

- Writer : Aakash Verma
[M.Sc.]
Satyam Prajapati
[M.Sc.]
Deepu Maurya
[M.Sc.]
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SYLLABUS

Unit I - Definition of a field, Examples $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ of fields, vector space: definition, examples and basic properties, subspaces, subspace generated by subset, Linear dependence and independence of a subset, Basic and dimensions of a vector space, Sum and direct sum of subspaces, Quotient spaces.

Unit II - Linear transformations and isomorphism of vector spaces, Null space, range space, rank and nullity of linear transformation, Fundamental theorem of homomorphism, Rank nullity theorem, $\text{Hom}(V, M)$ as a vector space, $\text{Dim Hom}(V, W) = \text{Dim } V \cdot \text{Dim } W$. Dual space V^* and transpose of a linear transformation, Annihilator of a subset of a vector space.

Unit III - Matrix representation of linear transformation, Algebra of matrices, Effect of change of basis of matrix representation, Equivalent and similar matrices, Rank and nullity of a matrix, System of linear equations, their consistency and inconsistency reduction of a matrix to normal form and algorithms to find the inverse (if it exists), Definition of Determinant and statement of properties (without proof), cofactor, Adjoint of a matrix and its properties.

Unit IV - Inner product space: Definition and examples, Cauchy Schwarz inequality, norm of a vector, Motion of angle, Orthogonality, Orthonormal set and basis, Gram Schmidt orthonormalization, Bessel's inequality, Orthogonal complement, Adjoint of a linear transformation (orthogonal Unitary), Hermitian, Skew Hermitian, Symmetric and Skew Symmetric matrices, Invariant Subspaces, Eigenvalues, Eigen vectors, Characteristic polynomials of a linear transformation, Cayley-Hamilton Theorem.

Unit V - Diagonalization of a matrix with distinct eigen values, Orthogonal (Unitary) reduction of real Symmetric (Hermitian) matrices.

Bilinear forms, Equivalence of bilinear forms, matrix, representation of Bilinear forms, Quadratic forms, Classification of Symmetric, Skew Symmetric bilinear forms over $\mathbb{R}$ and $\mathbb{C}$, Sylvesters law of inertia. Classification of curves and surfaces represented by equations of 2nd degree.

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Unit I

Q.1. Define basis of a vector space. Prove that a maximal linearly independent subset of a vector space is a basis of the vector spaces. (2014)

Ans. Basis:-

Let V be the vector space over the field F . A subset S of V is called basis of a vector space V over F .

if (i) $L(S) = V$

(ii) S is a linearly independent set.

Let $S = \{v_1, v_2, \dots, v_n\}$ be a maximal linear independent subset of V . Then S becomes a basis of V , if we prove that $L(S) = V$.

Let $v \in V$ be arbitrary. Since $v, v_1, v_2, \dots, v_n$ are $n+1$ vectors in V , they must be linearly dependent and so there exist scalars $\alpha, \alpha_1, \alpha_2, \dots, \alpha_n$; not all zero.

Such that $\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ (i)

if $\alpha = 0$ then equation (i) becomes -

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_n = 0 \quad (\because S \text{ is a linear independent set})$$

and $\alpha = 0$. This is a contradiction to the fact $\alpha, \alpha_1, \alpha_2, \dots, \alpha_n$ are not all zero.

Consequently, $\alpha \neq 0 \in F$ and so $\alpha^{-1} \in F$ and $\alpha^{-1}\alpha = 1$. From equation (i), we

$$\text{have } \alpha v = -\alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_n v_n$$

$$\text{or } \alpha^{-1}(\alpha v) = -\alpha^{-1}(\alpha_1 v_1) - \alpha^{-1}(\alpha_2 v_2) - \dots - \alpha^{-1}(\alpha_n v_n)$$

$$\text{or } v = (-\alpha^{-1}\alpha_1)v_1 + (-\alpha^{-1}\alpha_2)v_2 + \dots + (-\alpha^{-1}\alpha_n)v_n$$

$$\Rightarrow v \in L(S) \text{ and so } S \text{ is a basis of } V \text{ containing } n \text{ vectors.}$$

Q.2. If W is a subspace of a finite dimensional vector space V , prove that $\dim(V/W) = \dim V - \dim W$. (2012, 14)

Ans. We have to prove that $\dim(V/W) = \dim V - \dim W$

Proof:- Let $\dim W = m$

Let $S = \{a_1, a_2, \dots, a_m\}$ be a basis of W . Since S is linearly independent subset of V . Therefore S can be extended to form a basis of vector space V (By extension theorem).

Let $S_1 = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ be basis of V .

$$\therefore \dim V = m + n$$

consider $S_2 = \{W + b_1, W + b_2, \dots, W + b_n\}$

Claim:- S_2 is basis of V/W .

(i) S_2 is linearly independent.

$$\text{Let } \lambda_1(W + b_1) + \lambda_2(W + b_2) + \dots + \lambda_n(W + b_n) = W$$

$$\Rightarrow W + \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n = W$$

$$\Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n \in W \quad (\because W + a = W \Leftrightarrow a \in W)$$

$$\Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n = v_1 a_1 + v_2 a_2 + \dots + v_m a_m$$

$$\Rightarrow (-v_1)\lambda_1 + (-v_2)\lambda_2 + \dots + (-v_m)\lambda_m + \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n = 0$$

$$\Rightarrow v_1 = v_2 = \dots = v_m = \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

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Therefore S_2 is linear independent.

(ii) $V/W = [S_2]$

Let $W + x \in V/W$

$$\Rightarrow x \in V$$

$$\Rightarrow x = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m + \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_n b_n$$

$$\Rightarrow W + x = W + (\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m + \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_n b_n)$$

$$\Rightarrow W + x = W + \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_n b_n$$

$$(\because W + \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m = W)$$

$$\Rightarrow W + x = (W + \beta_1 b_1) + (W + \beta_2 b_2) + \dots + (W + \beta_n b_n)$$

$$= \beta_1 (W + b_1) + \beta_2 (W + b_2) + \dots + \beta_n (W + b_n)$$

$$\therefore V/W = [S_2]$$

$$\therefore S_2 \text{ is a basis of } V/W$$

$$\therefore \dim V/W = n$$

$$= m + n - m = \dim V - \dim W$$

$$\Rightarrow \dim V/W = \dim V - \dim W$$

Q.3. Define linearly dependent and linearly independent vectors of a vector space over a field. Let $S = \{a_1, a_2, \dots, a_n\}$ be a linearly independent subset of a vector space V over a field F and $a \in V$ but $a \notin L_S$. Then show that $\{a_1, a_2, \dots, a_n, a\}$ will be linearly independent. (2014)

Ans. Linearly dependent and linearly independent vectors of a vector space $V(F)$:-

Vectors $a_1, a_2, \dots, a_n$ are called linearly dependent if $\exists$ scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0$ in which atleast one $\lambda_i \neq 0$. Otherwise vectors are called linearly independent. Let

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n + \beta a = 0 \quad \dots \dots \dots (i) \quad \lambda_i, \beta \in F$$

Claim :- $\beta = 0$ in equation (i)

If possible let $\beta \neq 0$ in equation (i)

Multiply equation (i) by β^{-1} .

$$a = -(\beta^{-1} \odot \lambda_1) a_1 - (\beta^{-1} \odot \lambda_2) a_2 - \dots - (\beta^{-1} \odot \lambda_n) a_n$$

$$a = v_1 a_1 + v_2 a_2 + \dots + v_n a_n \quad (\because v_i = \beta^{-1} \odot \lambda_i)$$

which is a contradiction because 'a' does not belong to L_S . Therefore our assumption that $\beta \neq 0$ in equation (i) is wrong.

Hence $\beta = 0$ in equation (i)

From (i)-

$$\Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \quad (\because a_1, a_2, \dots, a_n \text{ are linearly independent})$$

$$\Rightarrow \{a_1, a_2, \dots, a_n, a\} \text{ are linearly independent.}$$

Q.4. If $W_1 = \{(x, y, z): x, y, z \in \mathbb{R} \text{ and } y + 2z = 0\}$ and $W_2 = \{(x, y, z): x, y, z \in \mathbb{R} \text{ and } x + y + z = 0\}$ are two subspaces of the vector space $\mathbb{R}^3$ ($\mathbb{R}$) then find

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- (i) a basis of W_1
 (iii) a basis of $W_1 \cap W_2$ and

Ans.(i) $0.x + 1.y + 2.z = 0$

$$A = [0, 1, 2]$$

$$\rho(A) = 1$$

Number of linear independent solution = $3 - 1 = 2$

$$y + 2z = 0$$

$$\Rightarrow y = -2z, x = x, z = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 0.z \\ 0.x + (-2)z \\ 0.x + z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} z$$

$$\dim W_1 = 2$$

Basis of $W_1 = \{(1, 0, 0), (0, -2, 1)\}$

(ii) $W_2 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x + y + z = 0\}$

$$x + y + z = 0$$

$$\Rightarrow x = y - z, y = y, z = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.x + (-1)y + (-1).z \\ 0.x + 1.y + 0.z \\ 0.x + 0.y + 1.z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$$\dim W_2 = 2$$

Basis of $W_2 = \{(-1, 1, 0), (-1, 0, 1)\}$

(iii) $W_1 \cap W_2 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } y + 2z = 0, x + y + z = 0\}$

We get $y = -2z, x = z, z = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.x + 0.y + 1.z \\ 0.x + 0.y + (-2).z \\ 0.x + 0.y + 1.z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z$$

$$\dim W_1 \cap W_2 = 1$$

Basis of $(W_1 \cap W_2) = \{(1, -2, 1)\}$

(iv) We know that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \\ = 2 + 2 - 1 = 3$$

Q.5. Define a vector space. Let V be a vector space over F and q be the zero vector of V . Prove that

$$(i) a0 = 0 \quad \forall a \in F \quad (ii) 0\alpha = 0 \quad \forall \alpha \in V$$

$$(iii) a(-\alpha) = -(a\alpha) = (-a)\alpha \quad \forall a \in F \text{ and } \forall \alpha \in V$$

Sol.

Vector Space:-

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Let $(F, \oplus, \odot)$ be a field and a system $(V, +, \cdot)$ consisting of

- (i) A non-empty set
 (ii) An internal binary operation $+$ in V
 (iii) An external binary operation $\cdot$ in V over F is called a vector space over a field F iff it satisfy the following properties -

(i) $(V, +)$ is an abelian group.

$$(ii) (a) \alpha(\beta a) = (\alpha \odot \beta)a \quad \forall \alpha, \beta \in F \text{ and } \forall a \in V$$

$$(b) 1a = a \quad \forall a \in V$$

$$(iii) (a) \alpha(a + b) = \alpha a + \alpha b \quad \forall \alpha, \beta \in F \text{ and } a, b \in V$$

$$(b) (\alpha \oplus \beta)a = \alpha a + \beta a \quad \forall \alpha, \beta \in F \text{ and } a \in V$$

$$(I) \alpha 0 = 0 \quad \forall \alpha \in F$$

Proof - We know that

$$0 + 0 = 0$$

$$\text{So } \alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$$

$$\Rightarrow \alpha 0 + 0 = \alpha 0 + \alpha 0$$

Hence $0 = \alpha 0$, by cancellation law in the group $(V, +)$.

$$(ii) 0a = 0 \quad \forall a \in V$$

Proof - Since $0 + 0 = 0$

$$\Rightarrow 0a = (0 + 0)a = 0a + 0a$$

$$\Rightarrow 0a + 0 = 0a + 0a \Rightarrow 0 = 0a$$

Hence $0a = 0$, by cancellation law in the group $(V, +)$.

$$(iii) \alpha(-a) = -(\alpha a) = (-\alpha)a \quad \forall \alpha \in F \text{ and } \forall a \in V$$

Proof - We know that

$$\alpha + (-\alpha) = 0$$

$$\Rightarrow \{\alpha + (-\alpha)\}a = 0.a = 0 \quad (\text{by eq.(ii) } 0a = 0)$$

$$\Rightarrow \alpha a + (-\alpha)a = 0$$

$$\text{Hence } (-\alpha)a = -(\alpha a)$$

$$\text{Similarly, } a + (-a) = 0$$

$$\Rightarrow \alpha\{a + (-a)\} = \alpha 0$$

$$\Rightarrow \alpha a + \alpha(-a) = 0 \quad (\text{by eq.(i) } 0 = \alpha 0)$$

$$\text{Hence } \alpha(-a) = -(\alpha a)$$

Q.6. Prove that the set of non-zero vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ of a vector space V over F is linearly dependent if some α_k $2 \leq k \leq n$, is a linear combination of vectors preceding it. (2015)

Other Related Questions -

Q. Let V be a vector space over a field F and $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V with each $v_i \neq 0$ ($1 \leq i \leq n$). Prove that S is linearly dependent

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over F if and only if $\exists k (2 \leq k \leq n)$ such that v_k is linear combination of $v_1, v_2, \dots, v_{k-1}$.

(2019)

Sol. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be linearly dependent. Now, the Singleton set $\{\alpha_1\}$ is linearly independent. Since $\alpha_1 \neq 0$. Consider $\{\alpha_1, \alpha_2\}$. If it is also linearly dependent, then consider $\{\alpha_1, \alpha_2, \alpha_3\}$. Proceeding thus, Let k be the first positive integer (where $2 \leq k \leq n$) for which $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is linearly independent.

Atmost $k = n$.

Hence $\exists$ scalars $a_1, a_2, \dots, a_k$ not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k = 0$. Now $a_k \neq 0$, for if $a_k = 0$, then $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$ will be linearly dependent contrary to the definition of k .

Hence $\alpha_k = a_1^{-1}a_1\alpha_1 - a_2^{-1}a_2\alpha_2 - \dots - a_{k-1}^{-1}a_{k-1}\alpha_{k-1}$ which proves that α_k is a linear combination of the preceding vectors $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$.

Q.7. Show that the vectors $(1, 2, 1), (2, 1, 0), (1, -1, 2)$ form a basis of $\mathbb{R}^3(\mathbb{R})$. (2015)

Sol. (i) S is $\ell.I.$ -

$$\text{Let } \alpha(1, 2, 1) + \beta(2, 1, 0) + \gamma(1, -1, 2) = 0$$

$$\alpha + 2\beta + \gamma = 0 \quad \dots\dots\dots(i)$$

$$2\alpha + \beta - \gamma = 0 \quad \dots\dots\dots(ii)$$

$$\alpha + 0\beta + 2\gamma = 0 \quad \dots\dots\dots(iii)$$

$$\Rightarrow \alpha = -2\gamma$$

So from eq.(i) and (ii)

$$2\beta - \gamma = 0$$

$$\beta = 5\gamma = 0$$

Solving them we get $\gamma = 0$ And so $\alpha = 0, \beta = 0$.

(ii) $\mathbb{R}^3 = [S]$

Let $(x, y, z) \in \mathbb{R}^3$.

$$\text{Let } (x, y, z) = \alpha(1, 2, 1) + \beta(2, 1, 0) + \gamma(1, -1, 2)$$

$$(x, y, z) = (\alpha + 2\beta + \gamma, 2\alpha + \beta - \gamma, \alpha + 2\gamma)$$

$$x = \alpha + 2\beta + \gamma \Rightarrow \alpha + 2\beta + \gamma = x$$

$$y = 2\alpha + \beta - \gamma \Rightarrow 2\alpha + \beta - \gamma = y$$

$$z = \alpha + 2\gamma \Rightarrow \alpha + 2\gamma = z$$

$$\alpha = \frac{\begin{vmatrix} x & 2 & 1 \\ y & 1 & -1 \\ z & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{-2x + 5y + 2z}{9}$$

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$$\beta = \frac{\begin{vmatrix} 1 & x & 1 \\ 2 & y & -1 \\ 1 & z & 2 \end{vmatrix}}{-9} = \frac{5x + y - 3z}{9}$$

$$\gamma = \frac{\begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 1 & 0 & z \end{vmatrix}}{-9} = \frac{x - 2y + 3z}{9}$$

$$\text{So, } (x, y, z) = \frac{(-2x + 5y + 2z)}{9}(1, 2, 1) + \frac{(5x + y - 3z)}{9}(2, 1, 0) + \frac{(x - 2y + 3z)}{9}(1, -1, 2)$$

Q.8. Prove that corresponding to each sub-space W_1 of a finite dimensional vector space $V(F)$, there exists a sub-space W_2 such that V is the direct sum of W_1 and W_2 . (2015)

Sol. Let $S = \{a_1, a_2, \dots, a_m\}$ be the basis of W_1 , i.e. $\dim W_1 = m$.

Since S is $\ell.I.$ subset of vector space V . Therefore it can be extended a basis of V .

Let $S_1 = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ be basis of V .

$$\therefore \dim V = m + n$$

Let $S_2 = \{b_1, b_2, \dots, b_n\}$ such that $W_2 = [S_2]$

Claim - S_2 is $\ell.I.$

Since S_2 is a subset of S_1 and S_1 is linearly independent. Therefore S_2 is also $\ell.I.$ (because a subset of $\ell.I.$ set is also $\ell.I.$).

Hence S_2 is basis of W_2 .

$$\text{i.e. } \dim W_2 = n$$

$$\therefore \dim V = \dim W_1 + \dim W_2$$

$$\dim V = \dim(W_1 \oplus W_2)$$

$$\Rightarrow V = W_1 \oplus W_2$$

Q.9. If W is a subspace of a vector space $V(F)$. Prove that the set V/W of all cosets $W + \alpha$, where α is an arbitrary element of V , is a vector space over F for the vector addition and scalar multiplication compositions defined as follows:

$$(W + \alpha) + (W + \beta) = W + (\alpha + \beta) \quad \forall \alpha, \beta \in V \text{ and}$$

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$$a(W + \alpha) = W + \alpha a \quad \forall a \in F \text{ and } \forall \alpha \in V.$$

(2015)

Sol. We have to show $\left(\frac{V}{W}, +, \cdot\right)$ is a vector space.

(i) $\left(\frac{V}{W}, +\right)$ is a commutative group.

(1) Closure property -

$$\text{Let } W + \alpha, W + \beta \in \frac{V}{W}, \quad \alpha, \beta \in V$$

$$(W + \alpha) + (W + \beta) = W + \alpha + \beta \in \frac{V}{W}$$

$\therefore$ '+' is an internal binary operation.

(ii) Associative property -

$$\text{Let } (W + \alpha), (W + \beta), (W + \gamma) \in \frac{V}{W}$$

$$\Rightarrow \{(W + \alpha) + (W + \beta)\} + (W + \gamma)$$

$$\Rightarrow \{W + \alpha + \beta\} + (W + \gamma)$$

$$\Rightarrow W + (\alpha + \beta) + \gamma$$

$$\Rightarrow W + \alpha + (\beta + \gamma) \quad (\text{Associative property of } V)$$

$$\Rightarrow (W + \alpha) + \{(W + \beta) + (W + \gamma)\}$$

(iii) Existence of identity -

$$\text{Let } W + \alpha \in \frac{V}{W}$$

$$\therefore W + (W + \alpha) = (W + 0) + (W + \alpha)$$

$$= W + (0 + \alpha)$$

$$W + (W + \alpha) = W + \alpha$$

$$\text{Identity} = W$$

(iv) Existence of inverse -

$$\forall (W + \alpha), \exists W + (-\alpha) \in \frac{V}{W}$$

$$\text{Such that } (W + \alpha) + (W + (-\alpha)) = W + \alpha + W + (-\alpha) \\ = W + \alpha + (-\alpha) = W + 0 = W$$

Therefore inverse of each element exists.

(v) Commutative property -

$$\text{let } W + \alpha, W + \beta \in \frac{V}{W}$$

$$\Rightarrow (W + \alpha) + (W + \beta) = W + \alpha + \beta = W + \beta + \alpha \quad (\because (V, +) \text{ is commutative})$$

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$$= (W + \beta) + (W + \alpha)$$

So $\left(\frac{V}{W}, +\right)$ is an abelian group.

$$(II) (i) a.(b.\alpha) \Rightarrow (a \odot b).\alpha \quad \forall a, b \in F, \forall \alpha \in V$$

$$\lambda\{\mu(W + \alpha)\} = \lambda(W + \mu\alpha) = W + \lambda(\mu\alpha)$$

$$= W + (\lambda \odot \mu).\alpha = (\lambda \odot \mu)(W + \alpha)$$

$$(ii) 1.\alpha = \alpha, \quad \forall \alpha \in V$$

$$1.(W + \alpha) = W + 1.\alpha \quad (\because 1.\alpha = \alpha, \alpha \in V)$$

$$= W + \alpha$$

$$(III) (i) a.(\alpha + \beta) = a.\alpha + a.\beta \quad \forall a \in F, \alpha, \beta \in V$$

$$\lambda\{(W + \alpha) + (W + \beta)\} = \lambda\{W + (\alpha + \beta)\}$$

$$= W + \lambda(\alpha + \beta) = W + (\lambda\alpha + \lambda\beta) = W + \lambda\alpha + W + \lambda\beta$$

$$= \lambda(W + \alpha) + \lambda(W + \beta)$$

$$(ii) (a \oplus b).\alpha = a.\alpha + b.\alpha \quad \forall a, b \in F, \forall \alpha \in V$$

$$(\lambda \oplus \mu).(W + \alpha) = W + (\lambda \oplus \mu).\alpha = W + \lambda\alpha + \mu\alpha$$

$$= W + \lambda\alpha + W + \mu\alpha = \lambda(W + \alpha) + \mu(W + \alpha)$$

So $\frac{V}{W}$ is a vector space.

Q.10. Let S be a generating set containing n elements of a vector space V over a field F. Prove that any linear independent subset T of V contains at most n elements. (2016)

Sol. Let V be a vector space over field F and $X = \{x_1, x_2, \dots, x_n\}$ be a

generating subset of V i.e. $V = [X]$

$$Y = \{y_1, y_2, \dots, y_m\} \text{ be linear independent subset of } V.$$

Claim :- $m \leq n$

$$\therefore V = [X] \text{ and } y_1 \in V$$

$$\therefore y_1 = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \quad \alpha_i \in F$$

$$\Rightarrow (-1)y_1 + \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0 \quad \dots \dots \dots (i)$$

Claim - At least one $\alpha_i \neq 0$ in eq.(i)

If possible let all $\alpha_i = 0$ in eq.(i)

$$\therefore (-1)y_1 = 0$$

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$$\Rightarrow y_1 = 0$$

Which is a contradiction because none of the element in a linear independent set is zero.

Therefore our assumption that all $\alpha_i = 0$ in eq.(i) is wrong. Hence $\exists$ atleast one $\alpha_i \neq 0$ in eq.(i)

$$\text{Let } \alpha_i \neq 0_f \Rightarrow \exists \alpha_i^{-1} \in F$$

Multiplying eq.(i) by α_i^{-1} we can write x_i as a linear combination of

$$\begin{aligned} & y_1, x_1, \dots, x_n \\ \therefore V &= [x_i], \text{ where } x_i = \{y_1, x_2, \dots, x_n\} \\ \therefore V &= [x_i] \text{ and } y_2 \in V \\ \Rightarrow y_2 &= \beta y_1 + \gamma_2 x_2 + \gamma_3 x_3 + \dots + \gamma_n x_n \\ \Rightarrow \beta y_1 + (-1)y_2 + \gamma_2 x_2 + \dots + \gamma_n x_n &= 0 \end{aligned} \quad \dots(ii)$$

Claim - At least one $\gamma_i \neq 0_f$ in eq. (ii)

If possible let all $\gamma_i = 0_f$ in eq.(ii)

$$\Rightarrow \beta y_1 + (-1)y_2 = 0$$

$$\Rightarrow \{y_1, y_2\} \text{ is l.d.}$$

$$\Rightarrow y = \{y_1, y_2, \dots, y_n\} \text{ is l.d. } (\because \text{super set of l.d. set also l.d.})$$

Which is a contradiction because y is linearly independent.

Therefore our assumption that all $\gamma_i = 0$ in equation ..(ii) is wrong. Hence atleast one $\gamma_i \neq 0_f$ in eq.(ii)

$$\text{Let } \gamma_i \neq 0_f \text{ in equation (ii)} \Rightarrow \gamma_i^{-1} \in F$$

Multiply equation (ii) by γ_i^{-1} we can write x_i as a linear combination of elements of

$$x_2 = \{y_1, y_2, \dots, x_3, \dots, x_n\}$$

$$\therefore V = [x_2]$$

Proceeding in this manner a question arise which one of x and y exhausted first.

Claim :- y will be exhausted before x if possible let x exhausted before y .

Using replacement process n time we get

$$x_n = \{y_1, y_2, \dots, y_n\}$$

$$V = [x_n] \text{ and } y_{n+1} \in V$$

$$\Rightarrow y_{n+1} = \delta_1 y_1 + \delta_2 y_2 + \dots + \delta_n y_n$$

$$y_1, y_2, \dots, y_n, y_{n+1} \text{ is linearly dependent}$$

$$\Rightarrow y = \{y_1, y_2, \dots, y_n, y_{n+1}, \dots, y_m\} \text{ is l.d. which is a contradiction.}$$

because y is linearly independent.

Therefore our assumption that $n < m$ wrong then $m \leq n$

Which proves that y can have n vectors.

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Q.11. Let V be a finite dimensional vector space over a field F . Let w_1 and w_2 be subspaces of V . Prove that

$$\dim(w_1 + w_2) = \dim w_1 + \dim w_2 - \dim(w_1 \cap w_2). \quad (2013, 16)$$

Sol. Let $\dim(w_1 \cap w_2) = \ell$

and let $s = \{a_1, a_2, \dots, a_\ell\}$ be a basis of $w_1 \cap w_2$.

since s is linearly independent subset of $w_1 \cap w_2$, therefore s is also L.I. subset of w_1 and w_2 . Therefore s can be extended to form a basis of w_1 and w_2 (by extension theorem)

Let $s_1 = \{a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_m\}$ be a basis of w_1 .

