = T(0) = 0$

$\Rightarrow$ $\alpha_1 T(V_1) + ... + \alpha_{K+1} T(V_{K+1}) = 0$ $(\because T(V_i) = C_i V_i)$

$\alpha_1 (C_1 V_1) + ... + \alpha_{K+1} (C_{K+1} V_{K+1}) = 0$ .....(ii)

Multiplying eq.(i) by the scalar C_{K+1} and substracting from eq.(ii), we get

$\alpha_1 (C_1 - C_{K+1}) V_1 + ... + \alpha_K (C_K - C_{K+1}) V_K = 0$

$\Rightarrow$ $\alpha_1 (C_1 - C_{K+1}) = 0, ..., \alpha_K (C_K - C_{K+1}) = 0$

Since $(V_1, V_2, ... V_K)$ is linearly independent.

$\Rightarrow$ $\alpha_1 = 0, \alpha_2 = 0, ..., \alpha_K = 0$, since $C_1, C_2, ... C_{K+1}$ are all distinct.

Putting these values in eq.(i) we get

$\alpha_{K+1} V_{K+1} = 0 \Rightarrow \alpha_{K+1} = 0$ $(\because V_{K+1} \neq 0)$

Thus $\alpha_1 = 0, ... \alpha_K = 0, \alpha_{K+1} = 0$ implies that the set $\{V_1, V_2, ...V_K, V_{K+1}\}$ is linearly independent. This complete the induction.

Hence S is linearly independent.

Since dim V = m, $\beta = \{V_1, V_2, ... V_m\}$ is an ordered basis of V which consist of eigen vectors of T. Hence T is diagonalizable. Since A linear operator T on a finite dimensional vector space V is diagonalizable if and only if there exists a basis of V consisting of eigen vectors of T.

Q.15. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Prove that there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ such that $P^t AP$ is a diagonal matrix. (2019)

Sol. Let $X_1, X_2, ... X_n$ be eigen vectors of A corresponding to its eigen values $\lambda_1, \lambda_2, ... \lambda_n$ respectively.

$\therefore$ $X_1, X_2, ... X_n$ are linearly independent and orthogonal to each other.

Normalizing the vectors

Let $Y_1 = \frac{X_1}{\|X_1\|}, Y_2 = \frac{X_2}{\|X_2\|}, ...Y_n = \frac{X_n}{\|X_n\|},$

Clearly $Y_1, Y_2, ...Y_n$ are also eigen vectors of A corresponding to eigen values $\lambda_1, \lambda_2, ... \lambda_n$ respectively.

$\therefore$ $AY_1 = \lambda_1 Y_1, AY_2 = \lambda_2 Y_2, ..., AY_n = \lambda_n Y_n,$ .....(i)

Let $P = (Y_1, Y_2, ... Y_n)$

clearly P is orthogonal matrix.

$$P^t = \begin{bmatrix} Y_1^t \\ Y_2^t \\ . \\ . \\ Y_n^t \end{bmatrix}_{n\times 1} \qquad A = [A]$$

$$P^tA = \begin{bmatrix} Y_1^t.A \\ Y_2^t.A \\ . \\ . \\ Y_n^t.A \end{bmatrix} = \begin{bmatrix} Y_1^t.A^t \\ Y_2^t.A^t \\ . \\ . \\ Y_n^t.A^t \end{bmatrix} \qquad (\because A \text{ is symmetric i.e. } A^t = A)$$

$$= \begin{bmatrix} (AY_1)^t \\ (AY_2)^t \\ . \\ . \\ (AY_n)^t \end{bmatrix} = \begin{bmatrix} (\lambda_1 Y_1)^t \\ (\lambda_2 Y_2)^t \\ . \\ . \\ (\lambda_n Y_n)^t \end{bmatrix} \qquad \text{from eq.(i)}$$

$$= \begin{bmatrix} \lambda_1 Y_1^t \\ \lambda_2 Y_2^t \\ . \\ . \\ \lambda_n Y_n^t \end{bmatrix} \qquad (\because (\lambda A)^t = \lambda A^t)$$

$$= \begin{bmatrix} \lambda_1 & O_F & ..... & O_F \\ O_F & \lambda_2 & ..... & O_F \\ . & & & \\ . & & & \\ O_1 & O_1 & ..... & \lambda_n \end{bmatrix} \begin{bmatrix} Y_1^t \\ Y_2^t \\ . \\ . \\ Y_n^t \end{bmatrix} = \text{Diag}[\lambda_1, \lambda_2, ...\lambda_n] P^{-1}$$

$\therefore \quad P^tAP = \text{Diag}[\lambda_1, \lambda_2, ...\lambda_n]$

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