And $s_2 = \{a_1, a_2, \dots, a_\ell, c_1, c_2, \dots, c_n\}$ be basis of w_2 .

$$\therefore \dim w_1 = \ell + m \text{ and } \dim w_2 = \ell + n$$

Consider $s_3 = \{a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n\}$

Claim :- s_3 is basis of $w_1 + w_2$.

(I) s_3 is L.I. -

$$\begin{aligned} \text{Let } \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_\ell a_\ell + \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_m b_m \\ + \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n = 0 \end{aligned} \quad \dots(1)$$

$$\Rightarrow \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n \in w_2 = (-\alpha_1) a_1 + (-\alpha_2) a_2 + \dots + (-\alpha_\ell) a_\ell \\ + (-\beta_1) b_1 + (-\beta_2) b_2 + \dots + (-\beta_m) b_m \in w_1$$

$$\Rightarrow \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n \in w_1 \cap w_2$$

$\Rightarrow \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n$ can be written as a linear combination of element of s .

$$\Rightarrow \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n = \delta_1 a_1 + \delta_2 a_2 + \dots + \delta_\ell a_\ell \text{ and } \delta_i \in F$$

$$\Rightarrow (-\delta_1) a_1 + (-\delta_2) a_2 + \dots + (-\delta_\ell) a_\ell + \gamma_1 c_1 + \gamma_2 c_2 + \dots + \gamma_n c_n = 0$$

$$\delta_1 = \delta_2 = \dots = \delta_\ell = \gamma_1 = \gamma_2 = \dots = \gamma_n = 0_f$$

$$(\because a_1, a_2, \dots, a_\ell, c_1, c_2, \dots, c_n \text{ are L.I.})$$

Putting the value of $\gamma_1, \gamma_2, \dots, \gamma_n$ in eq.(1) we get be

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_\ell a_\ell + \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_m b_m = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_\ell = \beta_1 = \beta_2 = \dots = \beta_m = 0$$

$$(\because a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_m \text{ are L.I.})$$

Hence s_3 is linearly independent.

$$(II) W_1 + W_2 = [s_3]$$

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$$\begin{aligned} \text{Let } x_1 + x_2 &\in W_1 + W_2 \\ \Rightarrow x_1 &\in W_1 \text{ and } x_2 \in W_2 \\ \Rightarrow x_1 &= \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_r a_r + \mu_1 b_1 + \mu_2 b_2 + \dots + \mu_m b_m \\ x_2 &= \theta_1 a_1 + \theta_2 a_2 + \dots + \theta_s a_s + \phi_1 c_1 + \dots + \phi_n c_n \end{aligned}$$

where $\lambda_1, \mu_1, \theta_1, \phi_1 \in F$

$$\begin{aligned} \therefore x_1 + x_2 &= (\lambda_1 + \theta_1) a_1 + (\lambda_2 + \theta_2) a_2 + \dots + (\lambda_r + \theta_r) a_r \\ &\quad + \mu_1 b_1 + \dots + \mu_m b_m + \phi_1 c_1 + \dots + \phi_n c_n \end{aligned}$$

$$\therefore W_1 + W_2 = [s_2]$$

Therefore s_2 is basis of $W_1 + W_2$.

$$\begin{aligned} \therefore \dim(W_1 + W_2) &= \ell + m + n = (\ell + m) + (\ell + n) - \ell \\ \Rightarrow \dim(W_1 + W_2) &= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \end{aligned}$$

Q.12. Let $w_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\}$ and $w_2 = \{(\lambda, 2\lambda, -3\lambda, 4\lambda) : \lambda \in \mathbb{R}\}$. Prove that w_1 and w_2 are subspaces of $\mathbb{R}^4$ and $\mathbb{R}^4$ is a direct sum of w_1 and w_2 . (2016)

Sol. Given that

$$\begin{aligned} W_1 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\} \\ \text{and } W_2 &= \{(\lambda, 2\lambda, -3\lambda, 4\lambda) : \lambda \in \mathbb{R}\} \end{aligned}$$

Therefore

$$\dim w_1 = 3 \text{ and } \dim w_2 = 1$$

Now

$$\begin{aligned} W_1 \cap W_2 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_3 = 0\} \\ &\quad \begin{aligned} x_2 &= 2x_1 \\ x_3 &= -3x_1 \\ x_4 &= 4x_1 \end{aligned} \end{aligned}$$

So from condition

$$x_1 - 4x_1 - 3x_1 = 0$$

$$\Rightarrow -6x_1 = 0 \Rightarrow x_1 = 0$$

$$\text{Therefore } x_2 = 0, x_3 = 0 \text{ and } x_4 = 0$$

$$\text{Hence } W_1 \cap W_2 = \{0\}$$

$$\begin{aligned} \text{Now } \dim V &= \dim W_1 + \dim W_2 = 3 + 1 = 4 \\ \text{therefore } \dim V &= \dim W_1 + \dim W_2 \end{aligned}$$

$$\text{and } W_1 \cap W_2 = \{0\}$$

$$\text{Hence } V = W_1 \oplus W_2$$

ie, w_1 and w_2 is direct sum.

Q.13. Define subspace generated by a subset of vector space. Find the equa-
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tions of the subspaces of $\mathbb{R}^3(\mathbb{R})$ generated by

$$(i) S_1 = \{(1, 1, 2), (2, 3, 5), (3, 5, 8)\}$$

$$(ii) S_2 = \{(2, 2, 4), (3, 3, 6)\}$$

Also obtain the dimension of the linear sum of the subspaces generated by S_1 and S_2 . (2012)

Sol. The subspace generated by a subset is the smallest subspace containing the subset. It is denoted by $[S]$.

$$(i) \text{ let } (x, y, z) \in \mathbb{R}^3$$

$$(x, y, z) = \alpha(1, 1, 2) + \beta(2, 3, 5) + \gamma(3, 5, 8)$$

$$x = \alpha + 2\beta + 3\gamma, y = \alpha + 3\beta + 5\gamma, z = 2\alpha + 5\beta + 8\gamma$$

$$\alpha = 3x - 2y + k, \beta = y - x - 2k, \gamma = k$$

$$(ii) (x, y, z) \in \mathbb{R}^3$$

$$(x, y, z) = \alpha(2, 2, 4) + \beta(3, 3, 6)$$

$$x = 2\alpha + 3\beta, y = 2\alpha + 3\beta, z = 4\alpha + 6\beta$$

$$\alpha = k, \beta = (x - 2k)/3, \gamma = y - 2z$$

$$\text{dimension of } S_1 = 2$$

$$\text{dimension of } S_2 = 1$$

$$\text{dimension of linear sum} = \dim S_1 + \dim S_2 = 2 + 1 = 3$$

Q.14. If W is a subspace of a finite dimensional vector space V , prove that there exists, a subspace, W' of V such that $V = W \oplus W'$. Is W' unique? Justify your answer. (2012)

Sol. Let $\dim W = m$. Let the set $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be a basis of W . Since S_1 is a linearly independent subset of V , therefore S_1 can be extended to form a basis of V . Let the set $S = \{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_r\}$ be a basis of V . Let W' be the subspace of V generated by the set $S_2 = \{\beta_1, \beta_2, \dots, \beta_r\}$. We shall prove that $V = W \oplus W'$.

For this we need to show that

$$V = W + W' \text{ and } W \cap W' = \{0\}$$

Let α be any element of V . Then

$$\alpha = \text{a linear combination of elements of } S$$

$$= \text{a linear combination of } S_1 + \text{a linear combination of } S_2$$

$$= \text{an element of } W + \text{an element of } W'$$

$$V = W + W'$$

Again let $\beta \in W \cap W'$. Then β can be expressed as a linear combination of S_1 and also as a linear combination of S_2 .

So we have

$$\Rightarrow \beta = a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_m \alpha_m = b_1 \beta_1 + b_2 \beta_2 + \dots + b_r \beta_r$$

$$\Rightarrow a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_m \alpha_m - b_1 \beta_1 - b_2 \beta_2 - \dots - b_r \beta_r = 0$$

$$\Rightarrow a_1 = 0, a_2 = 0, \dots, a_m = 0, b_1 = 0, b_2 = 0, \dots, b_r = 0$$

Since $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_r$ are linearly independent.

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$\therefore \beta = 0$ (zero vector)

Thus $W_1 \cap W_2 = \{0\}$

Hence $V = W \oplus W'$

Q.15. Define linearly dependent and linearly independent vector space. Let $\{a, b, c\}$ be a basis of $\mathbb{R}^3(\mathbb{R})$ and $d = \alpha a + \beta b + \gamma c$, $\alpha, \beta, \gamma \in \mathbb{R}$. Then prove that $\{a - d, b - d, c - d\}$ will be basis of $\mathbb{R}^3(\mathbb{R})$ if and only if $\alpha + \beta + \gamma \neq 1$. (2012)

Sol. **Proof.** Vectors $a_1, a_2, \dots, a_n$ are called linearly dependent if $\exists$ scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0$ in which atleast one $\lambda_i = 0_F$ otherwise vectors are called linearly independent.

Let $x, y, z \in \mathbb{R}$

$$\therefore x(a - d) + y(b - d) + z(c - d) = (0, 0, 0)$$

$$\Rightarrow x[a - (\alpha a + \beta b + \gamma c)] + y[b - (\alpha a + \beta b + \gamma c)] + z[c - (\alpha a + \beta b + \gamma c)] = (0, 0, 0)$$

$$\Rightarrow x\{(1 - \alpha)a - \beta b - \gamma c\} + y\{-\alpha a + (1 - \beta)b - \gamma c\} + z\{-\alpha a - \beta b + (1 - \gamma)c\} = (0, 0, 0)$$

$$\Rightarrow \{(1 - \alpha)x - \alpha y - \alpha z\}a + \{-\beta x + (1 - \beta)y - \beta z\}b + \{-\gamma x - \gamma y + (1 - \gamma)z\}c = (0, 0, 0)$$

Since a, b, c are linearly independent therefore $(1 - \alpha)x - \alpha y - \alpha z = 0$

$$\text{or } (\alpha - 1)x + \alpha y + \alpha z = 0$$

$$\text{and } \beta x + (\beta - 1)y + \beta z = 0$$

$$\text{and } \gamma x + \gamma y + (\gamma - 1)z = 0 \quad \dots\dots\dots(i)$$

$\{a - d, b - d, c - d\}$ will be linearly independent iff values of x, y, z will be zero, or system (i) has only trivial solution or iff coefficient matrix of system (i) is non-singular or iff

$$\begin{vmatrix} \alpha - 1 & \alpha & \alpha \\ \beta & \beta - 1 & \beta \\ \gamma & \gamma & \gamma - 1 \end{vmatrix} \neq 0$$

$$\text{or iff } \begin{vmatrix} \alpha + \beta + \gamma - 1 & \alpha + \beta + \gamma - 1 & \alpha + \beta + \gamma - 1 \\ \beta & \beta - 1 & \beta \\ \gamma & \gamma & \gamma - 1 \end{vmatrix} = 0$$

$$\text{or iff } (\alpha + \beta + \gamma - 1) \begin{vmatrix} 1 & 1 & 1 \\ \beta & \beta - 1 & \beta \\ \gamma & \gamma & \gamma - 1 \end{vmatrix} \neq 0$$

$$\text{or iff } \alpha + \beta + \gamma - 1 \neq 0$$

Q.16. If W is a subspace of a finite dimensional vector space V over a field, prove that:

$$\dim V/W = \dim V - \dim W,$$

and hence find the total number of cosets of W in V when W is a subspace of V over the field $\mathbb{Z}_3$ and $\dim V = 5$, $\dim W = 3$. (2012)

Ans. **First Part:-**

Go through Q.2.

Second Part:-

Given that $\dim V = 5$, $\dim W = 3$

Total number of cosets of subspace W in the vector space V = Total number of element in quotient space.

$$= (\text{No. of elements in a field } \mathbb{Z}_3)^{\dim \frac{V}{W}}$$

$$= (3)^{\dim V - \dim W} = (3)^{5 - 3} = 3^2 = 9$$

Q.17. Show that the vector space $\mathbb{R}^3(\mathbb{R})$ is the linear sum of a vector space. If $S = \{a_1, a_2, \dots, a_n\}$ is a subset of a vector space V over a field F , then prove that the following statement are equivalent:

I: S is linearly independent and S generates V .

II: S is a maximal linearly independent and S generates V . (2013)

Sol. Given that $W_1 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x - y + z = 0\}$

$$\text{and } W_2 = \left\{ (x, y, z) : x, y, z \in \mathbb{R} \text{ and } \frac{x}{2} = \frac{y}{1} = \frac{z}{1} \right\}$$

let $(x, y, z) \in \mathbb{R}^3(\mathbb{R})$

$$\Rightarrow (a, b, c) + (a', b', c') = (x, y, z)$$

$$\text{or } a + a' = x, b + b' = y, c + c' = z$$

$$\text{since } a - b + c = 0 \text{ and } a' = 2b' - 2c'$$

$$c - b + a' = x$$

$$z - c' - y + b' + a' = x$$

$$a' + b' - c' = x + y - z$$

$$2b' + b' - b' = x + y - z$$

$$b = \frac{x + y - z}{2}$$

$$a = x - (x + y - z) \Rightarrow a = -y + z$$

$$\Rightarrow a' = x + y - z$$

$$b = y - b' = y - \left(\frac{x + y - z}{2} \right) = \frac{-x + y + z}{2}$$

$$c' = \frac{x + y - z}{2}$$

$$c = z - c' = z - \left(\frac{x + y - z}{2} \right) = \frac{-x - y + 3z}{2}$$

$$(x, y, z) = (a, b, c) + (a', b', c')$$

$$= \left(-y + z, \frac{-x + y + z}{2}, \frac{-x - y + 3z}{2} \right) + \left(x + y - z, \frac{-x + y + z}{2}, \frac{-x - y + 3z}{2} \right)$$

$\in W_1 \qquad \qquad \qquad \in W_2$

$$\text{Let } (a, b, c) \in W_1 \cap W_2$$

$$a - b + c = 0 \qquad \frac{a}{2} = \frac{b}{1} = \frac{c}{-1}$$

$$\Rightarrow a - \frac{a}{2} + \frac{a}{2} = 0$$

$$a = 2b = 2c$$

$$\Rightarrow a = 0$$

$$\Rightarrow b = 0 \text{ and } c = 0$$

$$(a, b, c) = (0, 0, 0)$$

$$\Rightarrow W_1 \cap W_2 = \{0\}$$

Q.18. Define maximal linearly independent subset of a vector space. If $S = \{a_1, a_2, \dots, a_n\}$ is a subset of a vector space V over a field F . Then prove that the following statement are equivalent:

I : S is linearly independent and S generates V .

II : S is a maximal linearly independent subset of V .

(2013)

Ans. **Maximal linearly independent subset of a vector space :-**

A subset S of a vector space V over F is called maximal linearly independent subset of the vector space iff it satisfy the following conditions -

(1) S is linearly independent

(2) A proper super set of S is linearly independent.

I $\Rightarrow$ II

Let $S = \{a_1, a_2, \dots, a_n\} \subseteq V$ be a basis of V i.e. S is linearly independent and S generates V .

$$\Rightarrow V = [S]$$

$$\text{Let } x \in V$$

$$\therefore x = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n \qquad \alpha_i \in F \quad \dots (i)$$

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If possible let x has another expression i.e.

$$x = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n$$

From eq.(i) and (ii)

$$\beta_i \in F \quad \dots (ii)$$

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n$$

$$\Rightarrow (\alpha_1 - \beta_1) a_1 + (\alpha_2 - \beta_2) a_2 + \dots + (\alpha_n - \beta_n) a_n = 0$$

$\therefore a_1, a_2, \dots, a_n$ are linearly independent.

$$\Rightarrow \alpha_i - \beta_i = 0_F \quad \dots, \alpha_n - \beta_n = 0_F$$

$$\Rightarrow \alpha_i = \beta_i \quad \dots, \alpha_n = \beta_n$$

Therefore expression is unique.

II $\Rightarrow$ I :-

Let each $x \in V$ can be uniquely expressed as a linear combination of

S .

Claim:- S is linearly independent.

$$\text{Let } \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0_F a_1 + 0_F a_2 + \dots + 0_F a_n$$

($\because$ Expression is unique)

$$\Rightarrow \lambda_i = 0_F$$

$$\lambda_n = 0_F$$

S is linearly independent.

Therefore S is a basis of V i.e.

S is linearly independent and S generates V .

Q.19. Let $S_1 = \{a_1, a_2, \dots, a_n\}$ be linearly independent subset of a vector space V over a field F and $a \in V$ but $a \notin L_{S_1}$.

Then show that the set $S_2 = \{a, a_1 + \lambda_1 a, a_2 + \lambda_2 a, \dots, a_n + \lambda_n a\}$ where $\lambda_i \in F$ will be linearly independent. (2013)

Sol. $\because \{a_1, a_2, \dots, a_n\}$ is linearly independent and $a \notin L_{S_1}$. Therefore $\{a_1, a_2, \dots, a_n, a\}$ is also linearly independent.

Let

$$\mu_1 a + \mu_2 (a_1 + \lambda_1 a) + \mu_3 (a_2 + \lambda_2 a) + \dots + \mu_n (a_n + \lambda_n a) = 0_F$$

$$\Rightarrow \mu_1 a + \mu_2 a_1 + \mu_3 a_2 + \dots + \mu_n a_n + \mu_2 \lambda_1 a + \mu_3 \lambda_2 a + \dots + \mu_n \lambda_n a = 0$$

$$\Rightarrow \mu_1 a + \mu_2 a_1 + \mu_3 a_2 + \dots + \mu_n a_n + \lambda a (\mu_2 + \mu_3 + \dots + \mu_n) = 0$$

$$\Rightarrow [\lambda (\mu_2 + \mu_3 + \dots + \mu_n) + \mu_1] a + \mu_2 a_1 + \dots + \mu_n a_n = 0$$

$$\Rightarrow \mu_2 = 0 = \mu_3 = 0 \dots = \mu_n = 0$$

$$\text{and } \lambda (\mu_2 + \mu_3 + \dots + \mu_n) + \mu_1 = 0$$

(as $\{a, a_1, a_2, \dots, a_n\}$ are linearly independent)

$$\Rightarrow \lambda \cdot 0 + \mu_1 = 0 \quad \Rightarrow \mu_1 = 0$$

$$\Rightarrow \mu_1 = \mu_2 = 0 = \mu_3 = 0 \dots = \mu_n$$

Hence $S_2 = \{a, a_1 + \lambda_1 a, a_2 + \lambda_2 a, \dots, a_n + \lambda_n a\}$ are linearly independent.

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Q.20. Let V be a finitely generated vectors space over a field F . Prove that V contains a basis over F . (2017)

Sol. Let V be a finitely generated vector space over field F then $\exists$ finite subset

$$S = \{a_1, a_2, \dots, a_n\} \subseteq V \text{ such that}$$

$$V = [S] \text{ or } V = L(S)$$

$\therefore S$ is finite subset of V without loss of generality we can assume that none of the vector in S is zero vectors.

Now, if S is linearly independent then S will be a basis of V and then we are done. Assume that, if S is linearly dependent then $\exists$ a maximal linearly independent subset of S of S .

i.e. $S' = \{a_1, a_2, \dots, a_r\}$ is maximal linearly independent subset of S containing r vectors ($r < n$) of S and any $a_i \in S$ but $a_i \notin S'$.

Now, since $\{a_1, a_2, \dots, a_r\}$ are linearly independent and $\{a_1, \dots, a_n\}$ is maximal linearly independent then $\exists$ scalar $\alpha_1, \alpha_2, \dots, \alpha_r \in F$ such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_r a_r = 0 \quad \text{----- (i)}$$

where at least one $\alpha_i \neq 0$

$$\Rightarrow a_i \neq 0$$

otherwise if $\alpha_i = 0$, then by eq.(i)

$$\alpha_1 a_1, \dots, \alpha_r a_r = 0$$

where at least one $\alpha_i \neq 0$.

then S' is linearly independent which is a contradiction to $a_i \neq 0$.

and hence α_i will be linear combination of $\{a_1, \dots, a_r\}$ i.e. element of S' .

$$\Rightarrow a_i = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_r a_r$$

$$\Rightarrow V = [S']$$

Hence S' is a basis of V .

Q.21. Prove that $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^4$. Determine $\dim W$ and a subspace U of W such that $\mathbb{R}^4 = W \oplus U$. (2017)

Sol. Given that

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

Let $A, B \in W$

i.e. $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4) \in W$ and $\alpha, \beta \in \mathbb{R}$.

$\therefore A \in W$ and $B \in W$

$$\text{then } a_1 + a_2 + a_3 + a_4 = 0 \text{ and } b_1 + b_2 + b_3 + b_4 = 0$$

$$\Rightarrow \alpha(a_1 + a_2 + a_3 + a_4) = 0 \text{ and } \beta(b_1 + b_2 + b_3 + b_4) = 0$$

$$\Rightarrow (\alpha a_1 + \beta b_1) + (\alpha a_2 + \beta b_2) + (\alpha a_3 + \beta b_3) + (\alpha a_4 + \beta b_4) = 0$$

$$\Rightarrow (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3, \alpha a_4 + \beta b_4) \in W$$

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$$\Rightarrow \alpha A + \beta B \in W$$

$$\text{i.e. } \forall A, B \in W \Rightarrow \alpha A + \beta B \in W$$

Hence, W is a subspace of $\mathbb{R}^4$.

Clearly,

$$\text{Let } x_1 = -x_2 - x_3 - x_4$$

$$\text{Take } x_2 = 0, x_3 = 1, x_4 = -1$$

$$x_2 = 1, x_3 = -1, x_4 = 1$$

$$x_2 = -1, x_3 = 1, x_4 = 0$$

$$x_2 = -1, x_3 = 0, x_4 = 1$$

$$x_2 = 1, x_3 = 0, x_4 = -1$$

Let $S = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0), (-1, 0, 1), (1, 0, -1)\}$ which is linearly dependent because $(1, 0, -1) = -(-1, 0, 1)$

Let $S_1 = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0), (-1, 0, 1)\}$

No one is scalar multiple of other

Hence, $\dim W = 4$

$$\text{Given, } \mathbb{R}^4 = W \oplus U$$

$$\Rightarrow 4 = 4 + \dim U \Rightarrow \dim U = 0$$

Thus, the subspace U is generated by $(1, 0, 0, 0)$.

Q.22. Prove that union of two subspaces of a vector space is a subspace of the vector space if and only if one of them is contained in the other. Give an example to show that the union of two subspaces of a vector space may not be a subspace of the vector space. (2018)

Sol. Let w_1 and w_2 be subspace of vector space $V(F)$ and let $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$, then we have to show that $w_1 \cup w_2$ is subspace.

$$\therefore w_1 \cup w_2 = w_1 \quad \left\{ \text{If } w_2 \subseteq w_1 \right\}$$

$$\text{And } w_1 \cup w_2 = w_2 \quad \left\{ \text{If } w_1 \subseteq w_2 \right\}$$

Thus $w_1 \cup w_2$ is a subspace of vector space $V(F)$.

Conversely -

Let $w_1 \cup w_2$ is a subspace of vector space $V(F)$ then we have to

show that $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$.

Let, if possible $w_1 \not\subseteq w_2$ or $w_2 \not\subseteq w_1$.

let $a \in w_1 - w_2$ and $b \in w_2 - w_1$

$$\Rightarrow a \in w_1 \text{ and } a \notin w_2 \text{ and } b \in w_2 \text{ and } b \notin w_1$$

$$\text{If } a \in w_1 \Rightarrow a \in w_1 \cup w_2$$

$$\text{And } b \in w_2 \Rightarrow b \in w_1 \cup w_2$$

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$$a, b \in w_1 \cup w_2 \Rightarrow a + b \in w_1 \cup w_2 \quad \{\because w_1 \cup w_2 \text{ is subspace}\}$$

$$\Rightarrow a, b \in w_1 \cup w_2 \Rightarrow a + b \in w_1 \text{ or } a + b \in w_2$$

Case (I) When $a + b \in w_1$ -

$$\because a + b \in w_1, a \in w_1 \text{ and } b \notin w_1$$

$$\Rightarrow a \in w_2, -a \in w_2, \text{ and } b \notin w_1$$

$$\Rightarrow -a + (a + b) \in w_1 \text{ and } b \notin w_1$$

$$\Rightarrow (-a + a) + b \in w_1 \text{ and } b \notin w_1 \quad (\text{associativity})$$

$$\Rightarrow 0 + b \in w_1 \text{ and } b \notin w_1$$

$$\Rightarrow b \in w_1 \text{ and } b \notin w_1$$

which is self contradiction or which is not possible.

Case (II) When $a + b \in w_2$ -

$$\because a + b \in w_2, b \in w_2 \text{ and } a \notin w_2$$

$$\Rightarrow a + b \in w_2, -b \in w_2 \text{ and } a \notin w_2$$

$$\Rightarrow -b + (a + b) \in w_2 \text{ and } a \notin w_2$$

$$\Rightarrow a + (-b + b) \in w_2 \text{ and } a \notin w_2$$

$$\Rightarrow a + 0 \in w_2 \text{ and } a \notin w_2$$

$$\Rightarrow a \in w_2 \text{ and } a \notin w_2$$

which is self contradiction or which is not possible which is contradiction to an assumption $w_1 \not\subseteq w_2$ and $w_2 \not\subseteq w_1$ is wrong.

Hence, $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$.

Thus union of two subspace of vector space $V(F)$ is subspace iff one contain in other.

Example -

(1) Let $w_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$ and $w_2 = \{(0, y, 0) : y \in \mathbb{R}\}$ be subspace of $\mathbb{R}^3(\mathbb{R})$.

$$\because (1, 0, 0) \in w_1 \Rightarrow (1, 0, 0) \in w_1 \cup w_2$$

$$\text{And } (0, 1, 0) \in w_2 \Rightarrow (0, 1, 0) \in w_1 \cup w_2$$

$$(1, 0, 0) + (0, 1, 0) \Rightarrow (1, 1, 0) \notin w_1 \cup w_2$$

Thus, union is not subspace.

(2) Let, $V_4(\mathbb{Z}_2)$ be vector space.

$$\text{Let } w_1 = \{e, a\} \text{ and } w_2 = \{e, b\}$$

$\because w_1$ and w_2 is a subspace of $V_4(\mathbb{Z}_2)$.

but $w_1 \cup w_2 = \{e, a, b\}$ is not a subspace of $V_4(\mathbb{Z}_2)$.

Thus union is not subspace.

Q.23. Define a subspace of a vector space. If W is a subspace of $\mathbb{R}^3$ generated by $\{(1, 1, -2), (0, -1, 1), (3, -2, -1)\}$ then find the dimension of another sub-

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space W' of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W \oplus W'$.

Subspace: -

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(2018)

Let $(V, +, \cdot)$ be a vector space over field $(F, \oplus, \odot)$. A non empty subset W of $V(F)$ is called subspace of vector space $V(F)$ if W also forms V.S. over F .

Example -

(1) Q is subspace of $\mathbb{R}(Q)$.

(2) $\mathbb{R}$ is a subspace of $C(\mathbb{R})$.

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

$$R_2 \leftrightarrow R_3 \quad A = \begin{pmatrix} 1 & 1 & -2 \\ 3 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1 \quad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & -1 & -1 \end{pmatrix}$$

$$R_2 \rightarrow -\frac{1}{5}R_2 \quad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$R_3 \rightarrow R_3 + R_2 \quad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is an Echelon form.

Dimension of $W = 2$.

Hence $\mathbb{R}^3 = W \oplus W'$

$$\Rightarrow \dim \mathbb{R}^3 = \dim W + \dim W'$$

$$\Rightarrow 3 = 2 + \dim W'$$

$$\Rightarrow \dim W' = 3 - 2 = 1$$

So dimension of W' is 1.

Q.24. Let W be a subspace of a vector space V over a field F . If V is finite dimensional, then prove that

$$\dim(V/W) = \dim V - \dim W.$$

(2018)

Sol. Since W be subspace of vector space $V(F)$. Let $S_1 = (a_1, a_2, \dots, a_n)$ be the basis of W then

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$$\dim W = m$$

.....(1)

$\therefore S_1$ is linearly independent subset of V $\{S_1 \subset W \subseteq V \Rightarrow S_1 \subseteq V\}$
then by extension theorem S_1 is either basis of V or can be extended to form basis of V .

Let $S_2 = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ be basis of V then
 $\dim V = m+n$ (2)

Now let $S = \{W + b_1, W + b_2, \dots, W + b_n\} \subseteq V/W$

Claim -

S is basis of V/W -

$\therefore \{W + b_1, W + b_2, \dots, W + b_n\} \in V/W$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$
Such that

$$\alpha_1(W + b_1) + \alpha_2(W + b_2) + \dots + \alpha_n(W + b_n) = W \quad \text{.....(3)}$$

$$\Rightarrow (W + \alpha_1 b_1) + (W + \alpha_2 b_2) + \dots + (W + \alpha_n b_n) = W$$

[from scalar multiplication of a coset]

$$\Rightarrow W + \{\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n\} = W$$

[because of property of a coset]

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \in W$$

[because property $W + a \Leftrightarrow a \in W = W$]

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \in W \text{ and } S_1 \text{ is basis of } W.$$

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \in [S_1]$$

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_m a_m$$

$$\Rightarrow (-\beta_1)a_1 + (-\beta_2)a_2 + \dots + (-\beta_m)a_m + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = 0$$

$$\Rightarrow -\beta_1 = -\beta_2 = \dots = -\beta_m = 0 \text{ and } \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

[because $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$ being basis elements are linearly independent]

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Hence S is linearly independent.

$$(2) V/W = [S]$$

For this

$$\text{Let } W + x \in V/W \Rightarrow x \in V$$

$\therefore x \in V$ and S_2 is basis of V .

$$\Rightarrow x = \theta_1 a_1 + \theta_2 a_2 + \dots + \theta_m a_m + \phi_1 b_1 + \phi_2 b_2 + \dots + \phi_n b_n$$

[where θ_i 's and ϕ_j 's are unique elements]

$$\therefore W + x = W + (\theta_1 a_1 + \theta_2 a_2 + \dots + \theta_m a_m + \phi_1 b_1 + \phi_2 b_2 + \dots + \phi_n b_n)$$

$$w \in W$$

$$\Rightarrow W + x = (W + w) + (W + (\phi_1 b_1 + \phi_2 b_2 + \dots + \phi_n b_n))$$

$$\Rightarrow W + x = W + (W + \phi_1 b_1 + \phi_2 b_2 + \dots + \phi_n b_n)$$

$$\Rightarrow W + x = (W + \phi_1 b_1) + (W + \phi_2 b_2) + \dots + (W + \phi_n b_n)$$

$$W + x = \phi_1(W + b_1) + \phi_2(W + b_2) + \dots + \phi_n(W + b_n)$$

[from definition of scalar multiplication of coset]

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$$\Rightarrow V/W = [S]$$

From (1) and (2)

S is a basis of V/W

then $\dim V/W = n$

$$\Rightarrow \dim V/W = (m+n) - n$$

$$\Rightarrow \dim V/W = \dim V - \dim W$$

Q.25. Show that the vector space $\mathbb{R}^3$ over $\mathbb{R}$ is the sum of its sub-spaces

$$w_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0\} \text{ and } w_2 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x}{-1} = \frac{y}{1} = \frac{z}{1} \right\}.$$

Is this sum a direct sum? Justify your answer.

(2019)

Sol. Given that $w_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0\}$ and

$$w_2 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x}{-1} = \frac{y}{1} = \frac{z}{1} \right\}$$

Let $(x, y, z) \in \mathbb{R}^3(\mathbb{R})$

$$\Rightarrow (a, b, c) + (a', b', c') = (x, y, z)$$

$$\text{or } a + a' = x, b + b' = y, c + c' = z$$

$$\text{since } a + 2b + c = 0 \text{ and } a' = -b' = -c'$$

$$\text{Then } -2b - c + a' = x$$

$$\Rightarrow -2(y - b') - (z - c') + a' = x$$

$$\Rightarrow -2y + 2b' - z + c' + a' = x$$

$$\Rightarrow a' + 2b' + c' = x + 2y + z$$

$$\Rightarrow -b + 2b' + b' = x + 2y + z$$

$$\Rightarrow b' = \frac{(x + 2y + z)}{2}$$

$$\text{and } a' - 2a' - a' = x + 2y + z$$

$$a' = \frac{(x + 2y + z)}{-2} \text{ and } c' = \frac{(x + 2y + z)}{2}$$

$$\text{Now } a = x - a', b = y - b', c = z - c'$$

$$a = x - \frac{(x + 2y + z)}{-2}, b = y - \frac{(x + 2y + z)}{2}, c = z - \frac{(x + 2y + z)}{2}$$

$$a = \frac{+3x + 2y + z}{2}, b = \frac{-(x + z)}{2}, c = \frac{-(x + 2y - z)}{2}$$

$$(x, y, z) = (a, b, c) + (a', b', c')$$

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$$= \left(\frac{(3x+2y+z)}{2}, \frac{-(x+z)}{2}, \frac{-(x+2y-z)}{2} \right) \in w_1$$

$$+ \left(\frac{-(x+2y+z)}{2}, \frac{(x+2y+z)}{2}, \frac{(x+2y+z)}{2} \right) \in w_2$$

Let $(a, b, c) \in w_1 \cap w_2$

$$a + 2b + c = 0 \text{ and } \frac{a}{-1} = \frac{b}{1} = \frac{c}{1}$$

Then $a - 2a - a = 0$

$$\Rightarrow a = 0$$

$$a = -b = -c \Rightarrow b = 0 \text{ and } c = 0$$

$$w_1 \cap w_2 = \{0\}$$

yes sum is the direct sum.

Q.26. Determine the dimension of subspace of W of $\mathbb{R}^4$ generated by $S = (1, 1, 2, 0), (1, -1, 2, 0), (0, 1, 2, 1)$. (2019)

Sol.
$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$R_2 \rightarrow 2R_2 + R_2 \quad \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix}$$

Rank of this matrix is 3. So the dim. of space generated by vector is 3.

Q.27. For the subspace $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0, x_3 + x_4 = 0\}$ of the vector space $\mathbb{R}^4$ over $\mathbb{R}$, give a basis of the quotient space $\mathbb{R}^4/W$. (2019)

Sol.
$$W = \left\{ (x_1, x_2, x_3, x_4) : \begin{matrix} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 \end{matrix} \right\}$$

So $x_1 = -x_2$

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$$x_3 = -x_4$$

Therefore basis of W be $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Now this basis of can be extended to basis of $\mathbb{R}^4$.

and $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Will form a basis for $\mathbb{R}^4$.

So, the basis of $\frac{V}{W}$ is given by $\{w + \beta\}$ Where β belongs to basis of $\mathbb{R}^4$ not is basis of W .

$$\{W + \{1, 0, 0, 0\}, W + \{0, 0, 0, 1\}\} \text{ is form basis for } \frac{V}{W}.$$

$$\{(x_1, x_2, x_3, x_4) + (1, 0, 0, 0), (x_1, x_2, x_3, x_4) + (0, 0, 0, 0)(x_1, x_2, x_3, x_4) \in w.$$

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Unit II

Q.1. Let M_3 denote the vector space of 3×3 matrices over the field of real numbers. A map $T: M_3 \rightarrow M_3$ is defined by $T(A) = A - A^t \forall A \in M_3$. Then prove that T is linear transformation. Find a basis of $\text{Ker } T$ and hence find the nullity of T and rank of T . (2014)

Ans. Let A and $B \in M_3$ and $\alpha, \beta \in F$

$$\begin{aligned} T(\alpha A + \beta B) &= (\alpha A + \beta B) - (\alpha A + \beta B)^t \\ &= \alpha A + \beta B - \alpha A^t - \beta B^t \\ &= \alpha(A - A^t) + \beta(B - B^t) \end{aligned}$$

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$$

Hence T is linear transformation.

$$\text{Ker } T = \{A \in M_3 : T(A) = 0\}$$

$$= \{A \in M_3 : A - A^t = 0\}$$

$$= \{A \in M_3 : A^t = A\}$$

= Set of all symmetric matrix of M_3 .

$$\text{Nullity of } T = \dim \text{ker } T$$

$$= \frac{1}{2} \times 3 \times (3+1) = 6$$

Q.2. Prove or disprove -

(i) If $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ are two linear transformations such that $\text{Ker } T_1 = \text{Ker } T_2$ and $I_m T_1 = I_m T_2$, then $T_1 = T_2$. (2014)

Sol. If $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ are two linear transformation

such that $\text{ker } T_1 = \text{ker } T_2$ and $I_m T_1 = I_m T_2$.

Let $T_1(x, y) = x - y$ and $T_2(x, y) = y - x$

$$\text{Then } \text{Ker } T_1 = \left\{ (x, y) \in \mathbb{R}^2 \mid \begin{matrix} x-y=0 \\ x=y \text{ or } y=x \end{matrix} \right\}$$

$$\text{And } \text{Ker } T_2 = \left\{ (x, y) \in \mathbb{R}^2 \mid \begin{matrix} y-x=0 \\ y=x \text{ or } x=y \end{matrix} \right\}$$

$$\text{Ker } T_1 = \text{Ker } T_2$$

for all $a \in \mathbb{R}$, then there exist $(a, 0) \in \mathbb{R}^2$, such that $T_1(a, 0) = a - 0 = a$

Similarly for all $a \in \mathbb{R}$

$$T_2(0, a) = a - 0 = a$$

Here T_1 & T_2 are onto on $\mathbb{R}$.

$$I_m T_1 = I_m T_2 = \mathbb{R}$$

So, but $T_1 \neq T_2$

Hence this is disprove.

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Q.3. Prove or disapprove -

If $T: V_1 \rightarrow V_2$ is linear transformation such that $\rho(T) = 2$ and $\dim V_2 = 5$, then nullity of T will be 3. (2014)

Sol. Given that $T: V_1 \rightarrow V_2$ is a linear transformation such that $\rho(T) = 2$ and $\dim V_2 = 5$

By rank Nullity theorem, We have

$$\text{rank } T + \text{nullity of } (T) = \dim V_1$$

Let $T': V_2 \rightarrow V_1$ be linear transformation

By Rank-Nullity Theorem

For T' we have

$$\text{rank}(T') + \text{Nullity}(T') = \dim V_2$$

$$2 + \text{Nullity}(T') = 5$$

$$\text{Nullity of } (T') = 5 - 2 = 3$$

Hence it is true.

Q.4. If W is a subspace of $\mathbb{R}^4(\mathbb{R})$ spanned by subset $S = \{(2, 3, -2, 1), (4, 2, -1, 3), (2, -5, 4, 3), (4, 10, -7, 1)\}$, then dimension of the annihilator of W will be 2. (2014)

Sol. $S = \{(2, 3, -2, 1), (4, 2, -1, 3), (2, -5, 4, 3), (4, 10, -7, 1)\}$

$$\begin{bmatrix} 2 & 3 & -2 & 1 \\ 4 & 2 & -1 & 3 \\ 2 & -5 & 4 & 3 \\ 4 & 10 & -7 & 1 \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - R_1 \\ R_4 &\rightarrow R_4 - 2R_1 \end{aligned} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -4 & 3 & 1 \\ 0 & -8 & 6 & 2 \\ 0 & -4 & -3 & -1 \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_3 - 2R_2 \\ R_4 &\rightarrow R_4 + R_2 \end{aligned} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim W = 2$$

We know that $\dim W + \dim W^\circ = \dim V$

$$2 + \dim W^\circ = 4$$

$$\Rightarrow \dim W^\circ = 2$$

Hence Proved.

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Q.5. Let M_2 denote the vector space of 2×2 matrices over the field of real numbers. A linear transformation $T: M_2 \rightarrow M_2$ is defined by $T(A) = A \cdot B$

-B.A, $\forall A \in M_2$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, then find $I_m T$, Ker T and trace of T. (2014)

Ans. Standard basis of M_2 is

$$S = \{e_{11}, e_{12}, e_{21}, e_{22}\}$$

$$T(e_{11}) = e_{11}B - Be_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T(e_{11}) = 0 \cdot e_{11} + 1 \cdot e_{12} + (-1) \cdot e_{21} + 0 \cdot e_{22}$$

$$T(e_{12}) = e_{12}B - Be_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= (1) \cdot e_{11} + (1) \cdot e_{12} + (0) \cdot e_{21} + (-1) \cdot e_{22}$$

$$T(e_{21}) = e_{21}B - Be_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = (-1)e_{11} + (0)e_{12} + (-1)e_{21} + (1)e_{22}$$

Similarly $T(e_{22}) = e_{22}B - Be_{22}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (0)e_{11} + (-1)e_{12} + (1)e_{21} + (0)e_{22}$$

$$\text{matrix } X(T)_{S,S} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(T) = 0 + 1 + (-1) + 0 = 0$$

$$I_m T = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightarrow \text{Reduced in Echlon form}$$

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$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 + R_2}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echlon form

$$\text{Basis of } I_m(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\rho(T) = 2$$

$$\dim \ker T = \dim M_2 - \rho(T)$$

$$= 4 - 2 = 2$$

($\because$ Using R.N.T)

$$\text{Let } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$$

$$\Rightarrow T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} B - B \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b & a+3b \\ 2c+d & c+3d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow b = c, d = b + a, a = c$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & b+a \end{bmatrix} = \begin{bmatrix} a+0 \cdot b & 0 \cdot a + 1 \cdot b \\ 0 \cdot a + 1 \cdot b & 1 \cdot a + 1 \cdot b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} a + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} b$$

$$\ker T = \left\{ \begin{bmatrix} a & b \\ c & a+b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$\text{Basis of } \ker T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

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$$I_m T = \left\{ x \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$I_m T = \left\{ \begin{bmatrix} x & x+y \\ -y & -x \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Q.6. Find dual basis to the ordered basis

$X = \{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$ of $\mathbb{R}^3(\mathbb{R})$.

Sol. Given that $X = \{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$

Let $X^* = \{f_1, f_2, f_3\}$ be the dual basis of X .

Let $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = \alpha(1, -2, 3) + \beta(1, -1, 1) + \gamma(2, -4, 7) \quad \dots(i)$$

$$\Rightarrow \alpha + \beta + 2\gamma = x \quad \dots(ii)$$

$$-2\alpha - \beta - 4\gamma = y \quad \dots(iii)$$

$$3\alpha - \beta + 7\gamma = z \quad \dots(iv)$$

Solving eq. (ii), (iii) and (iv) we get

$$\alpha = -3x - 5y - 2z$$

$$\beta = 2x + y$$

$$\gamma = x + 2y + z$$

Putting the value α, β, γ in eq. (i) we get

$$f_1(x, y, z) = \alpha \odot f_1(1, -2, 3) \oplus \beta \odot f_1(1, -1, 1) \oplus \gamma \odot f_1(2, -4, 7)$$

$$f_1(x, y, z) = \alpha = -3x - 5y - 2z$$

Similarly $f_2(x, y, z) = \beta = 2x + y$

$$f_3(x, y, z) = \gamma = x + 2y + z$$

Q.7. Prove that a non-zero linear functional on a vector space V is injective if and only if $\dim V = 1$. (2014)

Sol. Let $f(\neq f_0): V \rightarrow F$ be a linear function and injective.

$$\Leftrightarrow \text{Ker } f = \{0\}$$

Using rank-nullity theorem for f

$$\Leftrightarrow \dim \text{Im } f + \dim \text{Ker } f = \dim V$$

$$\Leftrightarrow \dim \text{Im } f = \dim V \quad \dots(i)$$

Since image f is subspace of ' F '

$$\therefore \dim \text{Im } f \leq \dim f \quad (\because \dim \text{ of field is always } 1)$$

$$\leq 1$$

$$\therefore \dim \text{Im } f = 0, \text{ or } 1$$

$$\text{If } \dim \text{Im } f = 0,$$

$$\Rightarrow f = f_0 \text{ which is not possible.}$$

$$\therefore \dim \text{Im } f = 1$$

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Q.8. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W . Suppose that V is finite dimensional. Then prove that

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

Sol. We have prove that

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

(2015)

Proof - Let $\dim \text{ker } T = m$

Let $S = \{a_1, a_2, \dots, a_m\}$ be the basis of $\text{ker } T$.

Since S is l.i. subset of vector space V . Therefore it can be extended to form a basis of V .

Let $S_1 = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ be basis of V .
 $\dim V = m + n$

Since S_1 generates V . Therefore $T(S_1)$ generates $\text{Im } T$.

$$T(S_1) = \{T(a_1), T(a_2), \dots, T(a_m), T(b_1), T(b_2), \dots, T(b_n)\}$$

$$T(S_1) = \{O_w, O_w, \dots, O_w, T(b_1), T(b_2), \dots, T(b_n)\}$$

Let $S_2 = \{T(b_1), T(b_2), \dots, T(b_n)\}$ generates $\text{Im } T$.

Claim - S_2 is basis of $\text{Im } T$ -

For this it is sufficient to prove that S_2 is l.i.

$$\text{Let } \alpha_1 T(b_1) + \alpha_2 T(b_2) + \dots + \alpha_n T(b_n) = O_w$$

$$\Rightarrow T(\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n) = O_w \quad \{\because T \text{ is l.T.}\}$$

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \in \text{ker } T$$

$$\Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_m a_m$$

$$\Rightarrow (-\beta_1)a_1 + v(-\beta_2)a_2 + \dots + v(-\beta_m)a_m + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = 0 = O_v$$

$$\Rightarrow -\beta_1 = -\beta_2 = \dots = -\beta_m = \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Therefore S_2 is l.i. and S_2 is basis of $\text{Im } T$.

$$\therefore \dim \text{Im } T = n = m + n - m = \dim V - \dim \text{ker } T$$

$$\Rightarrow \dim V = \dim \text{Im } T + \dim \text{ker } T$$

Q.9. Prove that any two finite dimensional vector spaces V and W defined over the same field are isomorphic if and only if they have the same dimension. (2015)

Sol. We have to show that

$$V \approx W \Leftrightarrow \dim V = \dim W$$

Proof - Necessary part -

Let $V \approx W$

$\Rightarrow \exists$ a linear transformation

$$T: V \rightarrow W \text{ which is one-one and onto}$$

$$\Rightarrow \text{Ker } T = \{O_v\} \text{ and } \text{Im } T = W$$

Using Rank Nullity Theorem

$$\dim \text{Ker } T + \dim \text{Im } T = \dim V$$

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$$\dim V = \dim W = n$$

Sufficient part:-

Let $\dim V = \dim W = n$ (finite)

$$\Rightarrow X = \{x_1, x_2, \dots, x_n\}$$

$Y = \{y_1, y_2, \dots, y_n\}$ are the ordered basis of V and W respectively
define a map $T: V \rightarrow W$ as

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n$$

(i) T is linear transformation -

$$\text{Let } a = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \in V$$

$$\text{And } b = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n \in V$$

$$\lambda, \mu \in F$$

$$T(\lambda a + \mu b) = T(\lambda(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) + \mu(\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n))$$

$$= \{(\lambda \alpha_1 + \mu \beta_1) y_1 + (\lambda \alpha_2 + \mu \beta_2) y_2 + \dots + (\lambda \alpha_n + \mu \beta_n) y_n\}$$

$$= \lambda(\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n) + \mu(\beta_1 y_1 + \beta_2 y_2 + \dots + \beta_n y_n)$$

$$\Rightarrow T \text{ is linear transformation.}$$

(ii) T is one-one -

$$\text{Let } T(a) = T(b)$$

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = \beta_1 y_1 + \beta_2 y_2 + \dots + \beta_n y_n$$

$$\Rightarrow (\alpha_1 - \beta_1) y_1 + (\alpha_2 - \beta_2) y_2 + \dots + (\alpha_n - \beta_n) y_n = 0$$

$$\Rightarrow (\alpha_1 - \beta_1) = 0, (\alpha_2 - \beta_2) = 0, \dots, (\alpha_n - \beta_n) = 0$$

$$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n$$

$$\Rightarrow a = b$$

(iii) T is onto

$$\therefore T: V \rightarrow W \text{ is one-one.}$$

$$\Rightarrow \ker T = \{0_V\}$$

$$\Rightarrow \dim \ker T = 0$$

By Rank-Nullity Theorem

$$\dim \text{Im } T + \dim \ker T = \dim V$$

$$\Rightarrow \dim \text{Im } T = \dim V$$

$$\therefore T \text{ is onto.}$$

 $\therefore T$ is an isomorphism.

$$\text{Therefore } V \cong W$$

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Q.10. Define range space, null space of a linear transformation $T: V \rightarrow W$. Show that range of T is a subspace of W and null space of T is a subspace of V .

Ans. Range space of a linear transformation:-

(2015)

Let $T: V \rightarrow W$ be a linear transformation. The range of T , denoted by $R(T)$ or $T(V)$ is defined as $T(V) = \{T(x) : x \in V\}$.

Null Space:-

Let $T: V \rightarrow W$ be a linear transformation. The kernel of T , denoted by $\ker T$ or $N(T)$, is defined as

$$\ker T = \{x \in V : T(x) = 0 \in W\}$$

$\ker T$ is also called the null space of T .

Now we prove that range of T is a subspace of W and null space of T is a subspace of V .

By definition, range of T is

$$R(T) = \{T(x) : x \in V\}$$

Q.11. If A is a real symmetric matrix, then prove that there exists a real symmetric matrix B such that $B^3 = A$. (2014)

Sol. If A is symmetric over R .

Then it is diagonalizable

$$A = P D P^{-1} \quad \text{Such that } D \text{ is diagonal}$$

Let $B = P D_2 P^{-1}$ such that D_2 is a diagonal matrix whose values are third root of the matrix so we get.

$$B^3 = P D_2^3 P^{-1} = P D P^{-1} = A$$

Let $u_1, u_2 \in R(T)$ and $\alpha, \beta \in F$. Then $u_1 = T(x_1)$, $u_2 = T(x_2)$, for some $x_1, x_2 \in V$ we have.

$$= \alpha T(x_1) + \beta T(x_2)$$

$$= T(\alpha x_1 + \beta x_2) \quad (\text{as } T \text{ is C.T})$$

$$= T(x) \quad \text{where } x = \alpha x_1 + \beta x_2 \in V.$$

Since $T(x) \in R(T)$, so $\alpha u_1 + \beta u_2 \in R(T)$. Hence $R(T)$ is a subspace of W .

By definition, $\ker T = \{x \in V : T(x) = 0 \in W\}$

We know that $T(0) = 0$ and so $0 \in \ker T$ that is $\ker T$ is non-empty.

Let $x, y \in \ker T$ and $\alpha, \beta \in F$.

then $T(x) = T(y) = 0$, since T is a L.T.

$$\text{we have } T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$\Rightarrow \alpha x + \beta y \in \ker T$$

Hence $\ker T$ is a subspace of V .

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Q.12. Find the dual basis of the basis

$B = \{(1, -1, 3), (0, 1, -1), (0, 3, -2)\}$ for $\mathbb{R}^3(\mathbb{R})$.

Sol. Let $\alpha_1 = (1, -1, 3)$, $\alpha_2 = (0, 1, -1)$, $\alpha_3 = (0, 3, -2)$

Then $B = \{\alpha_1, \alpha_2, \alpha_3\}$

If $B' = \{f_1, f_2, f_3\}$ is dual basis of B,

$$\text{then } f_1(\alpha_1) = 1, f_1(\alpha_2) = 0, f_1(\alpha_3) = 0$$

$$f_2(\alpha_1) = 0, f_2(\alpha_2) = 1, f_2(\alpha_3) = 0$$

$$f_3(\alpha_1) = 0, f_3(\alpha_2) = 0, f_3(\alpha_3) = 1$$

Now to find explicit expressions for f_1, f_2, f_3

Let $(a, b, c) \in \mathbb{R}^3(\mathbb{R})$

$$\text{Let } (a, b, c) = x(1, -1, 3) + y(0, 1, -1) + z(0, 3, -2) \quad \dots (1)$$

$$= x\alpha_1 + y\alpha_2 + z\alpha_3$$

$$\text{Then } f_1(a, b, c) = x, f_2(a, b, c) = y$$

$$f_3(a, b, c) = z$$

Now to find the values of x, y, z from eq.(1) we have

$$x = a$$

$$-x + y + 3z = b$$

$$3x - y - 2z = 0$$

Solving these equations

$$x = a, y = 7a - 2b - 3c, z = b + c - 2a$$

$$\text{Hence } f_1(a, b, c) = a$$

$$f_2(a, b, c) = 7a - 2b - 3c$$

$$\text{and } f_3(a, b, c) = 2a + b - c$$

therefore $B' = \{f_1, f_2, f_3\}$ as a dual basis of B.

Q.13. Let V_1, V_2 and V_3 be finite dimensional vector spaces over a field F . Let $T_1: V_1 \rightarrow V_2$ and $T_2: V_2 \rightarrow V_3$ be linear transformations. Prove that $T_2 \circ T_1: V_1 \rightarrow V_3$ is a linear transformation and $\text{rank}(T_2 \circ T_1) \leq \min\{\text{rank}(T_1), \text{rank}(T_2)\}$. (2016, 19)

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(2015) Sol. Let v_1, v_2 and v_3 be finite dimensional vector space over a field F .
let $T_1: (v_1, +, \cdot) \rightarrow (v_2, +, \cdot)$ and

$$T_2: (v_2, +, \cdot) \rightarrow (v_3, +, \cdot) \text{ be } \ell.t$$

Then $T_2 \circ T_1: (v_1, +, \cdot) \rightarrow (v_3, +, \cdot)$ defined as

$$(T_2 \circ T_1)(x) = T_2(T_1(x)) \quad \forall x \in v_1$$

Claim :- $T_2 \circ T_1$ is a $\ell.t$.

let $a, b \in v_1$ and $\alpha, \beta \in F$

$$(T_2 \circ T_1)(\alpha a + \beta b) = T_2(T_1(\alpha a + \beta b))$$

$$= T_2(\alpha T_1(a) + \beta T_1(b)) \quad (\because T_1 \text{ is a } \ell.t)$$

$$= \alpha T_2(T_1(a)) + \beta T_2(T_1(b)) \quad (\because T_2 \text{ is a } \ell.t)$$

$$= \alpha(T_2 \circ T_1)(a) + \beta(T_2 \circ T_1)(b)$$

Now, we have to prove that

$$\text{rank}(T_2 \circ T_1) \leq \min\{\text{rank}(T_1), \text{rank}(T_2)\}$$

Proof :- (I) $\rho(T_2 \circ T_1) \leq \rho(T_1)$

$$\because T_1 V_1 \subseteq V_2$$

$$(\because T_1: v_1 \rightarrow v_2 \text{ is } \ell.t)$$

$$\Rightarrow T_2(T_1(v_1)) \subseteq T_2(v_2)$$

$$\Rightarrow (T_2 \circ T_1)(v_1) \subseteq T_2(v_2)$$

$$\Rightarrow \text{Im}(T_2 \circ T_1) \subseteq \text{Im } T_2$$

Since $\text{Im}(T_1 \circ T_2)$ and $\text{Im}(T_2)$ are subset of v_3 and $\text{Im}(T_2 \circ T_1) \subseteq \text{Im } T_2$

Therefore $\text{Im}(T_1 \circ T_2)$ is a subset of $\text{Im}(T_2)$

$$\Rightarrow \dim \text{Im}(T_2 \circ T_1) \leq \dim \text{Im } T_2$$

$$\text{rank}(T_2 \circ T_1) \leq \text{rank}(T_2)$$

$$\rho(T_2 \circ T_1) \leq \rho(T_2)$$

.....(1)

(II) Claim :- $\rho(T_2 \circ T_1) \leq \rho(T_1)$

Using rank nullity theorem for T_1 and $(T_1 \circ T_2)$

$$\rho(T_1) = \dim v_1 - \dim \ker T_1$$

.....(2)

$$\text{and } \rho(T_1 \circ T_2) = \dim V_1 - \dim \ker(T_2 \circ T_1)$$

.....(3)

Claim :- $\ker T_1 \subseteq \ker T_2 \circ T_1$

let $x \in \ker T_1$

$$\Rightarrow T_1(x) = 0_2$$

$$\Rightarrow T_2(T_1(x)) = T_2(0_2)$$

$$\Rightarrow T_2 \circ T_1(x) = 0_3$$

$$\Rightarrow x \in \ker(T_1 \circ T_2) \quad \therefore \ker T_1 \subseteq \ker(T_2 \circ T_1)$$

$\ker T_1$ and $\ker(T_1 \circ T_2)$ are subspaces of vector space v_1 and $\ker T_1 \subseteq \ker(T_2 \circ T_1)$

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$$\Rightarrow \text{Ker } T_1 \text{ is subspace of } \text{ker}(T_2 \circ T_1)$$

$$\Rightarrow \dim \text{ker } T_1 \leq \dim \text{ker}(T_2 \circ T_1) \quad \dots\dots(4)$$

from eq.(3) and eq.(4) we get

$$\rho(T_2 \circ T_1) \leq \rho(T_1) \quad \dots\dots(5)$$

$$\rho(T_2 \circ T_1) = \rho(T_1) \text{ if } T_2 \text{ is one - one}$$

ie. multiplication of a l.t. from left side by an injective l.t. does not change the rank

from (1) and (5)

$$\text{rank}(T_2 \circ T_1) \leq \min \{ \rho(T_1), \rho(T_2) \}$$

$$\rho(T_2 \circ T_1) \leq \{ \rho(T_1), \rho(T_2) \}$$

Q.14. State and prove Fundamental Theorem of Vector space homomorphisms.

Sol. Fundamental theorem of vector space homomorphism:-

Statement - A homomorphic image of a vector space is homomorphic to quotient space of the vector space.

$$\text{Let } T: V_1 \rightarrow V_2 \text{ be v.s homomorphism}$$

$$\text{Im}(T) = T(V_1)$$

$$\text{Claim - } \frac{V_1}{\text{ker } T} \simeq T(V_1)$$

$$\text{Let } \text{ker } T = k \Rightarrow \frac{V_1}{k} \simeq T(V_1)$$

define a relation

$$f: \left(\frac{V_1}{k} \right) \rightarrow T(V_1)$$

$$f(k+x) = T(x), \quad \forall x \in V_1$$

(I) f is a map :-

$$\text{let } k+x_1 \text{ and } k+x_2 \in \frac{V_1}{k}$$

$$\text{and } k+x_1 = k+x_2$$

$$\Rightarrow x_1 - x_2 \in k$$

$$\Rightarrow x_1 - x_2 \in \text{ker } T \quad (\because k = \text{ker } T)$$

$$\Rightarrow T(x_1 - x_2) = 0_2$$

$$\Rightarrow T(x_1) - T(x_2) = 0_2$$

$$\Rightarrow T(x_1) = T(x_2)$$

$$\Rightarrow f(k+x_1) = f(k+x_2)$$

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$\Rightarrow f$ is a map.

(II) f is a linear transformation.

$$\text{let } k+a \text{ and } k+b \in \frac{V_1}{k} \text{ and } \alpha, \beta \in F$$

$$F\{ \alpha(k+a) + \beta(k+b) \} = F\{ k + (\alpha a + \beta b) \}$$

$$= T(\alpha a + \beta b) \{ \because f(k+n) = T(x) \}$$

$$= \alpha.T(a) + \beta.T(b)$$

$$= \alpha.f(k+a) + \beta.f(k+b)$$

$\Rightarrow f$ is a linear transformation (l.t)

(III) f is one - one :-

$$\text{let } k+x_1 \text{ and } k+x_2 \in \frac{V_1}{k}$$

$$\Rightarrow f(k+x_1) = f(k+x_2)$$

$$\Rightarrow T(x_1) = T(x_2)$$

$$\Rightarrow T(x_1) + (-T(x_2)) = 0_2$$

$$\Rightarrow T(x_1) + T(-x_2) = 0_2$$

$$\Rightarrow T(x_1 - x_2) = 0_2 \quad (T \in a) = -T(-a)$$

$$\Rightarrow x_1 - x_2 \in \text{ker } T \quad (\because T \text{ is a l.t.})$$

$$\Rightarrow x_1 - x_2 \in k$$

$$\Rightarrow k+x_1 = k+x_2$$

$$\Rightarrow f \text{ is one - one}$$

(IV) f is onto :-

$$\text{for all } T(x) \in T(V_1)$$

$$\Rightarrow x \in V_1$$

$$\Rightarrow k+x \in \frac{V_1}{k}$$

$$\therefore f(k+x) = T(x)$$

$$\Rightarrow f \text{ is an onto map.}$$

$$\therefore f \text{ is an isomorphic}$$

$$\therefore \frac{V_1}{k} \simeq T(V_1)$$

$$\Rightarrow \frac{V_1}{\text{ker } T} \simeq T(V_1)$$

Q.15. Let V_1 and V_2 be finite dimensional vector spaces over a field and T :

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$V_1 \rightarrow V_2$ be a linear transformation. Define the transpose T' of T . Prove that $\text{rank}(T') = \text{rank}(T)$. (2016)

Sol. Transpose T' of T :-

Let V_1 and V_2 be finite dimensional vector space over a field F and $T: V_1 \rightarrow V_2$ be a linear transformation.

Then define a map $T': V_2^* \rightarrow V_1^*$ as

$$T'(f) = f \circ T, \forall f \in V_2^*$$

Now, we have to prove that

$$\text{rank}(T') = \text{rank}(T)$$

Since $\dim W + \dim W^0 = \dim V$

take $W = \text{Im } T$

($\because W$ is subspace of V_2)

$$\Rightarrow \dim(\text{Im } T) + \dim(\text{Im } T)^0 = \dim V_2$$

$$\dim(\text{Im } T) + \dim(\ker T') = \dim V_2 \quad \dots(1)$$

using rank-nullity Theorem of T'

$$\dim(\text{Im } T') + \dim(\ker T') = \dim V_2^* = \dim V_2 \quad \dots(2)$$

From eq.(1) and (2) we have

$$\text{rank}(T') = \text{rank } T$$

Q.16. Let V be a finite dimensional vector space over a field F . Let V^{**} denote the dual of dual space V^* of V . Prove that for each $v \in V$, the map $f_v: V^* \rightarrow F$ defined by $f_v(\phi) = \phi(v)$ ($\phi \in V^*$), is linear transformation and the map $\theta: V \rightarrow V^{**}$ defined by $\theta(v) = f_v$ is an isomorphism. (2016)

Sol. If $v \in V$ and $\phi \in V^*$, Then $\phi(v)$ is a unique element of F . Therefore the correspondence f_v defined by

$$f_v(\phi) = \phi(v) \forall \phi \in V^* \quad \dots(1)$$

is a function from V^* into F .

Let $a, b \in F$ and $\phi, \psi \in V^*$ Then

$$\begin{aligned} f_v(a\phi + b\psi) &= (a\phi + b\psi)(v) \\ &= (a\phi)(v) + (b\psi)(v) \\ &\quad \text{(by scalar multiplication of linear functionals)} \end{aligned}$$

$$a[f_v(\phi)] + b[f_v(\psi)] \quad \text{from eq.(1)}$$

Therefore f_v is a linear transformation on V^* and thus $f_v \in V^{**}$

Now, let θ be the function from V into V^{**} defined by

$$\theta(v) = f_v \forall v \in V$$

(I) θ is one - one :-

If $v, w \in V$, Then

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$$0(v) = 0(w)$$

$$\Rightarrow f_v = f_w \Rightarrow f_v(\phi) = f_w(\phi) \forall \phi \in V^*$$

$$\Rightarrow \phi(v) = \phi(w) \forall \phi \in V^*$$

from eq.(1)

$$\Rightarrow \phi(v) - \phi(w) = 0 \forall \phi \in V^*$$

$$\Rightarrow \phi(v - w) = 0 \forall \phi \in V^*$$

$$\Rightarrow v - w = 0$$

$$\Rightarrow v = w$$

$\therefore \theta$ is one - one

(II) θ is a linear transformation.

let $a, b \in F$ and $v, w \in V$. Then

$$\theta(av + bw) = f_{av+bw} \quad \text{(by definition of } \theta)$$

for every $\phi \in V^*$, we have

$$f_{av+bw}(\phi) = \phi(av + bw) \quad \text{(from eq.(I))}$$

$$= a\phi(v) + b\phi(w)$$

$$= af_v(\phi) + bf_w(\phi) \quad \text{(from eq.(II))}$$

$$= (af_v)^{(\phi)} + (bf_w)^{(\phi)}$$

$$= (af_v + bf_w)^{(\phi)}$$

$$f(av + bw) = af_v + bf_w = a\theta(v) + b\theta(w)$$

$$\text{Thus } \theta(av + bw) = a\theta(v) + b\theta(w)$$

$\therefore \theta$ is a linear transformation $\rightarrow$

From V^* into V^{**} we have $\dim V^* = \dim V^{**}$

Therefore θ is one - one implies that θ must also be onto

Hence θ is an isomorphism of V^* into V^{**} .

Q.17. Define the transpose of a linear transformation and annihilator of a non-empty subset of a vector space, and prove that:

(i) The transpose of a linear transformation is a linear transformation.

(ii) The annihilator of a non-empty subset of a vector space is a subspace of the dual space of the vector space.

(iii) kernel of the transpose of a linear transformation is the annihilator of the image of the linear transformation. (2012)

Ans. Transpose of a linear transformation :-

Let a linear transformation $T: V_1 \rightarrow V_2$.

Then define a map

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$$T^1: V_2^* \rightarrow V_1^* \quad \text{as}$$

$T(f) = foT \quad \forall f \in V_2^*$ is called a transpose of a linear transformation.

Annihilator of a subset of a vector space: -

Let 'S' be the subset of a V.S. 'V' over field 'F' which is non empty then annihilator of S is denoted by 'S^o' and is defined as

$$S^o = \{f \in V^* : f(s) = O_F \quad \forall s \in S\} \subseteq V^*$$

$$\neq \emptyset, \quad f_0 \in S^o$$

Claim - T' is a linear transformation

Let $f, g \in V_2^*$ and $\alpha, \beta \in F$

$$T(\alpha f + \beta g) = (\alpha f + \beta g) \circ T = (\alpha f) \circ T + (\beta g) \circ T$$

$$= \alpha(foT) + \beta(g \circ T) = \alpha T'(f) + \beta T'(g)$$

$\Rightarrow T'$ is a linear transformation.

(ii) **Claim S^o is a subspace of V^{*}.**

Let $f, g \in S^o$ and $\alpha, \beta \in F$

$$\Rightarrow f(s) = O_F \text{ and } g(s) = O_F \quad \forall s \in S \text{ and } \alpha, \beta \in F$$

$$\Rightarrow \alpha \odot f(s) = O_F \text{ and } \beta \odot g(s) = O_F$$

$$\Rightarrow (\alpha \cdot f)(s) = O_F \text{ and } (\beta \cdot g)(s) = O_F$$

$$\Rightarrow (\alpha f + \beta g)(s) = O_F \quad \forall s \in S$$

$$\Rightarrow (\alpha f + \beta g) \in S^o$$

$\therefore S^o$ is subspace of V^* .

(iii) **We have to prove that**

$$\text{Ker } T' = (\text{Im } T)^o$$

$$T: V_1 \rightarrow V_2 \text{ and } T^1: V_2^* \rightarrow V_1^*$$

let $f \in \text{Ker } T'$

$$\Leftrightarrow T(f) = f_0$$

$$\Leftrightarrow foT = f_0$$

$$\Leftrightarrow (foT)(x) = fo(x) = O_F$$

$$\Leftrightarrow f(T(x)) = O_F \quad \forall T(x) \in \text{Im } T$$

$$\Leftrightarrow f \in [I_m T]^o$$

Therefore

$$\text{Ker } T' = (\text{Im } T)^o$$

Q.18. (i) Define linear transformation. If $T: V_1 \rightarrow V_2$ is a linear transformation and W is a subspace of V_2 , then prove that $T^{-1}(W)$ will be subspace of V_1 .

(ii) Let $T: V_1 \rightarrow V_2$ be a surjective, linear transformation. Suppose that $\dim V_1 = 12$ and $\dim V_2 = 8$ and W be a subspace of V_2 such that $\dim W = 3$. Then

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find the dimension of $T^{-1}(W)$.

Sol. Linear transformation: -

Let $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ be two vector space over a field F .

Then a map $T: V_1 \rightarrow V_2$ is called a linear transformation iff

$$(i) T(a +_1 b) = T(a) +_2 T(b) \quad \forall a, b \in V_1$$

$$(ii) T(\alpha a) = \alpha T(a) \quad \forall \alpha \in F \text{ and } \forall a \in V_1$$

Let W is a subspace of V_2 . Then we have to prove that $T^{-1}(W)$ is a subspace of V_1 .

Let $C_1, d_1 \in T^{-1}(W)$ and $\lambda, \mu \in F$

$$\Rightarrow T(C_1), T(d_1) \in W \quad (\because x \in T^{-1}(V) \Leftrightarrow f(x) = V)$$

$$\Rightarrow \lambda T(C_1) +_2 \mu T(d_1) \in W \quad (\because W \text{ is subspace})$$

$$\Rightarrow T(\lambda C_1 +_1 \mu d_1) \in W$$

$$\Rightarrow \lambda C_1 +_1 \mu d_1 \in T^{-1}(W)$$

$$\Rightarrow T^{-1}(W) \text{ is a subspace of } V_1.$$

(ii) Since $T': T^{-1}(W) \rightarrow V_2$ is restriction of T over $T^{-1}(W)$ and T is onto map.

$$T'(T^{-1}(W)) = T[T^{-1}(W)] = W = I_m T' \quad \dots\dots\dots(1)$$

$$\text{Also } \text{Ker } T' = \text{Ker } T \cap T^{-1}(W) = \text{Ker } T \quad \dots\dots\dots(2) \quad (\because \text{Ker } T \subseteq T^{-1}(W))$$

Using Rank-Nullity theorem, we have

$$\dim \text{Ker } T' + \dim \text{Im } T' = \dim(T^{-1}(W))$$

$$\Rightarrow \dim \text{Ker } T + \dim W = \dim T^{-1}(W) \quad (\because \text{using eq.(1) and (2)})$$

$$\Rightarrow \dim V_1 - \dim \text{Im } T + \dim W = \dim T^{-1}(W) \quad (\text{Using R.N.T})$$

$$\Rightarrow \dim V_1 - \dim V_2 + \dim W = \dim T^{-1}(W) \quad (\because T^{-1} \text{ is onto } \Leftrightarrow \text{Im } T = V_2)$$

$$\Rightarrow 12 - 8 + 3 = \dim T^{-1}(W)$$

$$\Rightarrow \dim T^{-1}(W) = 7$$

Q.19. Define dual space of a vector space and prove that the dimension of the dual space of a finite dimensional vector space is equal to the dimension of the vector space. (2012)

Ans. Dual space of vector space: -

A linear transformation $f: V \rightarrow F$ is called linear function. If

$$\text{Hom}(V, F) = \{\text{all linear functional } V \rightarrow F\}$$

$$= V^*$$

and V^* is called dual space of vector space V .

We have to prove that

$$\dim V = \dim V^*$$

Proof - Construction of a basis of a dual space from the basis of vector space -

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Let $\dim V = n$ and
 let $X = \{x_1, x_2, \dots, x_n\}$ be basis of V
 and $X^* = \{f_1, f_2, \dots, f_n\} \subseteq V^*$
 Where $f_i : V \rightarrow F$ defined as -

$$f_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0_F & \text{if } i \neq j \end{cases}$$

Claim - X^* is a basis of V^* .

(i) X^* is a linear independent

$$\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n = f_0$$

$$(\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n)(x_i) = f_0(x_i)$$

$$\Rightarrow \alpha_1 \odot f_1(x_i) \oplus \alpha_2 \odot f_2(x_i) \oplus \dots \oplus \alpha_n \odot f_n(x_i) = 0_F$$

$$\Rightarrow \alpha_1 \cdot 1 \oplus \alpha_2 \cdot 0 \oplus \dots \oplus \alpha_n \cdot 0 = 0_F \quad \left(\because f_i(x_j) = 1 \text{ if } i = j \right)$$

$$= 0 \text{ if } i \neq j$$

$$\Rightarrow \alpha_i = 0_F$$

Similarly $\alpha_2 = \alpha_3 = \dots = \alpha_n = 0_F$

Therefore X^* is linearly independent.

(ii) $V^* = [X^*]$

Let $g \in V^* \Rightarrow g(x_i) = \beta_i$ where $i = 1, 2, 3, \dots, n$

Claim - $g = \beta_1 f_1 + \beta_2 f_2 + \dots + \beta_n f_n$
 let $x \in V$

$$\Rightarrow x = \gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n \quad \dots (1) \quad \forall x \in V$$

$$f_i(x) = f_i(\gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n)$$

$$\Rightarrow f_i(x) = \gamma_1 \odot f_i(x_1) \oplus \gamma_2 \odot f_i(x_2) \oplus \dots \oplus \gamma_n \odot f_i(x_n)$$

$$\Rightarrow f_i(x) = \gamma_1 \odot 1 \oplus \gamma_2 \odot 0 \oplus \dots \oplus \gamma_n \odot 0$$

$$\Rightarrow f_i(x) = \gamma_i$$

Similarly $f_j(x) = \gamma_j = \dots = f_n(x) = \gamma_n$

Consider R.H.S.

$$(\beta_1 f_1 + \beta_2 f_2 + \dots + \beta_n f_n)(x)$$

$$= \beta_1 \odot f_1(x) \oplus \beta_2 \odot f_2(x) \oplus \dots \oplus \beta_n \odot f_n(x)$$

$$= \beta_1 \odot \gamma_1 \oplus \beta_2 \odot \gamma_2 \oplus \dots \oplus \beta_n \odot \gamma_n \quad \dots (2)$$

Consider L.H.S.

$$g(x) = g(\gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n)$$

$$= \gamma_1 \odot g(x_1) \oplus \gamma_2 \odot g(x_2) \oplus \dots \oplus \gamma_n \odot g(x_n)$$

$$= \gamma_1 \odot \beta_1 \oplus \gamma_2 \odot \beta_2 \oplus \dots \oplus \gamma_n \odot \beta_n \quad \dots (3)$$

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From eq.(2) and (3)

$$g = \beta_1 f_1 + \beta_2 f_2 + \dots + \beta_n f_n$$

Hence $V^* = [X^*]$

$\therefore X^*$ is a basis of V^* and X^* is called dual basis of X .

Hence $\dim V = \dim V^*$

Note - $\dim(\text{Hom}(V_1, V_2)) = \dim V_1 \cdot \dim V_2$
 $\dim(\text{Hom}(V, F)) = \dim V \cdot \dim F$
 $\dim V^* = \dim V$

Q.20. Let M_2 be the set of all 2×2 real matrices. If $T : M_2 \rightarrow M_2$ is linear

transformation defined by $T(A) = A \cdot B - B \cdot A$, $\forall A \in M_2$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ then

find (i) $\ker T$ (ii) $\text{Im } T$ and (iii) Trace of T .

(2012)

Sol. Standard basis of M_2 is -

$$S = \{e_{11}, e_{12}, e_{21}, e_{22}\}$$

we have to find matrix $(T)_{ss}$ = ?

$$T(e_{11}) = e_{11} B - B \cdot e_{11}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$$

$$= 0 \cdot e_{11} + 4 \cdot e_{12} + (-1) \cdot e_{21} + 0 \cdot e_{22}$$

$$T(e_{12}) = e_{12} B - B \cdot e_{12}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= 1 \cdot e_{11} + 0 \cdot e_{12} + 0 \cdot e_{21} + (-1) \cdot e_{22}$$

$$T(e_{21}) = e_{21} B - B \cdot e_{21}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= -4 \cdot e_{11} + 0 \cdot e_{12} + 0 \cdot e_{21} + 4 \cdot e_{22}$$

$$T(e_{22}) = e_{22} B - B \cdot e_{22}$$

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$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

$$= 0.e_{11} + 4.e_{12} + 1.e_{21} + 0.e_{22}$$

$$\text{mat}(T)_{ss} = \begin{bmatrix} 0 & 1 & -4 & 0 \\ 4 & 0 & 0 & 4 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & 0 \end{bmatrix}$$

trace of matrix = 0

$$\text{Im}(T) = \text{span} \left\{ \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \text{basis of Im}(T)$$

$$\rho(T) = 2$$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$

$$T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+b & 4a+2b \\ 2c+d & 4c+2d \end{bmatrix} - \begin{bmatrix} 2a+4c & 2b+4d \\ a+2c & b+2d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b-4c & 4a-4d \\ d-a & 4c-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow b-4c=0, \quad d-a=0, \quad 4(a-d)=0, \quad 4c-b=0$$

Solving these equations $a=d, b=4c$
So let $a=\lambda, c=\mu$
 $a=\lambda, d=\lambda, b=4\mu, c=\mu$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda & 4\mu \\ \mu & \lambda \end{bmatrix}$

basis of $\ker T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \right\}$

$$\dim(\ker T) = 2$$

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Q.21. Define kernel of a linear transformation. Find a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ such that $\ker T$ and $\text{Im} T$ are respectively spanned by $\{(1, 1, 1), (1, 0, 0, 1)\}$ and $\{(1, 1, 0), (1, 0, 1)\}$.

Ans. Kernel of a linear transformation: -

(2013)

Let $U(F)$ and $V(F)$ be two vector spaces and let T be a linear transformations from U into V . Then the kernel of T or null space of T written as $N(T)$ is the set of all vectors α in U such that $T(\alpha) = 0$.

Thus $N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\}$

Let $S = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$ extending to $\{(1, 1, 1, 1), (1, 0, 0, 1)\}$ in basis of $\mathbb{R}^4$.

$\{(1, 1, 1, 1), (1, 0, 0, 1), e_3, e_4\}$ are linear independent.

$T(1, 1, 1, 1) = (0, 0, 0)$ and $T(e_3) = (1, 1, 0)$

$T(1, 0, 0, 1) = (0, 0, 0)$ and $T(e_4) = (1, 0, 1)$

Let $(x, y, z, u) \in \mathbb{R}^4$

$$\therefore (x, y, z, u) = \alpha(1, 1, 1, 1) + \beta(1, 0, 0, 1) + \gamma e_3 + \delta e_4$$

$$T(x, y, z, u) = \alpha T(1, 1, 1, 1) + \beta T(1, 0, 0, 1) + \gamma T(e_3) + \delta T(e_4)$$

$$= \alpha(0, 0, 0) + \beta(0, 0, 0) + \gamma(1, 1, 0) + \delta(1, 0, 1)$$

$$T(x, y, z, u) = (\gamma + \delta, \gamma, \delta)$$

Q.22. Define transpose of a linear transformation and prove that the rank of the transpose of a linear transformation is equal to the rank of the linear transformation.

(2013)

Ans. Transpose of a linear transformation: -

Let a linear transformation $T: V_1 \rightarrow V_2$. Then define a map

$T^*: V_2^* \rightarrow V_1^*$ as

$T^*(f) = f \circ T \quad \forall f \in V_2^*$ is called transpose of linear transformation.

Now we have to show that

$$\text{Rank}(T^*) = \text{Rank}(T), \text{ i.e. } \rho(T^*) = \rho(T)$$

Since $\dim W + \dim W^\circ = \dim V$

Take $W = \text{Im} T$

$$\Rightarrow \dim(\text{Im} T) + \dim(\text{Im} T)^\circ = \dim V_2$$

$$\dim(\text{Im} T) + \dim(\ker T^*) = \dim V_2 \quad \dots\dots\dots(i)$$

$$(\because (\text{Im} T)^\circ = (\ker T^*))$$

Using Rank Nullity Theorem of T^* .

$$\dim(\text{Im} T^*) + \dim(\ker T^*) = \dim V_1^* = \dim V_2 \quad \dots\dots\dots(ii)$$

from eq.(i) and (ii) we get

$$\text{Rank}(T^*) = \text{Rank}(T)$$

Q.23. If a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is defined by $T[(x, y, z, u)] = (x-y+z+u, x+2z-u, x+y+3z-3u) \quad \forall (x, y, z, u) \in \mathbb{R}^4$, then find (i) a basis of $\ker T$ and (ii) a basis of $\text{Im} T$.

(2013)

Sol.

$T(x, y, z, u) = (x-y+z+u, x+2z-u, x+y+3z-3u)$ the matrix form can

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be written as

$$[T] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

Changing echalon form

$$\xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \xrightarrow{R_3 - R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{basis of } \text{Im} T = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

Let $(x, y, z) \in \ker T$, then

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + z + u = 0$$

$$y + z - 2u = 0$$

Let $u = k, z = \lambda$.

$$y = 2k - \lambda$$

$$x = y - z - u = 2k - \lambda - \lambda - k = k - 2\lambda$$

so the basis, $\ker T$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Q.24. Define the annihilator of a non-empty subset of a vector space and prove that the sum of the dimensions of a sub-space of a finite dimensional vector space and its annihilator, is equal to the dimension of the vector space (2013)

Sol. Annihilator of a subset of a vector space: -

Let 'S' be the subset of a vector space 'V' over field 'F'

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which is non empty, then annihilator of S is denoted by 'S^o' and is defined as -

$$S^o = \{f \in V^* : f(s) = 0_F \quad \forall s \in S\} \subseteq V^*$$

Now we prove that

$$\dim W + \dim W^o = \dim V$$

let $\dim W = r$ and

$X_1 = \{x_1, x_2, \dots, x_r\}$ be the basis of W. Since X_1 is linearly independent subset of vector space V therefore it can be extended to form a basis of vector space 'V' (By Extension Theorem). Let

$$X_2^* = \{f_1, f_2, \dots, f_r, f_{r+1}, \dots, f_n\} \subseteq V^*$$

be the dual basis of X_2 and basis of V^* .

Consider $S = \{f_{r+1}, f_{r+2}, \dots, f_n\}$

Claim - $S \subseteq W^o$

Let $x \in W$

$$\Rightarrow x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_r x_r \quad \text{where } \alpha_i \in F$$

$$\Rightarrow f_{r+1}(x) = \alpha_1 \odot f_{r+1}(x_1) \oplus \alpha_2 \odot f_{r+1}(x_2) \oplus \dots \oplus \alpha_r \odot f_{r+1}(x_r)$$

$$\Rightarrow f_{r+1}(x) = 0_F$$

$$\Rightarrow f_{r+1} \in W^o$$

Similarly $f_{r+2}, \dots, f_n \in W^o$

$$\Rightarrow S \subseteq W^o$$

(I) S is linearly independent -

$S \subseteq X_2^*$ and X_2^* is linearly independent.

Therefore S is also linearly independent.

(II) $W^o = [S]$ -

$$\text{Let } g \in W^o \Rightarrow g(x_i) = 0_F, \quad 1 \leq i \leq r$$

$$\therefore g \in V^*$$

$$\Rightarrow g = \sum_{i=1}^n g(x_i) f_i$$

$$g = g(x_1) f_1 + g(x_2) f_2 + \dots + g(x_r) f_r + g(x_{r+1}) f_{r+1} + \dots + g(x_n) f_n$$

$$\Rightarrow g = g(x_{r+1}) f_{r+1} + \dots + g(x_n) f_n$$

g is a linear combination of $f_{r+1}, \dots, f_n$

$$\therefore W^o = [S]$$

$\therefore S$ is a basis of W^o .

$$\dim W^o = n - r$$

$$\Rightarrow \dim W^o = \dim V - \dim W$$

$$\Rightarrow \dim W^o + \dim W = \dim V$$

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Q.25. Define an isomorphism between two vector spaces over the same field. Prove that two finite dimensional vector spaces over a field F are isomorphic if and only if they have same dimension. (2017)

Ans. Let V_1 and V_2 be two V.S. over the same field then we say that V_1 is isomorphic to V_2 if $\exists$ a l.t. $T: V_1 \rightarrow V_2$ which is one-one onto.

Q.26. Define the annihilator of a non-empty subset of a vector space over a field. Determine the annihilator of the subset $S = \{(x, y, z) \in \mathbb{R}^3 : xy = 1\}$ of the real vector space $\mathbb{R}^3$. (2017)

Sol. Let V be a V.S. over field F and $f: V \rightarrow F$ be linear functional. Let S be a non-empty subset of vector space V . Then the annihilator of S is denoted by S° and defined as -

$$S^\circ = \{f \in V^* : f(s) = 0, \forall s \in S\}$$

where, V^* is dual basis of V .

given that $S = \{(x, y, z) \in \mathbb{R}^3 : xy = 1\}$

$$\text{Let } x = \frac{1}{y}$$

$$\text{Take, } x = 1, y = 1, z = 0$$

$$x = -1, y = -1, z = 0$$

$$\text{Let } S = \{(1, 1, 0), (-1, -1, 0)\}$$

then S is l.d.

we take $S = \{(1, 1, 0), (0, 1, 0), (0, 0, 1)\}$

$$\text{Let } a_1 = (1, 1, 0), a_2 = (0, 1, 0), a_3 = (0, 0, 1)$$

define $f_i: \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f_1(a_1) = 1, f_1(a_2) = 0, f_1(a_3) = 0$$

$$f_2: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ and } f_3: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ as}$$

$$f_2(a_1) = 0, f_2(a_2) = 1, f_2(a_3) = 0, f_3(a_1) = 0, f_3(a_2) = 0, f_3(a_3) = 1.$$

Let $(x, y, z) \in \mathbb{R}^3$ and S is basis of $\mathbb{R}^3$.

$$(x, y, z) = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = \beta_1(1, 1, 0) + \beta_2(0, 1, 0) + \beta_3(0, 0, 1)$$

$$\Rightarrow (x, y, z) = (\beta_1, \beta_1 + \beta_2, \beta_3)$$

$$\Rightarrow \beta_1 = x, \beta_1 + \beta_2 = y \Rightarrow \beta_2 = y - x, \beta_3 = z$$

$$\text{then } f_1(x, y, z) = x, f_2(x, y, z) = y - x, f_3(x, y, z) = z$$

Then $\{f_1, f_2, f_3\}$ is dual basis of S and this is annihilator of S .

Q.27. State and prove the fundamental theorem of vector space homomorphisms. (2018)

Sol. **Fundamental theorem of a vector space homomorphism:-**
A homomorphic image of a vector space is isomorphic to a quotient space of the vector space.

Proof:-

Let $T: V \rightarrow V_1$ be a vector space homomorphism.

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$\text{Im}(T) = T(V_1)$ (homomorphic image of V_1)

$$\text{Claim: } \frac{V_1}{\text{Ker } T} \approx T(V_1) :-$$

define a relation -

$$f: \left(\frac{V_1}{K}\right) \rightarrow T(V_1)$$

$$f(k+x) \rightarrow T(x)$$

$$\left[\frac{V}{W} \Rightarrow w+x\right]$$

$$\forall x \in V_1$$

i) " f " is a map -

$$\text{Let } k+x_1 \text{ and } k+x_2 \in V_1/K$$

$$\text{and } k+x_1 = k+x_2$$

$$\Rightarrow x_1 - x_2 \in K \quad \left(\begin{array}{l} w+a = w+b \\ \Leftrightarrow a-b \in w \end{array} \right)$$

$$\Rightarrow x_1 - x_2 \in \text{Ker } T$$

$$\Rightarrow T(x_1) - T(x_2) = O_2$$

$$\Rightarrow f(k+x_1) = f(k+x_2)$$

$$\Rightarrow T(x_1 - x_2) = O_2$$

$$\Rightarrow T(x_1) = T(x_2)$$

herefore " f " is a map.

ii) " f " is a linear transformation -

$$\text{Let } k+a \text{ and } k+b \in V_1/K \text{ and } \alpha, \beta \in F$$

$$\{\alpha(k+a) + \beta(k+b)\}$$

$$\{k + (\alpha + \beta)a\}$$

$$f(\alpha a + \beta b)$$

$$(\because w + \alpha a = a(w + \alpha))$$

$$(\because f(k+x) \rightarrow T(x))$$

$$= \alpha T(a) + \beta T(b) = \alpha f(k+a) + \beta f(k+b)$$

" f " is a linear transformation.

iii) " f " is one-one -

$$\text{Let } k+x_1 \text{ and } k+x_2 \in V_1/K \text{ and } f(k+x_1) = f(k+x_2)$$

$$\Rightarrow T(x_1) = T(x_2)$$

$$\Rightarrow T(x_1) - T(x_2) = O_2$$

$$\Rightarrow T(x_1 - x_2) = O_2$$

$$\Rightarrow x_1 - x_2 \in \text{Ker } T$$

$$\Rightarrow K+x_1 = K+x_2$$

$$\Rightarrow T(x_1) + (-T(x_2)) = O_2$$

$$\Rightarrow T(x_1) + (-T(x_2)) = O_2$$

$$\Rightarrow T(x_1) + (-T(x_2)) = O_2$$

$$(\because a-b \in W \Leftrightarrow w+a = w+b)$$

iv) " f " is onto -

$$\forall T(x) \in T(V_1)$$

$$x \in V_1$$

$$f(k+x) = T(x)$$

$$f \text{ is an onto map.}$$

$$f \text{ is an isomorphism.}$$

$$\frac{V_1}{K} \approx T(V_1) \Rightarrow \frac{V_1}{\text{Ker } T} \approx T(V_1)$$

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Q.28. Define rank and nullity of linear transformation. Let T be a linear operator on a finite dimensional vector space V such that $\text{rank}(T^2) = \text{rank}(T)$. Prove that $\text{range}(T) \cap \text{Kernel}(T) = \{0\}$ (20)

Sol. Let $T: V \rightarrow V$ be a linear transformation of finite dimensional vector space over field F such that

$$\rho(T^2) = \rho(T)$$

Then we have to prove that

$$\text{Ker } T \cap \text{Im } T = \{0\}$$

Let $T: V \rightarrow V$ be a linear transformation

$T^2: V \rightarrow V$ be also a linear transformation using Rank - Nullity theorem we have

$$\dim \ker T + \dim \text{Im } T = \dim V \quad \dots\dots\dots(i)$$

$$\text{And } \dim \ker T^2 + \dim \text{Im } T^2 = \dim V \quad \dots\dots\dots(ii)$$

respectively then from eq.(i) and (ii)

$$\dim \ker T + \dim \text{Im } T = \dim \ker T^2 + \dim \text{Im } T^2$$

$$\Rightarrow \dim \ker T = \dim \ker T^2 \quad \{ \because \rho(T^2) = \rho(T) \}$$

$$\Rightarrow \ker T = \ker T^2 \Rightarrow \ker T^2 = \ker T \quad \{ \because \ker T \text{ is a subspace of } \ker T^2 \}$$

$$\Rightarrow x \in \ker T^2 \Rightarrow x \in \ker T$$

$$\Rightarrow T^2(x) = 0 \Rightarrow T(x) = 0 \quad \dots\dots\dots(iii)$$

$$\text{Let } a \in \ker T \cap \text{Im } T \Rightarrow a \in \ker T \text{ and } a \in \text{Im } T$$

$$\Rightarrow T(a) = 0 \text{ and } \exists b \in V \text{ such that } T(b) = a$$

$$\Rightarrow T(a) = 0 \text{ and } \exists b \in V \text{ such that } T(T(b)) = T(a)$$

$$\Rightarrow T(a) = 0 \text{ and } \exists b \in V \text{ such that } T^2(b) = T(a)$$

$$\Rightarrow T^2(b) = T(a) = 0 \Rightarrow T^2(b) = 0$$

$$\Rightarrow T(b) = 0 \quad \{ \because T^2(x) = 0 \Rightarrow T(x) = 0 \}$$

$$\Rightarrow a = 0 \quad \{ \because \text{Using (ii)} \}$$

$$\text{Hence } \text{Ker } T \cap \text{Im } T = \{0\}$$

Q.29. Define the transpose of a linear transformation. Prove that the rank of the transpose of a linear transformation T between finite dimensional vector space over the same field is equal to the rank of T . (20)

Sol. Transpose of a linear transformation: -

Let $V_1(F)$ and $V_2(F)$ be two vector space and $T: V_1 \rightarrow V_2$ be a linear transformation. And V_1^* and V_2^* is dual space of V_1 and V_2 respectively.

Let $f: V_2 \rightarrow F$ be linear functional

$\therefore$ Composition of two linear map is also a linear map.

$\therefore f \circ T: V_1 \rightarrow F$ be a linear functional.

Theorem: -

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If $T: V_1 \rightarrow V_2$ is linear transformation,

then $\rho(T^*) = \rho(T)$

Proof: -

$\therefore T: V_1 \rightarrow V_2$ is a linear transformation then by (R - N) Rank - Nullity theorem.

$$\dim \ker T + \dim \text{Im } T = \dim V$$

$$\Rightarrow \dim \ker T + \rho(T) = \dim V_1 \quad \dots\dots\dots(i)$$

and $T^*: V_2^* \rightarrow V_1^*$ is also a linear transformation, then by R-N theorem

$$\dim \ker T^* + \dim \text{Im } T^* = \dim V_1^*$$

$$\Rightarrow \dim \ker T^* + \rho(T^*) = \dim V_2^* \quad \dots\dots\dots(ii)$$

$$\therefore \dim V_2^* = \dim V_2 \Rightarrow \dim \ker T^* + \rho(T^*) = \dim V_2 \quad \dots\dots\dots(iii)$$

Since, we know that $\text{Im } T$ is a subspace of V_2 .

$$\therefore \dim \text{Im } T + \dim [\dim T]^0 = \dim V_2 \quad \dots\dots\dots(iii)$$

from eq.(ii) and (iii)

$$\dim \ker T^* + \rho(T^*) = \dim \text{Im } T + \dim [\dim T]^0$$

$$\Rightarrow \dim \ker T^* + \rho(T^*) = \rho(T) + \dim [\dim T]^0 \quad \dots\dots\dots(iv)$$

$$\text{Claim - } \ker T^* = [\dim T]^0$$

Let $f \in \ker T^*$

$$\Leftrightarrow T^*(f) = f_0 \quad \text{(zero linear function and by definition of Ker } T^*)$$

$$\Leftrightarrow f \circ T = f_0 \quad \text{(from definition of } T^*)$$

$$\Leftrightarrow (f \circ T)(x) = f_0(x) \quad \text{(by definition of equal map } x \in V_1)$$

$$\Leftrightarrow f[T(x)] = 0 \quad \{ \forall x \in V_1 \}$$

$$\Leftrightarrow f[T(x)] = 0 \quad \{ \forall T(x) \in \text{Im } T = T(V_1) \}$$

$$\Leftrightarrow f \in [\text{Im } T]^0 \quad \text{(by definition of Annihilator)}$$

$$\text{Hence, } \ker T^* = [\text{Im } T]^0$$

$$\Rightarrow \dim \ker T^* = \dim [\text{Im } T]^0 \quad \dots\dots\dots(v)$$

from eq.(iv) and (v)

$$\dim \ker T^* + \rho(T^*) = \rho(T) + \dim [\text{Im } T]^0$$

$$\Rightarrow \rho(T^*) = \rho(T)$$

Q.30. Let V_1 and V_2 be vector spaces over the same field F and $T: V_1 \rightarrow V_2$ be a surjective linear transformation. Suppose that W is a subspace of V_2 of dimension 2. If $\dim V_1 = 7$ and $\dim V_2 = 5$, then find $\dim T^{-1}(W)$. (2018)

Sol. Let us define a map-

$$T^*: T^{-1}(W) \rightarrow W$$

$$\therefore T[T^{-1}(W)] = W \quad \{ \because T \text{ is a surjective map} \}$$

Since T^* is restriction of linear transformation T over subspace $T^{-1}(W)$

$$\therefore T^*[T^{-1}(W)] = W$$

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$$\Rightarrow \text{Im } T^1 = W \Rightarrow \dim \text{Im } T^1 = \dim W$$

$$\Rightarrow \dim \text{Im } T^1 = 2 \quad (\because \dim W = 2)$$

Since $\ker T^1 = \ker T \cap T^{-1}(W)$

$\therefore$ we know that,

$$\ker T \subseteq T^{-1}(W), \forall W \text{ subspace of } V_2$$

$$\therefore \ker T^1 = \ker T$$

$\therefore$ From Rank-nullity theorem,

$$\dim \ker T + \dim \text{Im } T = \dim V_1$$

$$\Rightarrow \dim \ker T = \dim V_1 - \dim \text{Im } T$$

$$\dim \ker T = \dim V_1 - \dim \text{Im } T = 7 - 5 = 2$$

Again by Rank Nullity theorem-

$$\dim \ker T^1 + \dim \text{Im } T^1 = \dim [T^{-1}(W)]$$

$$\Rightarrow 2 + 2 = \dim [T^{-1}(W)]$$

$$\Rightarrow \dim [T^{-1}(W)] = 4$$

Q.31. Define the dual space of vector space. Find the dual basis of the space of $\mathbb{R}^3$ relative to the ordered basis $S = \{(1, 1, 0), (1, 2, 0), (1, 1, 1)\}$ of $\mathbb{R}^3$.

Sol. Dual space:-

A linear transformation $f: V \rightarrow F$ is called linear function if,

$$\text{Hom}(V, F) = \{\text{all linear functional } V \rightarrow F\} = V^*$$

and V^* is called dual space of vector space V .

$$\text{Here } S = \{(1, 1, 0), (1, 2, 0), (1, 1, 1)\}$$

Let $S^* = \{f_1, f_2, f_3\}$ be the dual basis of S .

$$\text{Let } (x, y, z) \in \mathbb{R}^3$$

$$(x, y, z) = \alpha(1, 1, 0) + \beta(1, 2, 0) + \gamma(1, 1, 1) \quad \dots\dots\dots (1)$$

$$\Rightarrow \alpha + \beta + \gamma = x \quad \dots\dots\dots (2)$$

$$\alpha + 2\beta + \gamma = y \quad \dots\dots\dots (3)$$

$$\gamma = z \quad \dots\dots\dots (4)$$

From eq.(2) and (3) we get

$$\beta = y - x$$

Putting the value of γ and β in eq.(2) we get,

$$\alpha + y - x + z = x$$

$$\Rightarrow \alpha = 2x - y - z$$

Now, Putting the value of α, β, γ in eq. (1)

$$f_1(x, y, z) = \alpha \odot f_1(1, 1, 0) \oplus \beta \odot f_1(1, 2, 0) \oplus \gamma \odot f_1(1, 1, 1)$$

$$f_1(x, y, z) = \alpha$$

$$f_1(x, y, z) = 2x - y - z$$

$$\text{Similarly } f_2(x, y, z) = \beta = y - x$$

$$f_3(x, y, z) = \gamma = z$$

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Q.32. Define the annihilator of non-empty subset of a vector space. If V is a finite dimensional vector space over a field F and W is a subspace of V , then prove that:

$$\dim W + \dim W^0 = \dim V,$$

Where W^0 is the annihilator of W .

Sol. Annihilator of a Non-empty subset of a vector space b :- (2019)

Let ' S ' be the subset a vector space ' V ' over field F which is non-empty then annihilator of S is denoted by ' S^0 ' and is defined as -

$$S^0 = \{f \in V^* : f(s) = O_F \forall s \in S\} \subseteq V^*$$

$$\neq \phi \quad f_0 \in S^0$$

Claim : S^0 is sub space of V^* .

Let $f, g \in S^0$ and $\alpha, \beta \in F$

$$\Rightarrow f(s) = O_F \text{ and } g(s) = O_F \quad \forall s \in S \text{ and } \alpha, \beta \in F$$

$$\Rightarrow \alpha \odot f(s) = O_F \quad \text{and} \quad \beta \odot g(s) = O_F$$

$$\Rightarrow (\alpha \cdot f)(s) = O_F \quad \text{and} \quad (\beta \cdot g)(s) = O_F$$

$$\Rightarrow (\alpha \cdot f + \beta \cdot g)(s) = O_F \quad \forall s \in S$$

$$\Rightarrow \alpha \cdot f + \beta \cdot g \in S^0$$

$\therefore S^0$ is subspace of V^* .

Now we prove that

$$\dim W + \dim W^0 = \dim V$$

Let $\dim W = r$ and $X_1 = \{x_1, x_2, \dots, x_r\}$

be the basis of W .

Since X_1 is linearly independent subset of vector space V therefore it can be extended to form a basis of V . V (By extension theorem).

Let $X_2 = \{f_1, f_2, \dots, f_r, f_{r+1}, \dots, f_n\} \subseteq V^*$ be the dual basis of X_1 and basis of V^* .

Consider $S = \{f_{r+1}, f_{r+2}, \dots, f_n\}$

Claim:-

$$S \subseteq W^0$$

Let $x \in W$

$$\Rightarrow x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_r x_r \quad \text{where } \alpha_i \in F$$

$$\Rightarrow f_{r+1}(x) = \alpha_1 \odot f_{r+1}(x_1) \oplus \alpha_2 \odot f_{r+1}(x_2) \oplus \dots + \alpha_r \odot f_{r+1}(x_r)$$

$$\Rightarrow f_{r+1}(x) = O_F \quad \Rightarrow f_{r+1} \in W^0$$

Similarly $f_{r+2}, \dots, f_n \in W^0$

$$\Rightarrow S \subseteq W^0$$

Now we prove that S is a basis of W^0 .

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(I) S is linearly independent

$S \subseteq X_1^*$ and X_1^* is linear independent. Therefore S is also linear independent. ($\because$ A subset of L.I. set in L.I.)

(II) $W^0 = [S]$

$$\text{Let } g \in W^0 \Rightarrow g(x_i) = 0_{\mathbb{F}}, \quad 1 \leq i \leq r$$

$$\therefore g \in V^* \Rightarrow g = \sum_{i=1}^n g(x_i) f_i$$

$$g = g(x_1)f_1 + g(x_2)f_2 + \dots + g(x_r)f_r + g(x_{r+1})f_{r+1} + \dots + g(x_n)f_n$$

$$g = g(x_{r+1})f_{r+1} + \dots + g(x_n)f_n$$

$\Rightarrow$ g is a linear combination of $f_{r+1}, \dots, f_n$.

$\therefore W^0 = [S]$

$\therefore$ S is a basis of W^0

$$\dim W^0 = n - r$$

$$\dim W^0 = \dim V - \dim W \Rightarrow \dim W^0 + \dim W = \dim V$$

Q.33. Define an elementary matrix. Write

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{R})$$

as a product of elementary matrices.

Sol. Elementary Matrix:-

A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix (or E-matrix). Given that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{R})$$

Now $A = I_3 A$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_2 \rightarrow R_2 - 2R_1 \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_3 \rightarrow R_3 / 2 \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_1 \rightarrow R_1 - R_3 \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$C_2 \rightarrow C_2 - C_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 1 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \rightarrow R_2 / 4$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$C_3 \rightarrow C_3 - \frac{1}{4}C_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/4 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$\text{So } A = E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1} E_6^{-1} E_5^{-1}$$

$$\text{Or } A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

Unit III

Q.1 If a linear transformation associated with a square matrix is nonsingular, then prove that the linear transformations associated with the matrix will be non singular. (2014, 16)

Sol. Let $A = [a_{ij}]$ be $n \times n$ matrix over a field and $T_1 : V \rightarrow V$ be a linear transformation associated with 'A' related to ordered basis -

$$X = \{x_1, x_2, \dots, x_n\} \text{ of } V$$

$$\therefore T_1(x_j) = \sum_{i=1}^n a_{ij} x_i \quad \forall j \quad \dots\dots\dots(1)$$

Let $T_2 : V \rightarrow V$ be another linear transformation associated with matrix A relative to basis,

$$X' = \{x'_1, x'_2, \dots, x'_n\} \text{ of } V.$$

$$\therefore T_2(x_j) = \sum_{i=1}^n a_{ij} x'_i \quad \forall j \quad \dots\dots\dots(2)$$

Let T_1 be non-singular.

Claim - T_2 is non-singular

$$\Rightarrow X = \{x_j : j = 1, 2, \dots, n\} \quad X = \{x_i : i = 1, 2, \dots, n\}$$

$$\begin{array}{ccc} V & \xrightarrow{T_1} & V \\ \downarrow L_1 & & \uparrow L_2 \\ V & \xrightarrow{T_2} & V \end{array} \quad L_1(x_j) = x'_j \quad L_2(x'_j) = x_j$$

L_1 and L_2 are transition map which is one-one and onto that is non singular. Now we will show that

$$L_2 \circ T_1 \circ L_1^{-1} = T_2$$

$$L_2 \circ T_1 \circ L_1^{-1}(x_j) = L_2 \circ T_1(L_1^{-1}(x_j)) = L_2(T_1(x'_j))$$

$$= (L_2 T_1(x'_j)) = L_2 \left(\sum_{i=1}^n a_{ij} x_i \right)$$

$$= \sum_{i=1}^n a_{ij} L_2(x_i) = \sum_{i=1}^n a_{ij} x'_i = T_2(x_j)$$

$\therefore$ from equality of map

$$L_2 \circ T_1 \circ L_1^{-1} = T_2$$

Therefore, T_1 is non-singular.

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- $\Rightarrow L_2 \circ T_1 \circ L_1^{-1}$ is non-singular.
 $\Rightarrow L_2 \circ T_1$ is non-singular.
 $\Rightarrow T_1$ is non-singular.

($\because L_1$ is non-singular)

($\because L_2$ is non-singular)

Q.2. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

and hence write A^{-1} as the product of elementary matrices.

Sol. Given that

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow (R_1 + R_2) - R_3$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_3$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & -3 & -3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

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$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow (R_1 + R_2) + R_3$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -6 & -5 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 3R_3$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow (R_1 + R_2) - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.3. Define row-rank and determinantal rank of matrix and prove that row-rank of a matrix is equal to the determinantal rank of matrix. (2014)

Ans. Row rank:-

Let $A = [a_{ij}]$ be $n \times n$ matrix over field F .

$$\text{Therefore } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Rows of A -

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$$R_1 = (a_{11}, a_{12}, \dots, a_{1n}) \in F^n$$

$$R_2 = (a_{21}, a_{22}, \dots, a_{2n}) \in F^n$$

$$R_m = (a_{m1}, a_{m2}, \dots, a_{mn}) \in F^n$$

$$S_1 = \{R_1, R_2, \dots, R_m\} \in F^n$$

$L(S_1) = [S_1] = \{\text{all possible linear combination of element of } S_1\}$

$[S_1]$ is a subspace of F^n and is called as row-space of 'A' and dimension

of row space of 'A' is called row rank of A.

Determinantal-Rank of a matrix:-

An $m \times n$ matrix over a field F is said to be of determinantal rank 'd' iff there exist a $d \times d$ submatrix D of A such that

$|D| \neq 0_F$ and determinants of all submatrices $(d+1) \times (d+1)$ is zero.

Let 'A' be $m \times n$ matrix over a field 'F' and 'd' be the determinantal rank of A and 'r' be rank of matrix.

Claim:- $d = r$

Since 'd' is determinantal rank of A.

$\Rightarrow$ there exists $d \times d$ submatrix (D) of A i.e. $D \neq 0_F$

$\Rightarrow$ D is non singular.

$\Rightarrow \rho(D) = d$ (Because the rank of non-singular matrix is equal to order of matrix.)

$\Rightarrow$ all d-rows of D are linear independent.

$\Rightarrow$ Those d-rows of matrix 'A' which are contributing the elements of D will be linear independent.

$\Rightarrow d \leq r$ (i)

Let $\text{rank}(A) = r$

$\Rightarrow$ row rank of A = r

$\Rightarrow$ There are 'r' rows in 'A' which are linear independent. Consider a submatrix 'B' having r-rows of 'A' which are L.I.

$\Rightarrow$ row rank (B) = r

$\Rightarrow$ Column rank of B = r

$\Rightarrow \exists B_1$ of order $r \times r$ submatrices of B (hence submatrix of A) by those r-columns of B formed by those columns B which are L.I.

$\Rightarrow \rho(B_1) = r$

$\Rightarrow B_1$ is non-singular.

$\Rightarrow |B_1| \neq 0_F$

$\Rightarrow r \leq d$ (ii)

From eq.(i) and (ii)

$$r = d$$

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Q.4. Define adjoint of a square matrix. If A is nxn ($n \geq 2$) matrix, then

prove that $\text{adj adj } (|A|)^{n-2} A$.

(2014)

Sol. Case I - When $n = 2$ -

Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be 2×2

$$\text{adj } A = \begin{bmatrix} \alpha & -\beta \\ -\gamma & \delta \end{bmatrix}$$

$$\text{adj adj } A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = A = (|A|)^{n-2} A$$

Case II - If $n > 2$ or $n \geq 3$

(i) If A is singular: -

$$\rho(A) \leq n-1 \quad \text{and} \quad |A| = 0_F$$

$$\Rightarrow \rho(\text{adj } A) \leq 1 \quad \text{and} \quad |A| = 0_F$$

$$\Rightarrow \rho(\text{adj } A) < n-1 \Rightarrow \rho(\text{adj adj } A) = 0$$

$$\Rightarrow \text{adj adj } A = 0 = 0_F \cdot A = (|0_F|)^{n-2} A$$

$$\Rightarrow \text{adj adj } A = (|A|)^{n-2} A$$

(ii) If A is Non-singular: -

Using adjoint theorem for adj A

$$\text{adj } A \cdot \text{adj adj } A = |A| \text{In}$$

$$\Rightarrow A(\text{adj } A \text{ adj adj } A) = A(|A| \text{In})$$

$$\Rightarrow |A| \text{In}(\text{adj adj } A) = |A| \text{In}(A \text{In})$$

$$\Rightarrow |A|(\text{adj adj } A) = |A| \text{In} \cdot A$$

$$\Rightarrow |A|(\text{adj adj } A) = (|A|)^{n-1} A$$

$$|A|^{-1} A(|A| \text{ adj adj } A) = |A|^{-1} (|A|)^{n-1} A$$

$$\Rightarrow \text{adj adj } A = (|A|)^{n-2} A$$

Q.5. Reduce the matrix

$$a = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix}$$

(2015)

Sol. $a = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix} \quad R_3 \rightarrow R_3 - (R_1 + R_2)$

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$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \rightarrow R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \rightarrow C_4 - 2C_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_3 \rightarrow C_3 - 3C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \rightarrow C_4 + C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_4 \rightarrow C_4 - C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is normal form of matrix a.

Hence $\rho(a) = 3$

Q.6. Let $V = \mathbb{R}^3$ and $T: V \rightarrow V$ be a linear transformation defined by $T(a, b, c) = (a+c, -2a+b, -a+2b+c)$.

Find the matrix of T with respect to the basis.

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$\{(1, 0, 1), (-1, 1, 1), (0, 1, 1)\}$.

Sol. Given that $v = \mathbb{R}^3$

and $T: V \rightarrow V$ be a linear transformation defined by

$$T(a, b, c) = (a + c, -2a + b, -a + 2b + c) \quad \dots(1)$$

Let $S = \{\alpha_1 = (1, 0, 1), \alpha_2 = (-1, 1, 1), \alpha_3 = (0, 1, 1)\}$

from eq.(1) we have

$$T(\alpha_1) = T(1, 0, 1) = (2, -2, 0)$$

$$T(\alpha_2) = T(-1, 1, 1) = (0, 3, 4)$$

$$T(\alpha_3) = T(0, 1, 1) = (1, 1, 3)$$

let $(2, -2, 0) = x\alpha_1 + y\alpha_2 + z\alpha_3$

$$(2, -2, 0) = x - y, y + z, x + y + z$$

implies $x - y = 2, y + z = -2, x + y + z = 0$

After solving we get

$$x = 2, y = 0 \text{ and } z = -2$$

$$\therefore T(1, 0, 1) = (2, -2, 0) = 2\alpha_1 - 2\alpha_3$$

Similarly $(0, 3, 4) = (x - y, y + z, x + y + z)$

implies $x - y = 0, y + z = 3, x + y + z = 4$

Solving we get

$$x = 1, y = 1 \text{ and } z = 2$$

$$T(-1, 1, 1) = (0, 3, 4) = \alpha_1 + \alpha_2 + 2\alpha_3$$

Similarly $(1, 1, 3) = (x - y, y + z, x + y + z)$

$$x - y = 1, y + z = 1, x + y + z = 3$$

Solving we get, $y = 1, x = 2, z = 0$

$$T(0, 1, 1) = (1, 1, 3) = 2\alpha_1 + \alpha_2$$

Hence the matrix of T w.r.t

$$\text{basis } S = \{\alpha_1 = (1, 0, 1), \alpha_2 = (-1, 1, 1), \alpha_3 = (0, 1, 1)\}$$

is $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

Q.7. Show that $x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30$ are consistent and solve them.

Solve.

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 7z = 30$$

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$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2, \rho(A^*) = 2$$

Hence it is constant

$$x + y + z = 6$$

$$y + 2z = 8$$

.....(1)

let $x = k$

then $y + z = 6 - k$

$$y + 2z = 8$$

$$-z = 6 - k - 8$$

$$-z = -k - 2 \Rightarrow z = k + 2$$

Putting the value of x, y, z in ..(1)

$$k + y + k + 2 = 6 \Rightarrow y = 4 - 2k$$

Hence $x = k, y = 4 - 2k, z = k + 2$

Hence there is infinite no. of Solution.

Q.8. Define the adjoint of a square matrix and prove that

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$$

(2015)

Sol. Adjoint of a Square matrix -

Let $A = (a_{ij})$ be an $n \times n$ square matrix, let C_{ji} be the co-factor of a_{ij} then the transpose of the matrix $[C_{ji}]^t$ is called the adjoint of the matrix A and is written as $\text{Adj } A$. Let A and B are non singular. Therefore A, B is also non-singular

By adjoint theorem for A, B

$$(A.B)\text{adj}(A.B) = |A.B|I_n$$

.....(1)

Consider

$$(A.B)(\text{adj } B, \text{adj } A)$$

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$$\begin{aligned}
 &= A(B \operatorname{adj} B) \operatorname{adj} A \\
 &= A(|B| \operatorname{In}) \operatorname{adj} A && \text{(by adjoint theorem)} \\
 &= |B|(A \operatorname{In}) \operatorname{adj} A \\
 &= |B|(A \operatorname{adj} A) \\
 &= |B|(|A| \operatorname{In}) && \text{(by adjoint theorem)} \\
 &= (|B| \odot |A|) \operatorname{In} \\
 &= (|A| \odot |B|) \operatorname{In} && \text{(commutative property of field)} \\
 &= |AB| \operatorname{In} && \dots(2)
 \end{aligned}$$

From eq. (1) and (2)

$$\begin{aligned}
 (A.B) \operatorname{adj} (A.B) &= (A.B) (\operatorname{adj} B \operatorname{adj} A) \\
 (A.B)^{-1} (A.B) \operatorname{adj} (A.B) &= (A.B)^{-1} (A.B) (\operatorname{adj} B \operatorname{adj} A) \\
 \operatorname{adj} (A.B) &= \operatorname{adj} B \operatorname{adj} A
 \end{aligned}$$

Q.9. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$. Find the basis of $\ker T$. (2012)

Ans. Given that

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$$

since $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation associated to A with respect to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$

Therefore

$$\begin{aligned}
 T(1,0,0) &= 2(1,0) + 1(0,1) = (2,1) \\
 T(0,1,0) &= (1,0) + 1(0,1) = (0,1) \\
 T(0,0,1) &= -1(1,0) + (-1)(0,1) = (-1,-1) \\
 T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1) \\
 T(x,y,z) &= x(2,1) + y(0,1) + z(-1,-1) \\
 T(x,y,z) &= (2x-2, x+y-z)
 \end{aligned}$$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = 0\}$$

$$\text{ie- } \begin{aligned} 2x - z &= 0 \\ x + y - z &= 0 \end{aligned}$$

$$\text{Let } z = k \text{ then } x = +\frac{k}{2}$$

$$\frac{k}{2} + y - k = 0$$

$$y = \frac{k}{2}$$

$$\text{So } \ker T = \{(k/2, k/2, k) : k \in \mathbb{R}\}$$

$$\text{Basis of } \ker T = \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\}$$

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or basis of $\ker T = \{(1, 1, 2)\}$

Q.10. Define row-space of a matrix. Find a basis of the row-space of the matrix:

$$A = \begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & -3 & 1 \\ -1 & -2 & -2 & 2 \\ 4 & 6 & 4 & -6 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Hence obtain the rank of A .

(2012)

Sol. Row space of a matrix:-

Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix over the field F . The row vectors of A are the vectors $\alpha_1, \dots, \alpha_m \in V_n(F)$ defined by

$$\alpha_i = (a_{i1}, a_{i2}, \dots, a_{in}) \quad 1 \leq i \leq m$$

The row space of A is the subspace of $V_n(F)$ spanned by these vectors.

$$A = \begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & -3 & 1 \\ -1 & -2 & -2 & 2 \\ 4 & 6 & 4 & -6 \\ 1 & 3 & 4 & -3 \end{bmatrix} \xrightarrow{\substack{R_1 \rightarrow 2R_1 - R_2 \\ R_2 \rightarrow R_1 + R_2 \\ R_3 \rightarrow 2R_1 + R_2 \\ R_4 \rightarrow R_1 - 2R_2 \\ R_5 \rightarrow 2R_1 - R_2}} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & -2 & -6 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -8 & -6 \\ 0 & 2 & 2 & -6 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \rightarrow 2R_2 - R_3 \\ R_3 \rightarrow R_2 + R_3}} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -8 & -6 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_3} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & -8 & -6 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

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$$\xrightarrow{R_3 \rightarrow 2R_3 + R_2} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{\begin{matrix} R_3 \rightarrow R_3/2 \\ R_6 \rightarrow R_6/6 \end{matrix}} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\begin{matrix} R_1 \rightarrow R_1 + R_6 \\ \text{and interchanging} \\ \text{rows} \end{matrix}} \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1/2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank A = 3

Q.11. Define adjoint of a square matrix and prove that
(i) If A is an n-square non singular matrix, then

$$\text{adj adj } A = (|A|)^{n-2} \cdot A$$

(ii) If A and B are non-singular matrices of the same order, then
 $\text{adj}(A \cdot B) = \text{adj } B \cdot \text{adj } A$

Sol. Adjoint of a square matrix:-

Let $A = [a_{ij}]$ be $n \times n$ matrix over a field F and C_j be the matrix of co-factors of elements of 'A' then the transpose of matrix of the co-factors

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elements of A is called as adjoints of 'A' and it is denoted as $A \text{ adj } A$ or $\text{adj } A$.

(i) We have to prove that

$$\text{adj adj } A = (|A|)^{n-2} A$$

If A is an n-square non singular matrix then using adjoint theorem, we get

$$\text{adj } A \text{ adj adj } A = |A| \text{ adj } A$$

$$\Rightarrow A \cdot (\text{adj } A \cdot \text{adj adj } A) = A(|A| \text{ adj } A)$$

$$\Rightarrow |A| \text{ adj } A \cdot \text{adj adj } A = |A| \text{ adj } A \cdot A$$

$$\Rightarrow |A| \text{ adj adj } A = |A| \text{ adj } A$$

$$\Rightarrow |A| \text{ adj adj } A = (|A|)^{n-1} A$$

$$\Rightarrow \text{adj adj } A = (|A|)^{n-2} A$$

(ii) We have to prove that

$$\text{adj}(A \cdot B) = \text{adj } B \cdot \text{adj } A$$

Proof -

Since A and B are non-singular matrix therefore A.B is also non-singular

By adjoint theorem for A.B

$$(A \cdot B) \text{ adj}(A \cdot B) = |A \cdot B| \text{ In} \quad \dots\dots\dots(1)$$

$$\text{Conside } (A \cdot B) (\text{adj } B \cdot \text{adj } A) = A \cdot (B \text{ adj } B) \text{ adj } A$$

$$= A(|B| \text{ In}) \text{ adj } A = |B| (A \cdot \text{In}) \text{ adj } A$$

$$= |B| (A \cdot \text{adj } A) = |B| (|A| \text{ In})$$

$$= (|B| \odot |A|) \text{ In} = (|A| \odot |B|) \text{ In}$$

$$= |A \cdot B| \text{ In} \quad \dots\dots\dots(2)$$

From eq.(1) and (2) we get

$$(A \cdot B) \text{ adj}(A \cdot B) = (A \cdot B) (\text{adj } B \cdot \text{adj } A)$$

$$\Rightarrow \text{adj}(A \cdot B) = \text{adj } B \cdot \text{adj } A$$

Q.12. Define trace of a square matrix. If A and B are $m \times n$ and $n \times m$ matrices, respectively, over a field, then prove that A.B and B.A have the same trace. Hence prove that similar matrices have the same trace. (2012)

Sol. Trace of a square matrix:-

Let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \quad \text{be } n \times n$$

square matrix over field $(F, \oplus, \odot)$. Then element $\alpha_{11} \oplus \alpha_{22} \oplus \dots \oplus \alpha_{nn}$ is

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called trace of matrix and denoted by 'tr(A)'.

Now we have to prove that $\text{tr}(A.B) = \text{tr}(B.A)$

Let $A = [\alpha_{ij}]$ be $m \times n$ matrix

and $B = [\beta_{jk}]$ be $n \times m$ matrix.

where $A.B = [\gamma_{ik}]_{m \times m}$

$$\text{where } \gamma_{ik} = \sum_{j=1}^n \alpha_{ij} \odot \beta_{jk}$$

$$\therefore \text{trace}(A.B) = \gamma_{11} \oplus \gamma_{22} \oplus \dots \oplus \gamma_{mm}$$

$$= \sum_{i=1}^m \gamma_{ii} = \sum_{i=1}^m \left(\sum_{j=1}^n \alpha_{ij} \odot \beta_{ji} \right)$$

$$= \sum_{j=1}^n \sum_{i=1}^m \alpha_{ij} \odot \beta_{ji} \quad \dots\dots\dots(1)$$

$$B = [\beta_{jk}]_{n \times m}$$

$$A = [\alpha_{ij}]_{m \times n}$$

$$B.A = [\theta_{jk}]_{n \times n}$$

$$\text{where } \theta_{jk} = \sum_{i=1}^m \beta_{ji} \odot \alpha_{ik}$$

$$\text{tr}(B.A) = \sum_{j=1}^n \theta_{jj} = \sum_{j=1}^n \left(\sum_{i=1}^m \beta_{ji} \odot \alpha_{ij} \right)$$

$$= \sum_{j=1}^n \sum_{i=1}^m \beta_{ji} \odot \alpha_{ij} \quad \dots\dots\dots(2)$$

from eq.(i) and (ii)

$$\text{tr}(A.B) = \text{tr}(B.A)$$

we have to show that

Similarly matrices has same trace. Let A and B are similar matrices then

$$B = PAP^{-1}$$

$$\text{tr} B = \text{tr}(PAP^{-1}) = \text{tr}(PP^{-1}A)$$

$$= \text{tr}(I_n A)$$

$$\text{tr} B = \text{tr} A$$

Q.13. Find the basis of the solution space of the following system of linear equations over the field of real numbers:

$$2x + z - u + v = 0, x + z - u - v = 0, 6x + y + 4z + v = 0.$$

Sol.

We have given that

$$2x + z - u + v = 0$$

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$$x + z - u - v = 0$$

$$6x + y + 4z + v = 0$$

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 1 \\ 6 & 1 & 4 & 1 & 0 \end{bmatrix}$$

Reduce in echlon form we get

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -2 & 6 & 5 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

Basis of Ker T

$$= \{(1, 0, 1, -1), (0, 1, -2, 6, 5), (0, 0, -1, 1, -1)\}$$

Q.14. Define trace of square matrix. If A is a real matrix such that $\text{trace}(A.A^t) = 0$, then prove that $A = 0$. (2013)

Ans. Trace of a square matrix: -

Let A be a square matrix of order n over a field F. The sum of elements of A lying along the principal diagonal is called the trace of A, i.e. if A

$$= [a_{ij}]_{n \times n} \text{ then trace } A = \sum_{i=1}^n a_{ii}.$$

$$\text{Let } A = [a_{ij}]_{n \times n}.$$

$$\text{Then } A^t = [b_{ij}]_{n \times n}, \text{ where } b_{ij} = a_{ji}.$$

Now $A^t.A$ is a matrix of the type $n \times n$.

Let $A^t.A = [C_{ij}]_{n \times n}$. Then

C_{ii} = the sum of the products of A^t and the i^{th} column of A.

$$\begin{aligned} &= b_{i1}a_{1i} + b_{i2}a_{2i} + \dots + b_{in}a_{ni} \\ &= a_{1i}a_{i1} + a_{2i}a_{i2} + \dots + a_{ni}a_{in} \quad [\because b_{ij} = a_{ji}] \\ &= a_{ii}^2 + a_{2i}^2 + \dots + a_{ni}^2 \end{aligned}$$

$$\text{Now trace}(A^t.A) = \sum_{i=1}^n C_{ii}$$

$$= \sum_{i=1}^n \{a_{1i}^2 + a_{2i}^2 + \dots + a_{ni}^2\}$$

= The sum of the squares of all the element of A.

Now the elements of A are real numbers.

Therefore $\text{trace}(A^t.A) = 0$

$\Rightarrow$ The sum of the squares of all the elements of A is zero.

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- ⇒ Each element of A is zero.
 ⇒ A is null matrix i.e. $A = 0$.

Q.15. Define determinantal rank of a matrix. If A is an n-square matrix of rank n-1, then prove that the rank of adjoint of A is 1. (2013)

Ans. Determinantal rank of matrix:-

An $m \times n$ matrix over a field F is said to be of determinantal rank 'd' iff there exists a $d \times d$ submatrix D of A such that $|D| \neq 0_F$ and determinants of all submatrix $(d+1) \times (d+1)$ is zero.

Let $\rho(A) = n-1 < n$

⇒ A is singular matrix and $\exists$ atleast one co-factor of element of A different from zero.

$$\Rightarrow |A| = 0_F \text{ and } \text{Adj} A \neq 0$$

$$\Rightarrow A \cdot \text{adj} A = 0 \text{ and } \text{adj} A \neq 0 \quad (\because A \cdot \text{adj} A = |A| I_n = 0 \cdot I_n = 0)$$

$$\Rightarrow \rho(A \cdot \text{adj} A) = 0 \text{ and } \rho(\text{adj} A) \neq 0$$

$$\therefore \rho(A \cdot \text{adj} A) \geq \rho(A) + \rho(\text{adj} A) - n$$

$$\Rightarrow 0 \geq \rho(A) + \rho(\text{adj} A) - n \text{ and } \rho(\text{adj} A) \neq 0$$

$$\Rightarrow 0 \geq n-1 + \rho(\text{adj} A) - n \text{ and } \rho(\text{adj} A) \neq 0$$

$$\Rightarrow \rho(\text{adj} A) \leq 1 \text{ and } \rho(\text{adj} A) \neq 0$$

$$\Rightarrow \rho(\text{adj} A) = 0 \text{ or } 1 \text{ and } \rho(\text{adj} A) \neq 0$$

Q.16. Can we express $(1, 4, 5) \in \mathbb{R}^3(\mathbb{R})$ as a linear combination of $(1, 1, 2)$, $(2, 3, 5)$ and $(3, 5, 8)$ in $\mathbb{R}^3$? If yes, in how many ways? (2013)

Sol. Let $(1, 4, 5) = x(1, 1, 2) + y(2, 3, 5) + z(3, 5, 8)$

$$\text{Then } x + 2y + 3z = 1$$

$$x + 3y + 5z = 4$$

$$2x + 5y + 8z = 5$$

$$A^* = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 4 \\ 2 & 5 & 8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A^*) = 2$ (infinite solution)

Therefore $(1, 4, 5) \in \mathbb{R}^3(\mathbb{R})$ can be expressed as a linear combination of $\{(1, 1, 2), (2, 3, 5) \text{ and } (3, 5, 8)\}$ in infinite number of ways.

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Q.17. Find the value of k for which the system of linear equation $kx + y + z = 1$, $x + ky + z = k$, $x + y + kz = k^2$ over the field of real numbers does not have a solution.

Sol. $AX = B$

(2013)

$$\text{Where } A^* = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

$$\text{Then } A^* = \begin{bmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & k \\ 1 & 1 & k & k^3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & k & k^3 \\ 1 & k & 1 & k \\ k & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - kR_1 \end{matrix} \begin{bmatrix} 1 & 1 & k & k^3 \\ 0 & k-1 & 1-k & k-k^3 \\ 0 & 1-k & 1-k^2 & 1-k^4 \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{k-1} R_2 \begin{bmatrix} 1 & 1 & k & k^3 \\ 0 & 1 & -1 & -k(k+1) \\ 0 & 1-k & 1-k^2 & 1-k^4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - (1-k)R_2 \begin{bmatrix} 1 & 1 & k & k^3 \\ 0 & 1 & -1 & -k(k+1) \\ 0 & 0 & (k+2)(1-k) & (1+k)(1-k^3) \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{1-k} R_3 \begin{bmatrix} 1 & 1 & k & k^3 \\ 0 & 1 & -1 & -k(k+1) \\ 0 & 0 & (k+2) & (1+k)(1+k+k^2) \end{bmatrix}$$

for $k = -2$

$\rho(A^*) = 3$ and $\rho(A) = 2$

Since $\rho(A^*) \neq \rho(A)$

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Therefore in this case solution does not exist.

Q.18. Define non-singular matrix. Prove that an n-square matrix is non-singular if and only if the rank of the matrix is n. (2013)

Sol. Non-singular Matrix:-

A square matrix over a field is said to be non-singular iff linear transformation associated with matrix is non-singular.

Let 'A' be $n \times n$ matrix over field F and $TA: F^n \rightarrow F^n$ be a linear transformation associated with matrix A.

A is a non-singular $\Leftrightarrow$ TA is non-singular

$\Leftrightarrow$ TA is one-one

$\Leftrightarrow \ker TA = \{O_n\}$

$\Leftrightarrow \dim \ker TA = 0$

$\Leftrightarrow \dim \operatorname{Im} TA + \dim \ker TA = \dim F^n$

$\Leftrightarrow \rho(TA) + 0 = n$

$\Leftrightarrow \rho(A) = n$

Q.19. Define an elementary matrix. Write

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \in M_3(\mathbb{R}).$$

(2017)

Sol. The matrix associated with elementary linear transformation is known as elementary matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \in M_3(\mathbb{R})$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{3} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow \left(-\frac{1}{2}\right)R_2 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

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$$R_1 \rightarrow R_1 - R_2 \quad \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + \frac{3}{2}R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - \frac{5}{2}R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & -5/6 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$$

I^3

A^{-1}

A as the product of elementary matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.20. Find the values of a for which the following system of linear equations over $\mathbb{R}$ is consistent:

$$x + 3y - z = 1, 2x + 7y + az = 3, x + ay - 7z = 0$$

(2017)

Sol. Given system

$$x + 3y - z = 1,$$

$$2x + 7y + az = 3,$$

$$x + ay - 7z = 0$$

$$AX = B$$

Where $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & a \\ 1 & a & -7 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Augmented matrix

$$A^* = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 7 & a & 3 \\ 1 & a & -7 & 0 \end{bmatrix}$$

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$$\begin{array}{l} R_1 \rightarrow R_1 - 2R_2 \\ R_2 \rightarrow R_2 - R_3 \end{array} \rightarrow \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & a+2 & 3 \\ 0 & a-3 & -6 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - (a-3)R_2 \rightarrow \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & a+2 & 3 \\ 0 & 0 & 6-(a+2)(a-3) & 0 \end{bmatrix}$$

$$\begin{aligned} (a+2)(a-3) &= 6 & \Rightarrow & a^2 - a - 6 = 6 \\ \Rightarrow a^2 - a - 12 &= 0 & \Rightarrow & a^2 - 4a + 3a - 12 = 0 \\ \Rightarrow a(a-4) + 3(a-4) &= 0 & \Rightarrow & a = 4, -3 \end{aligned}$$

If $a = 4, -3$ then system is consistent with infinite solution and if $a \neq 4, -3$ the system is consistent with unique solution.

Q.21. Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

and write A^{-1} as the product of elementary matrices.

(2018)

Sol. $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ we get-

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$ we get-

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$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}; \quad I_3 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_1 \rightarrow (-1)R_1$

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}; \quad I_3 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 5R_3$ and $R_2 \rightarrow R_2 + 2R_3$ we get-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad I_3 = \begin{bmatrix} -2 & -6 & 5 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

$= I_3$

$= A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} -2 & -6 & 5 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

A and A^{-1} as product of elementary matrices i.e.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Q.22. Reduce the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

to normal form and find its rank.

Sol. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$R_2 \leftrightarrow R_1$$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 2 & -3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1, R_4 \rightarrow R_4 - 6R_1$$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$C_2 \rightarrow C_2 + C_1, C_3 \rightarrow C_3 + 2C_1, C_4 \rightarrow C_4 + 4C_1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

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$$R_3 \rightarrow R_3 - 4R_2, R_4 \rightarrow R_4 - 9R_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

(2018)

$$C_3 \rightarrow C_3 + 6C_2, C_4 \rightarrow C_4 + 3C_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{33} R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 66R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 \rightarrow C_4 - \frac{2}{3} C_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 1} \end{bmatrix}$$

$$\rho(A) = \text{order of } I_3$$

$$\rho(A) = 3$$

Q.23.

Define the adjoint of a square matrix. If A and B are non-singular

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matrices of the same order over the same field F , then prove that $\text{adj}(AB) = \text{adj}(B) \text{adj}(A)$. (2018)

Sol. Adjoint of a square matrix:-

Let, $A = [a_{ij}]$ be $n \times n$ matrix over field F and $[c_{ij}]$ be the matrix of co-factor of element A then the transpose of matrix of co-factor of element will be called as adjoint of the matrix A and it is denoted by $\text{adj}(A)$ or $\text{adj } A$ i.e.,

adj a definition Transpose of matrix of co-factor of elements of A

If A and B are non-singular matrix of same order over it then

$$\text{adj}(A.B) = \text{adj } B . \text{adj } A$$

Proof:-

Since, A and B are non-singular.

$\therefore A.B$ is also non-singular.

$$\therefore (A.B)(\text{adj } A.B) = |A.B| I_n \quad \dots\dots(i)$$

$$\text{Now, } (A.B)(\text{adj } B . \text{adj } A) = A(B \text{adj } B) \text{adj } A$$

$$= A(|B| . I_n) \text{adj } A = |B| (A . \text{adj } A)$$

$$= |B| \odot (|A| . I_n) = (|A| \odot |B|) I_n$$

$$= |A.B| I_n \quad \dots\dots(ii)$$

From eq.(i) and eq.(ii)

$$(A.B)(\text{adj } A.B) = (A.B)(\text{adj } B . \text{adj } A)$$

$$\Rightarrow (A.B)^{-1} (A.B)(\text{adj } A.B) = (A.B)^{-1} (A.B)(\text{adj } B . \text{adj } A)$$

$$\Rightarrow \text{adj } A.B = \text{adj } B . \text{adj } A$$

Q.24. Let the rank of the coefficient matrix of a non-homogeneous system of m linear equations in n variables be p . If the system is consistent, the show that the number of linearly independent solutions of the system is $n - p + 1$ (2018)

Sol. Let $AX = B$

Let Y be a solution of $AX_0 = 0$

$$AY = 0 \quad \dots\dots(i)$$

Let X_0 be a particular solution of eq.(i)

$$AX_0 = B \quad \dots\dots(ii)$$

Claim:-

$(X_0 + Y)$ will be a solution of eq.(i)

For $A(X_0 + Y) = AX_0 + AY = B + 0 = B$

So $(X_0 + Y)$ will be a solution of eq.(i)

Let $Y_1, Y_2, \dots, Y_{n-p}$ be linearly independent solution of homogeneous system $AX = 0$

Claim:- $X_0, X_0 + Y_1, X_0 + Y_2, \dots, X_0 + Y_{n-p}$ are solutions of the system

$\{X_0 + Y_0 + Y_1, \dots, X_0 + Y_{n-p}\}$ is linearly independent subset of F^n .

For this we have to show that X_0 is not a linear combination of $Y_1, Y_2, \dots, Y_{n-p}$.

$$\Rightarrow X_0 = \alpha_1 Y_1 + \dots + \alpha_{n-p} Y_{n-p}$$

$$\Rightarrow AX_0 = \alpha_1 (AY_1) + \dots + \alpha_{n-p} (AY_{n-p})$$

$$\Rightarrow AX_0 = 0$$

Which is wrong as X_0 is particular solution of $A.X = B$

Therefore, our assumption that X_0 is linear combination of $Y_1, Y_2, \dots, Y_{n-p}$ is wrong. So X_0 is not linear combination of $Y_1, Y_2, \dots, Y_{n-p}$.

$\therefore \{Y_1, Y_2, \dots, Y_{n-p}\}$ is linearly independent subset of F^n and X_0 is not linear combination of $\{Y_1, Y_2, \dots, Y_{n-p}\}$.

So, $\{X_0, Y_1, Y_2, \dots, Y_{n-p}\}$ is also linearly independent.

$$\Rightarrow \{X_0, X_0 + Y_1, \dots, X_0 + Y_{n-p}\} \text{ is also linearly independent.}$$

Q.25. Define the row reduced Echelon form of a matrix. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} \in M_{3 \times 4}(\mathbb{R})$$

Find a non-singular matrix $P \in M_3(\mathbb{R})$ such that PA is in row reduced Echelon form. (2019)

Sol. Row-Reduced Echelon form of a Matrix:-

An $m \times n$ matrix R is called a row reduced echelon matrix if:

(i) Every row of R which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The first non-zero entry in each non-zero row is equal to 1.

(iii) If the first non-zero entry in row i appears in column k_i are zero.

(iv) If r is the number of non zero rows, then $k_1 < k_2 < \dots < k_r$.

$$\text{Given that } A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} \in M_{3 \times 4}(\mathbb{R})$$

$$A = IA$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_2 \rightarrow R_2 - 2R_1, \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 1 \\ -1 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$R_1 \rightarrow 5R_1 - R_2 \quad \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 1 \\ 0 & 0 & 15 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -1 & 5 \end{pmatrix} A$$

So
$$P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -1 & 5 \end{pmatrix}$$

Q.27. Define $\text{adj}(A)$, the adjoint of a matrix, $A \in M_n(F)$, where F is a field and prove that $A \cdot \text{adj}(A) = \det(A)I_n = \text{adj}(A) \cdot A$. (2019)

Ans. Adjoint(A):-

Let $A = [\alpha_{ij}]$ be $n \times n$ matrix over a field F and $[C_{ij}]$ be the matrix of cofactor of elements of A . The transpose of the matrix of cofactor of elements of A is called an adjoint of 'A' and is denoted by ' $\text{adj}A$ '.

Now, we prove that

$$A \cdot \text{adj}A = \text{adj}A \cdot A = |A| \cdot I_n$$

Proof: Let $A = [\alpha_{ij}]$ be $n \times n$ matrix over a field F therefore

$$\text{adj}[C_{jk}] \text{ where } C_{jk} = C_{kj} \quad \dots\dots\dots(i)$$

$$A \cdot \text{adj}A = [\gamma_{jk}]$$

Where $\gamma_{jk} = \sum_{i=1}^n \alpha_{ij} \odot C_{jk} = \sum_{i=1}^n \alpha_{ij} \odot C_{kj} = \delta_{jk} \odot |A|$ using eq.(i)

Therefore

$$\begin{aligned} A \cdot \text{adj}A &= \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \dots & \gamma_{nn} \end{bmatrix} \\ &= \begin{bmatrix} 1 \odot |A| & 0 \odot |A| & \dots & 0 \odot |A| \\ 0 \odot |A| & 1 \odot |A| & \dots & 0 \odot |A| \\ \vdots & \vdots & \ddots & \vdots \\ 0 \odot |A| & 0 \odot |A| & \dots & 1 \odot |A| \end{bmatrix} \\ &= |A| I_n \end{aligned}$$

Similarly $\text{adj}A \cdot A = |A| I_n$

From eq.(ii) and (iii)

$$A \cdot \text{adj}A = \text{adj}A \cdot A = |A| I_n$$

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Q.28. Define the row-rank and the column-rank of a matrix. Prove that for any field F and $A \in M_{m \times n}(F)$, $(\text{row-rank})(A) = (\text{column-rank})(A)$. (2019)

Sol. Row-rank and Column-rank of matrix:-

Let $A = [\alpha_{ij}]$ be $n \times n$ matrix over field F

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}$$

Rows of A -

$$R_1 = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}) \in F^n$$

$$R_2 = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2n}) \in F^n$$

.

.

$$R_m = (\alpha_{m1}, \alpha_{m2}, \dots, \alpha_{mn}) \in F^n$$

$$S_1 = \{R_1, R_2, \dots, R_m\} \in F^n$$

$$L(S_1) = [S_1] = \{\text{all possible linear combination of element of } S_1\}$$

$[S_1]$ is a subspace of F^n and is called as row space of A and dimension of row-space of A is called row-rank of A.

Column of A is

$$C_1 = (\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1}) \in F^m$$

$$C_2 = (\alpha_{12}, \alpha_{22}, \dots, \alpha_{m2}) \in F^m$$

.

.

$$C_m = (\alpha_{1m}, \alpha_{2m}, \dots, \alpha_{mm}) \in F^m$$

$$S_2 = \{C_1, C_2, \dots, C_m\} \in F^m$$

$$L(S_2) = \{\text{all possible linear combination of elements of } S_2\}$$

is a subspace of F^m and is called column space of A and dimension of column space of A is called column rank of A.

Now - We prove that

$$(\text{Row-rank})(A) = (\text{Column-rank})(A)$$

Let
$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1j} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{bmatrix}$$

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$$R_1 = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1j}, \dots, \alpha_{1n}) \in F^n$$

$$R_2 = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2j}, \dots, \alpha_{2n}) \in F^n$$

$$R_m = (\alpha_{m1}, \alpha_{m2}, \dots, \alpha_{mj}, \dots, \alpha_{mn}) \in F^n$$

$$C_j = \{\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{mj}\}$$

Claim - Column-rank of A $\leq$ Row-rank of A

Let dimension of row-rank A = r

$\Rightarrow$ Number of row rank A = r

$\Rightarrow$ Number of elements in a basis of row-space = r

Let S = $\{X_1, X_2, \dots, X_r\}$ be the basis of row-space.

$$\therefore X_1 = (\beta_{11}, \beta_{12}, \dots, \beta_{1j}, \dots, \beta_{1n})$$

$$X_2 = (\beta_{21}, \beta_{22}, \dots, \beta_{2j}, \dots, \beta_{2n})$$

$$X_r = (\beta_{r1}, \beta_{r2}, \dots, \beta_{rj}, \dots, \beta_{rn})$$

$\therefore R_1, R_2, \dots, R_n$ belongs to row-space of A.

$\therefore$ It can be uniquely expressed as a linear combination of elements of S.

$$R_1 = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1j}, \dots, \alpha_{1n}) = (\lambda_{11}x_1 + \lambda_{12}x_2 + \dots + \lambda_{1r}x_r)$$

$$R_2 = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2j}, \dots, \alpha_{2n}) = (\lambda_{21}x_1 + \lambda_{22}x_2 + \dots + \lambda_{2r}x_r)$$

$$R_m = (\alpha_{m1}, \alpha_{m2}, \dots, \alpha_{mj}, \dots, \alpha_{mn}) = (\lambda_{m1}x_1 + \lambda_{m2}x_2 + \dots + \lambda_{mr}x_r)$$

Now Jth column of A

$$C_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix} = \begin{bmatrix} \lambda_{11} \odot \beta_{1j} \oplus \lambda_{12} \odot \beta_{2j} \oplus \dots \oplus \lambda_{1r} \odot \beta_{rj} \\ \lambda_{21} \odot \beta_{1j} \oplus \lambda_{22} \odot \beta_{2j} \oplus \dots \oplus \lambda_{2r} \odot \beta_{rj} \\ \vdots \\ \lambda_{m1} \odot \beta_{1j} \oplus \lambda_{m2} \odot \beta_{2j} \oplus \dots \oplus \lambda_{mr} \odot \beta_{rj} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \vdots \\ \lambda_{m1} \end{bmatrix} \odot \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{rj} \end{bmatrix} \oplus \begin{bmatrix} \lambda_{12} \\ \lambda_{22} \\ \vdots \\ \lambda_{m2} \end{bmatrix} \odot \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{rj} \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} \lambda_{1r} \\ \lambda_{2r} \\ \vdots \\ \lambda_{mr} \end{bmatrix} \odot \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{rj} \end{bmatrix}$$

$$= \beta_{1j}Y_1 + \beta_{2j}Y_2 + \dots + \beta_{rj}Y_r \quad \forall j$$

which show that all column of A is linear combination of elements $Y_1, Y_2, \dots, Y_r$ which are elements of F^n .

$\therefore \{Y_1, Y_2, \dots, Y_r\}$ generates column space of A.

$\Rightarrow$ dimension of column space of A is always less or equal to r.

i.e. dimension of column space of A $\leq r \leq$ dim row space of A

$\Rightarrow$ dim of column space of A $\leq$ dim of row space of A

$\Rightarrow$ dim of row space of A $\leq$ dim of column of A

from eq.(i) and (ii)

Column rank of A = Row rank of A.

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Unit IV

Q.1. Using Gram-Schmidt orthonormalization process orthonormalize the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3(\mathbb{R})$. (2014)

Sol. Let $S = \{b_1, b_2, b_3\}$

$$a_1 = \frac{b_1}{\|b_1\|} = \frac{(0, 1, 1)}{\sqrt{2}} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$c_2 = b_2 - \langle b_2, a_1 \rangle a_1 = (1, 0, 1) - \frac{1}{\sqrt{2}} \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1, 0, 1) - \left(0, \frac{1}{2}, \frac{1}{2}\right) = \left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$a_2 = \frac{c_2}{\|c_2\|} = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1 + \frac{1}{4} + \frac{1}{4}}} = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{6}{4}}}$$

$$= \left(\frac{1}{\sqrt{3/2}}, \frac{-1/2}{\sqrt{3/2}}, \frac{1/2}{\sqrt{3/2}}\right)$$

$$c_3 = b_3 - \langle b_3, a_1 \rangle a_1 - \langle b_3, a_2 \rangle a_2$$

$$c_3 = (1, 1, 0) - \frac{1}{\sqrt{2}} \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{3/2}} - \frac{1}{2\sqrt{3/2}} + 0\right) \left(\frac{1}{\sqrt{3/2}}, \frac{-1/2}{\sqrt{3/2}}, \frac{1/2}{\sqrt{3/2}}\right)$$

$$= (1, 1, 0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{2\sqrt{3/2}}\right) \left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right)$$

$$= (1, 1, 0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{2\sqrt{3/2}}\right) \left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right)$$

$$= (1, 1, 0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right) = \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$$

$$a_3 = \frac{c_3}{\|c_3\|} = \frac{\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)}{\sqrt{\frac{4}{9} + \frac{16}{9} + \frac{4}{9}}} = \left(\frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}}, \frac{2}{\sqrt{24}}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

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therefore normalization basis is $\{a_1, a_2, a_3\}$ B.Sc. IInd - Math I - Unit IV - 84

$$= \left\{ \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3/2}}, \frac{-1}{2\sqrt{3/2}}, \frac{1}{2\sqrt{3/2}}\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right\}$$

Q.2. Define adjoint T^* of a linear transformation T on an inner product space $(V, \langle \cdot, \cdot \rangle)$ and prove that

$\text{Ker}(T^* \circ T) = \text{Ker } T$ and hence prove that $\rho(T^* \circ T) = \rho(T) = \rho(T \circ T^*)$. (2014)

Sol. Adjoint of a linear transformation:-

Let $T: (V, \langle \cdot, \cdot \rangle) \rightarrow (V, \langle \cdot, \cdot \rangle)$ be a linear transformation. Then we defined a map.

$$T^*: (V, \langle \cdot, \cdot \rangle) \rightarrow (V, \langle \cdot, \cdot \rangle) \text{ as}$$

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle, \forall x, y \in V$$

Then the map T^* is called adjoint of T .

Claim : $\text{Ker}(T^* \circ T) = \text{Ker } T$

Let $x \in \text{Ker}(T^* \circ T)$

$$\begin{aligned} \Rightarrow (T^* \circ T)(x) &= 0 & \Rightarrow T^*(T(x)) &= 0 \\ \Rightarrow \langle x, (T^* \circ T)(x) \rangle &= 0 & \Rightarrow \langle x, T^*(T(x)) \rangle &= 0 \\ \Rightarrow \|T(x)\|^2 &= 0 & \Rightarrow T(x) &= 0 \\ \Rightarrow x &\in \text{Ker } T \end{aligned}$$

$$\text{Ker}(T^* \circ T) \subseteq \text{Ker } T \quad \text{.....(i)}$$

let $x \in \text{Ker } T$

$$\begin{aligned} \Rightarrow T(x) &= 0 & \Rightarrow T^*(T(x)) &= 0 \\ \Rightarrow (T^* \circ T)(x) &= 0 & \Rightarrow x &\in \text{Ker}(T^* \circ T) \end{aligned}$$

$$\therefore \text{Ker } T \subseteq \text{Ker}(T^* \circ T)$$

From eq.(i) and (ii)

$$\text{Ker}(T^* \circ T) = \text{Ker } T \quad \text{.....(iii)}$$

Using Rank-Nullity Theorem for T and $(T^* \circ T)$

$$\dim \text{Ker } T + \dim \text{Im } T = \dim V \quad \text{.....(iv)}$$

$$\dim \text{Ker}(T^* \circ T) + \dim \text{Im}(T^* \circ T) = \dim V \quad \text{.....(v)}$$

from eq.(iii), (iv) and (v)

$$\text{we get } \rho(A) = \rho(T^* \circ T)$$

Q.3. Prove that eigen values of a Hermitian Matrix are all real. (2014)

Sol. Suppose A is a Hermitian matrix, λ an eigen value of A and X a corresponding eigen vector. Then

$$AX = \lambda X \quad \text{.....(i)}$$

Pre multiplying both side by X^H , we get

$$X^H AX = \lambda X^H X \quad \text{.....(ii)}$$

Taking conjugate transpose of both sides of eq.(ii), we get

$$(X^H AX)^H = (\lambda X^H X)^H$$

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$$\text{or } X^0 A^0 (X^0)^0 = \bar{\lambda} X^0 (X^0)^0$$

$$\text{or } X^0 A X = \bar{\lambda} X^0 X$$

[$\because (X^0)^0 = X$ and $A^0 = A$, A being Hermitian]

From eq. (ii) and (iii) we get

$$\lambda X^0 X = \bar{\lambda} X^0 X \Rightarrow (\lambda - \bar{\lambda}) X^0 X = 0$$

But X is not a zero vector,

Therefore $X^0 X \neq 0$

Hence $\lambda - \bar{\lambda} = 0$, so that $\lambda = \bar{\lambda}$ and consequently λ is real.

Q.4. Prove that eigen vectors of a Hermitian matrix corresponding to distinct eigen values of the matrix are orthogonal. (2014)

Sol. Given that A is Hermitian.

$$\text{i.e. } A^0 = A$$

Also, we have

$$A x_1 = \lambda_1 x_1 \text{ and } A x_2 = \lambda_2 x_2$$

$$\text{or } \lambda_1 x_1^0 (x_2^0)^0 = \lambda_1 x_1^0 x_2 \quad [\bar{\lambda}_1 \text{ is real} \Rightarrow \lambda_1^0 = \lambda_1]$$

$$\text{or } \lambda_1 x_1^0 x_2 = \lambda_2 x_1^0 x_2 \Rightarrow (\lambda_1 - \lambda_2) x_1^0 x_2 = 0$$

Since $\lambda_1 - \lambda_2 \neq 0$,

we have $x_1^0 x_2 = 0$

$\Rightarrow x_1$ and x_2 are orthogonal with respect to each other.

Q.5. Prove that the determinant of a skew-symmetric matrix of odd order is zero. (2014)

Sol. Suppose that n is an odd integer and let A be an $n \times n$ skew symmetric matrix.

Thus we have $A^T = -A$

$$\text{Then } \det(A) = \det(A^T) = \det(-A) \quad \text{.....(i)}$$

$$= (-1)^n \det(A) = -\det(A) \quad \text{from eq. (i)}$$

$$\text{Therefore } 2 \det(A) = 0 \text{ and } \det(A) = 0 \quad \text{Since n is odd}$$

Q.6. If A is real skew-symmetric matrix such that $A^2 + I_n = 0$, then prove that A is of even order. (2014)

Ans. Given that A is real skew symmetric matrix such that $A^2 + I_n = 0$

$$\Rightarrow A^2 = -I_n$$

$$|A|^2 = (-1)^n$$

$$|A|^2 = -1$$

Only if |A| is pure imaginary.

In case of skew symmetric matrices. Since every skew-symmetric matrix of odd

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order is always singular.

$$|A|^2 = \pm 1$$

$$|A| \neq 0$$

Hence A is of even order.

Q.7. Define inner product on a vector space $V(F)$. Let $V(C)$ be the vector space of all continuous complex-valued functions on the unit interval $0 \leq t \leq 1$. If $f(t), g(t) \in V$, Let us define

$$f(t) \cdot g(t) = \int_0^1 f(t) \overline{g(t)} dt$$

Show that $(f(t), g(t))$ is an inner product on $V(C)$.

(2015)

Sol. Inner product space:-

Let V be a vector space over field of complex number (or real no.) Then an inner product in V is a map i.e. $\langle \cdot, \cdot \rangle : V \times V \rightarrow C$ which assigns each ordered pair (a, b) of $V \times V$ to a unique scalar (complex no.)

$\langle a, b \rangle$ or a.b satisfying the following conditions

$$(1) \langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle \quad \forall a, b, c \in V$$

$$(2) \langle \lambda a, b \rangle = \lambda \langle a, b \rangle \quad \forall \lambda \in C, \forall a, b \in V$$

$$(3) \langle a, b \rangle = \overline{\langle b, a \rangle}, \quad \forall a, b \in V$$

$$(4) \langle a, a \rangle \geq 0 \text{ and } \langle a, a \rangle = 0 \text{ iff } a = 0 \quad \forall A \in V \text{ and } (v, \langle \cdot, \cdot \rangle) \text{ is called an inner product space}$$

$$\text{Given that } (f(t), g(t)) = \int_0^1 f(t) \overline{g(t)} dt \quad 0 \leq t \leq 1 \quad \text{.....(1)}$$

We have to show that all the properties of an inner product hold in eq. (i)

(1) Conjugate symmetry:-

We have

$$(g(t), f(t)) = \int_0^1 g(t) \overline{f(t)} dt \quad \text{from eq. (1)}$$

$$\therefore \overline{(g(t), f(t))} = \overline{\int_0^1 g(t) \overline{f(t)} dt} = \int_0^1 \overline{g(t) \overline{f(t)}} dt$$

$$= \int_0^1 \overline{g(t)} f(t) dt = \int_0^1 f(t) \overline{g(t)} dt = (f(t), g(t))$$

(2) Linearity:-

Let a, b $\in C$ and $h(t) \in V$.

$$\text{Then } [af(t) + bg(t), h(t)] = \int_0^1 (af(t) + bg(t)) \overline{h(t)} dt$$

$$= a \int_0^1 f(t) \overline{h(t)} dt + b \int_0^1 g(t) \overline{h(t)} dt$$

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$$= a(f(t), h(t)) + b(g(t), h(t))$$

Non-negativity:-
We have

$$(f(t), f(t)) = \int_0^1 f(t) \overline{f(t)} dt = \int_0^1 |f(t)|^2 dt \quad \dots (2)$$

Since $[f(t)]^2 \geq 0$ for every t lying in the closed interval $[0, 1]$. Therefore eq. (2)

$$\geq 0. \text{ Thus } (f(t), f(t)) \geq 0$$

$$\text{Also } (f(t), f(t)) = 0$$

$$\Rightarrow \int_0^1 [f(t)]^2 dt = 0$$

$$\Rightarrow [f(t)]^2 = 0 \text{ for every } t \text{ lying in } [0, 1]$$

$$\Rightarrow f(t) = 0 \text{ for every } t \text{ lying in } [0, 1]$$

Hence the product defined eq. (1) is an inner product on V (c).

Q.8. If α, β are vectors in an inner product space V , prove that

$$(i) \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

$$(ii) \|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2). \quad (2015)$$

Sol. (i). $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ (triangle inequality)

Proof:- We have

$$\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta)$$

$$= (\alpha, \alpha + \beta) + (\beta, \alpha + \beta)$$

$$= (\alpha, \alpha) + (\alpha, \beta) + (\beta, \alpha) + (\beta, \beta)$$

$$= \|\alpha\|^2 + (\alpha, \beta) + (\alpha, \beta) + \|\beta\|^2$$

$$= \|\alpha\|^2 + (\gamma + \bar{\gamma}) + \|\beta\|^2$$

$$= \|\alpha\|^2 + 2 \operatorname{Re} \gamma + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2|\gamma| + \|\beta\|^2$$

$$= \|\alpha\|^2 + 2|(\alpha, \beta)| + \|\beta\|^2$$

$$= \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2 \text{ (by Cauchy - Schwarz inequality)}$$

Where $\gamma = (\alpha, \beta) \in \mathbb{C}$

$$\therefore (\operatorname{Re} \gamma \leq |\gamma|)$$

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$$= \|\alpha\|^2 + \|\beta\|^2$$

$$\therefore \|\alpha + \beta\|^2 \leq (\|\alpha\| + \|\beta\|)^2$$

$$\text{Hence } \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

$$(ii) \|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$$

(Parallelogram law)

Proof:- We have

$$\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta)$$

$$= (\alpha, \alpha + \beta) + (\beta, \alpha + \beta)$$

$$= (\alpha, \alpha) + (\alpha, \beta) + (\beta, \alpha) + (\beta, \beta) \quad \dots (1)$$

$$\text{And } \|\alpha - \beta\|^2 = (\alpha - \beta, \alpha - \beta)$$

$$= (\alpha, \alpha - \beta) - (\beta, \alpha - \beta)$$

$$= (\alpha, \alpha) - (\alpha, \beta) - [(\beta, \alpha) - (\beta, \beta)]$$

$$= (\alpha, \alpha) - (\alpha, \beta) - (\beta, \alpha) + (\beta, \beta) \quad \dots (2)$$

Adding eq. (1) and (2) we get

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\alpha, \alpha) + 2(\beta, \beta)$$

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$$

Q.9. State and prove Bessel's inequality for an inner product space. (2015)

Ans. Bessel's inequality:-

If V is an inner product space and if $\{w_1, w_2, \dots, w_n\}$ is an orthonormal set in V . Then

$$\sum_{i=1}^n |(w_i, v)|^2 \leq \|(v)\|^2, \text{ for all } v \in V$$

Proof:- Let $v \in V$ be arbitrary consider the vector

$$x = v - \sum_{i=1}^n \alpha_i w_i \quad \text{where } \alpha_i = (v, w_i), \quad \dots (1)$$

$$\text{Then } (x, x) = \left(v - \sum_{i=1}^n \alpha_i w_i, v - \sum_{i=1}^n \alpha_i w_i \right)$$

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$$\begin{aligned}
 &= (v, v) - \left(v, \sum_{j=1}^n \alpha_j w_j \right) - \left(\sum_{i=1}^n \alpha_i w_i, v \right) + \left(\sum_{i=1}^n \alpha_i w_i, \sum_{j=1}^n \alpha_j w_j \right) \\
 &= \|v\|^2 - \sum_{j=1}^n \alpha_j (v, w_j) - \sum_{i=1}^n \alpha_i (w_i, v) + \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j (w_i, w_j) \\
 &= \|v\|^2 - \sum_{j=1}^n (\overline{(v, w_j)})(v, w_j) - \sum_{i=1}^n (v, w_i)(\overline{(v, w_i)}) + \sum_{i=1}^n \alpha_i \bar{\alpha}_i \\
 &\quad \left[\because (w_i, w_j) = 0 \text{ for } i \neq j \text{ and } (w_i, w_i) = 1 \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \|v\|^2 - \sum_{i=1}^n |(v, w_i)|^2 - \sum_{i=1}^n |(v, w_i)|^2 + \sum_{i=1}^n |(v, w_i)|^2 \\
 &= \|v\|^2 - \sum_{i=1}^n |(v, w_i)|^2 = \|v\|^2 - \sum_{i=1}^n |(v, w_i)|^2 \quad (\because |z| = |\bar{z}|) \\
 &\|x\|^2 = \|v\|^2 - \sum_{i=1}^n |(w_i, v)|^2 \quad \dots\dots\dots(2)
 \end{aligned}$$

Since $\|x\|^2 \geq 0$,
so by eq.(2)

$$\|v\|^2 - \sum_{i=1}^n |(w_i, v)|^2 \geq 0 \Rightarrow \|v\|^2 \geq \sum_{i=1}^n |(w_i, v)|^2$$

Hence $\sum_{i=1}^n |(w_i, v)|^2 \leq \|v\|^2, \forall v \in V$

Q.10. Define eigen values and eigen vectors of a square matrix A. Prove that the eigen vectors corresponding to distinct eigen values of a matrix are linearly independent. (2015)

Ans. Eigen values and eigen vectors of a square matrix:-
To show eigen vector corresponding to distinct eigen values of a matrix are linearly independent.
Let v_1, v_2 be eigen vectors corresponding to eigen values λ_1, λ_2 respectively.

To show that the vectors v_1, v_2 are C.I. consider linear combination
Where c_1 and c_2 are complex numbers,
We have to show that $c_1 = c_2 = 0$

By definition of eigen values and eigen vectors

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$$\begin{aligned}
 Av_1 &= \lambda_1 v_1 \text{ and } Av_2 = \lambda_2 v_2 \\
 0 &= A.0 = A(c_1 v_1 + c_2 v_2) \\
 &= c_1 Av_1 + c_2 Av_2 \\
 0 &= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2
 \end{aligned}$$

Multiplying by λ_2 in eq. (1) we get $\dots\dots(2)$

Subtracting (1) from (3) we get $\dots\dots(3)$

$$\begin{aligned}
 0 &= (c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2) - (c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2) \\
 0 &= c_1 (\lambda_2 - \lambda_1) v_1 \\
 \Rightarrow c_1 (\lambda_2 - \lambda_1) &= 0 \quad (\text{Since } v_1 \neq 0)
 \end{aligned}$$

Since λ_1 and λ_2 are distinct we have

$$c_1 = 0$$

Putting c_1 in eq.(1) we get

$$c_2 = 0$$

(since $v_2 \neq 0$)

Therefore, the values of c_1 and c_2 are both zero, and hence the eigen vector v_1, v_2 are linearly independent.

Q.11. Prove that two eigen vectors of Skew-Hermitian matrix corresponding to two distinct eigen values of the matrix are orthogonal. (2012)

Sol. Let A is skew Hermitian matrix. Then iA is Hermitian.

Let u and v be eigen values and λ and μ be eigenvectors respectively.

$$Au = \lambda u, Av = \mu v$$

$$(iA)u = (i\lambda)u, (iA)v = (i\mu)v$$

Take $(i\lambda)u^*v = (i\lambda u)^*v = ((iA)u)^*v$

$$= u^*(iA)^*v = u^*(-i)A^*v = (-i)u^*(-A)v \quad \{\because A^* = -A\}$$

$$= i u^*Av = i u^*\mu v \quad \{\because Av = \mu v\}$$

$$\Rightarrow (i\lambda)u^*v = i\mu u^*v$$

$$\Rightarrow i(\lambda - \mu)u^*v = 0 \Rightarrow u^*v = 0$$

$\Rightarrow u$ and v are orthogonal to each other.

Q.12. If $S = \{x_1, x_2, \dots, x_n\}$ is an orthogonal subset of an inner product space $(V, \langle \cdot, \cdot \rangle)$, then for $x \in V$ prove that

$$\sum_{i=1}^n |\langle x, x_i \rangle|^2 = \|x\|^2$$

(2012)

if and only if S is a basis of V.

Sol. Sufficient Part :-

Let $X = \{x_1, x_2, \dots, x_n\}$ be an orthogonal basis of $(V, \langle \cdot, \cdot \rangle)$.

$$x \in V \Rightarrow x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

$$\langle x, x \rangle = \langle \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n, \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \rangle$$

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$$\begin{aligned}
 &= \alpha_1 \bar{\alpha}_1 + \alpha_2 \bar{\alpha}_2 + \dots + \alpha_n \bar{\alpha}_n \\
 &= |\alpha_1|^2 + |\alpha_2|^2 + \dots + |\alpha_n|^2 \\
 \|x\|^2 &= |\langle x, x_1 \rangle|^2 + |\langle x, x_2 \rangle|^2 + \dots + |\langle x, x_n \rangle|^2
 \end{aligned}$$

Necessary Part: -

$$\text{Let } \|x\|^2 = \sum_{i=1}^n |\langle x, x_i \rangle|^2 \quad \dots\dots\dots(i)$$

Then we will show that x will be basis of IPS $(V, \langle \cdot, \cdot \rangle)$.

$\therefore X$ is an orthonormal set which is linearly independent.

Let $x \in V$ and

$$y = x - \langle x, x_1 \rangle x_1 - \dots - \langle x, x_n \rangle x_n \quad \dots\dots\dots(ii)$$

$$\|y\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, x_i \rangle|^2 = 0 \quad (\text{using eq.(i)})$$

$$\Rightarrow y = 0$$

$$\Rightarrow x = \langle x, x_1 \rangle x_1 + \dots + \langle x, x_n \rangle x_n$$

$$\Rightarrow x = \alpha_1 x_1 + \dots + \alpha_n x_n$$

$$\therefore V = [X]$$

Therefore X is basis.

Q.13. Define eigen value of a square matrix. Let $A = [a_{ij}]$ be an n -square real matrix such that $|a_{ij}| \leq m, \forall i = 1, 2, \dots, n$ and λ be an eigen value of A . Then prove that

$$|\lambda| \leq mn \quad (2012)$$

Ans. Eigen Value of a square matrix: -

Let $A = [a_{ij}]_{n \times n}$ be any n -rowed square matrix and λ an indeterminate. Then the equation $|A - \lambda I| = 0$ is called the characteristic equation of A and the roots of this equation are called the eigen values of the matrix A .

Given that $A = [a_{ij}]$ be an n -square real matrix such that $|a_{ij}| \leq m, \forall i = 1, 2, \dots, n$ we know that

Trace of matrix = Sum of eigen values

$$\text{Then } \sum a_{ii} \leq \frac{m + m + \dots + m}{n - \text{time}} \leq |mn|$$

$$\text{or trace of matrix} \leq |mn|$$

$$\text{or sum of eigen value} \leq |mn|$$

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$$\text{or } |\lambda| \leq mn.$$

(Hence Proved)

Q.14. Define the orthogonal components $S^\perp$ of a non-empty subset S of an inner product space. If $\{(1, 1, 2, 0), (1, 1, 0, 3)\}$ generates the subspace W of Euclidean space $\mathbb{R}^4$ then find an orthogonal basis of $W^\perp$. (2013)

Sol. Orthogonal complement of a non empty set of an inner product space: -

Let $S \neq \phi$ be a subset of an IPS $(V, \langle \cdot, \cdot \rangle)$. Then orthogonal complement of " S " in " V " is collection of all those elements of " V " which are orthogonal elements of " V " which are orthogonal to every element of " S " and it is denoted by " $S^\perp$ ".

$$\text{i.e. } S^\perp \text{ def } \{x \in V : \langle x, s \rangle = 0, \forall s \in S\}$$

Since $S = \{(1, 1, 2, 0), (1, 1, 0, 3)\}$ is linear independent.

$\therefore S$ is a basis of W .

$$\dim W = 2$$

$$\text{Since } \dim V = \dim W + \dim W^\perp$$

$$\dim \mathbb{R}^4 = 2 + \dim W^\perp$$

$$4 = 2 + \dim W^\perp$$

$$\Rightarrow \dim W^\perp = 2$$

Let $(x, y, z, u) \in W^\perp$

$$\langle (x, y, z, u), (1, 1, 2, 0) \rangle = 0$$

$$\text{and } \langle (x, y, z, u), (1, 1, 0, 3) \rangle = 0$$

$$\Rightarrow x + y + 2z = 0$$

$$x + y + 3u = 0$$

$$AX = 0$$

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

The above equation are equivalent to

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y + 2z = 0$$

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$$2z + 3u = 0 \Rightarrow z = -\frac{3}{2}u$$

$$x = -y - 3u$$

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -1y - 3u \\ y \\ 0y + \frac{3}{2}u \\ u \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 3/2 \\ 1 \end{bmatrix} u$$

Therefore linear independent solution is $(-1, 1, 0, 0)$ and $(-3, 0, 3/2, 1)$ or $(-1, 1, 0, 0), (-6, 0, 3, 2)$.

Therefore $S_1 = \{(-1, 1, 0, 0), (-6, 0, 3, 2)\}$ is basis of $W^\perp$.

Now orthonormalize above basis by Gram-Schmidt orthonormalization process we get.

$$S_2 = \left\{ \frac{1}{\sqrt{2}}(-1, 1, 0, 0), \frac{1}{\sqrt{49}}(-6, 0, 3, 2) \right\}$$

Q.15. Define eigen values and eigen vectors of a linear operator on a vector space. Find eigen values and corresponding eigen-vectors of the linear operator T on a vector space $\mathbb{R}_2[x]$ of all real polynomial of degree at most 2, given by $T(a_0 + a_1x + a_2x^2) = (3a_0 - 2a_1) - (2a_0 - 3a_1)x + 4a_2x^2 \forall (a_0 + a_1x + a_2x^2) \in \mathbb{R}_2[x]$.

Sol. Eigen value and eigen vectors of a linear operator: -

For a given linear operator $T: V \rightarrow V$, a non zero vector X and a constant scalar λ are called an eigen vector and its eigen values respectively when $T(X) = \lambda X$.

Let $\{1, x, x^2\}$ be basis of $\mathbb{R}_2[x]$.

$$\text{Then } T(1) = 3 - 2x = 3 \cdot 1 - 2x + 0 \cdot x^2$$

$$T(x) = -2 + 3x = -2 + 3x + 0 \cdot x^2 = -2 \cdot 1 + 3 \cdot x + 0 \cdot x^2$$

$$T(x^2) = 4x^2 = 0 \cdot 1 + 0 \cdot x + 4 \cdot x^2$$

matrix of T

$$= \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = A \text{ (say)}$$

Then characteristics equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$(4 - \lambda)[(3 - \lambda)^2 - 4] = 0$$

$$\Rightarrow (4 - \lambda)(9 + \lambda^2 - 6\lambda - 4) = 0 \Rightarrow (4 - \lambda)(\lambda^2 - 6\lambda + 5) = 0$$

$$(4 - \lambda)(\lambda - 5)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 1, 4, 5$$

Eigen vector corresponding to eigen values $\lambda = 1$.

$$(A - I)X = 0 \quad \text{where } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 3-1 & -2 & 0 \\ -2 & 3-1 & 0 \\ 0 & 0 & 4-1 \end{bmatrix} X = 0 \quad \text{i.e.} \quad \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} X = 0$$

$$2x_1 - 2x_2 = 0$$

$$-2x_1 + 2x_2 = 0$$

$$3x_3 = 0$$

$$\Rightarrow x_3 = 0$$

$$2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$$\text{Then } X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly we find eigen vectors corresponding to eigen values $\lambda = 4$ and $\lambda = 5$.

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ respectively.}$$

Q.16. Define adjoint of a linear operator. Find the adjoint of the linear operator T_A on $M_n(F)$ (Set of all $n \times n$ matrices over the field F) given by $T_A(X) = AX \forall X \in M_n(F)$, where A is a fixed matrix in $M_n(F)$. (2013)

Sol. Adjoint of a linear operator: -

Let $T: (V, \langle \cdot, \cdot \rangle) \rightarrow (V, \langle \cdot, \cdot \rangle)$ be a linear transformation. Then we defined a map $T^*: (V, \langle \cdot, \cdot \rangle) \rightarrow (V, \langle \cdot, \cdot \rangle)$ as

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \forall x, y \in V$$

Then the map T^* is called adjoint of T .

Given that

$$T(X) = AX, \quad \forall X \in M_n$$

We have

$$\langle T(x), y \rangle = \langle Ax, T^*(y) \rangle \quad \forall x, y \in M_n(F) \quad \dots (i)$$

$$\text{Also } \langle T(x), y \rangle = \langle Ax, y \rangle = m(AXY^*)$$

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$$= \text{tr}(XY^*A) = \text{tr}(X(A^*Y)^*) = \langle X, A^*Y \rangle \quad \dots\dots\dots(ii)$$

From eq.(i) and (ii) we get

$$T^*(Y) = A^*Y \quad \text{or} \quad T^*(X) = A^*X$$

Q.17. If T^* is the adjoint of a linear operator T on the inner product space $(V, \langle \cdot, \cdot \rangle)$ over the field of complex numbers, then prove that $\text{rank of } T^* = \text{rank of } T$. (2013)

Sol. First we prove that

$$\text{Im } T^* = (\text{Ker } T)^\perp \quad \dots\dots\dots(i)$$

$$T: V \rightarrow V; \quad T^*: V \rightarrow V$$

$$\dim W + \dim W^\perp = \dim V \quad (W \text{ is subspace of } V)$$

$$\dim \text{Ker } T + \dim (\text{Ker } T)^\perp = \dim V$$

$$\Rightarrow \dim \text{Ker } T + \dim (\text{Im } T^*) = \dim V \quad (\text{using (i)})$$

$$\Rightarrow \dim \text{Ker } T + \rho(T^*) = \dim V \quad \dots\dots\dots(ii)$$

Using Rank-Nullity theorem for T we have

$$\Rightarrow \dim \text{Ker } T + \rho(T) = \dim V \quad \dots\dots\dots(iii)$$

From eq.(ii) and (iii)

$$\rho(T^*) = \rho(T)$$

Q.18. If S is a real skew-symmetric matrix such that $S^2 + I = 0$, then Prove that S is an orthogonal matrix of even order. (2013)

Sol. S is real skew symmetric matrix then $-S = S^t$

$$\text{or } SS^t = -S.S \quad \Rightarrow \quad SS^t = -S^2$$

$$\Rightarrow SS^t = -S^2 = I \quad \left\{ \text{given } S^2 + I = 0 \Rightarrow S^2 = -I \right\}$$

$$\Rightarrow SS^t = I \quad \Rightarrow \quad S \text{ is orthogonal.}$$

$$\text{Since } S^2 + I = 0 \quad \Rightarrow \quad S^2 = -I_n$$

$$(\det(S))^2 = (-1)^n$$

L.H.S. is square quantity and must be positive because of S have real entries. Then n must be even. (Proved)

Q.19. State and prove the Bessel's inequality for an inner product space. (2018)

Sol. Bessel's Inequality:-

If $\{x_1, x_2, \dots, x_n\}$ is an orthonormal subset of an inner product space $(V, \langle \cdot, \cdot \rangle)$, then any $x \in V$, we have -

$$\langle x, x_1 \rangle^2 + \langle x, x_2 \rangle^2 + \dots + \langle x, x_n \rangle^2 \leq \|x\|^2$$

$$\text{Or } \sum_{i=1}^n |\langle x, x_i \rangle|^2 \leq \|x\|^2$$

Proof:-

Let us consider

$$y = x - \langle x, x_1 \rangle x_1 - \langle x, x_2 \rangle x_2 - \dots - \langle x, x_n \rangle x_n$$

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$$= x - \sum_{i=1}^n \langle x, x_i \rangle x_i$$

$$\Rightarrow y = x - \sum_{i=1}^n \alpha_i x_i \quad \dots\dots\dots(i) \quad \text{where } \alpha_i = \langle x, x_i \rangle$$

$$\begin{aligned} \text{Now } \|y\|^2 &= \langle y, y \rangle = \left\langle x - \sum_{i=1}^n \alpha_i x_i, x - \sum_{i=1}^n \alpha_i x_i \right\rangle \\ &= \langle x, x \rangle - \left\langle x, \sum_{i=1}^n \alpha_i x_i \right\rangle - \left\langle \sum_{i=1}^n \alpha_i x_i, x \right\rangle + \left\langle \sum_{i=1}^n \alpha_i x_i, \sum_{i=1}^n \alpha_i x_i \right\rangle \end{aligned}$$

$$= \|x\|^2 - \sum_{i=1}^n \alpha_i \langle x, x_i \rangle - \sum_{i=1}^n \alpha_i \langle x, x_i \rangle + \sum_{i=1}^n \alpha_i \bar{\alpha}_i \langle x_i, x_i \rangle$$

$$\Rightarrow \|y\|^2 = \|x\|^2 - \sum_{i=1}^n \overline{\langle x, x_i \rangle} \langle x, x_i \rangle - \sum_{i=1}^n \langle x, x_i \rangle \overline{\langle x, x_i \rangle} + \sum_{i=1}^n \langle x, x_i \rangle \overline{\langle x, x_i \rangle}$$

$$\left\{ \begin{array}{l} \because \alpha_i = \langle x, x_i \rangle \\ \text{And } \langle x, x_i \rangle = \overline{\langle x, x_i \rangle} = 1 \end{array} \right.$$

$$\Rightarrow \|y\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, x_i \rangle|^2$$

$$\text{Since } \|y\|^2 \geq 0$$

$$\Rightarrow \|x\|^2 - \sum_{i=1}^n |\langle x, x_i \rangle|^2 \geq 0 \Rightarrow \sum_{i=1}^n |\langle x, x_i \rangle|^2 \leq \|x\|^2$$

Q.20. Let W be a subspace of a finite dimensional inner product space V . Prove that $V = W \oplus W^\perp$. (2018)

Sol. **Proof:-**

Since, W is a subspace of a finite dimensional I.P.S. $(V, \langle \cdot, \cdot \rangle)$.

$\therefore W$ must be a finite dimensional and passes an orthonormal basis.

Let $S_1 = \{a_1, a_2, \dots, a_m\}$ be an orthonormal basis of W .

$$\therefore \dim W = m \quad \dots\dots\dots(ii)$$

Since, S_1 is a linearly independent subset of V . So, by extension theorem it can be extended to form an orthonormal basis of V .

Let $S_2 = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ is an orthogonal basis of V .

$$\text{i.e. } \dim V = m + n \quad \dots\dots\dots(ii)$$

Claim- $b_1, b_2, \dots, b_n \in W^\perp$

Since S_2 is an orthonormal basis of V .

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$b_1, b_2, \dots, b_n$ are orthonormal to $a_1, a_2, \dots, a_m$.
 $\Rightarrow b_1, b_2, \dots, b_n$ are orthogonal to every linear combination of $a_1, a_2, \dots, a_m$.
Hence $b_1, b_2, \dots, b_n$ are orthogonal to every element of W (or S_1).
Hence $b_1, b_2, \dots, b_n \in W^\perp$.
Now for $V = W \oplus W^\perp$.
We have to show that

$$(i) V = W + W^\perp \quad (ii) W \cap W^\perp = \{0\}$$

$$(i) V = W + W^\perp$$

Let $x \in V$

$$\Rightarrow x = \langle x, a_1 \rangle a_1 + \langle x, a_2 \rangle a_2 + \dots + \langle x, a_m \rangle a_m + \langle x, b_1 \rangle b_1 + \langle x, b_2 \rangle b_2 + \dots + \langle x, b_n \rangle b_n$$

$$\Rightarrow x = \underbrace{\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m}_{\in W} + \underbrace{\beta_1 b_1 + \beta_2 b_2 + \dots + \beta_n b_n}_{\in W^\perp}$$

$\Rightarrow$ In every $x \in V$ can be written as one element of W plus one element of $W^\perp$.

$$\Rightarrow V = W + W^\perp$$

$$(ii) W \cap W^\perp = \{0\}$$

$$\text{Let } x \in W \cap W^\perp \Rightarrow x \in W \text{ and } x \in W^\perp$$

$$\Rightarrow x \perp x \Rightarrow \langle x, x \rangle = 0$$

$$\Rightarrow x = 0 \quad \therefore W \cap W^\perp = \{0\}$$

$$\text{thus } V = W \oplus W^\perp$$

(Proved)

Q.21. Let $S = \{(1, 1, 0), (1, 3, 0), (1, 1, 1)\} \subseteq \mathbb{R}^3$. Use Gram-Schmidt process to orthonormalize S . (2019)

$$\text{Sol. } S = \{(1, 1, 0), (1, 3, 0), (1, 1, 1)\} \subseteq \mathbb{R}^3$$

$$\text{Let } V_1 = (1, 1, 0), V_2 = (1, 3, 0), V_3 = (1, 1, 1)$$

$$\text{Let } w_1 = V_1 = (1, 1, 0) \Rightarrow \|w_1\|^2 = (w_1, w_1) = 1 + 1 + 0 = 2$$

$$\therefore \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{Let } w_2 = V_2 - \frac{(V_2, w_1)}{\|w_1\|^2} w_1 \quad \dots\dots\dots(i)$$

$$\text{from eq (i)} \quad (V_2, w_1) = (V_2, V_1) = 1 + 3 + 0 = 4$$

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$$w_2 = (1, 3, 0) - \frac{1}{2}(1, 1, 0)4 = (1, 3, 0) - (2, 2, 0) = (-1, 1, 0)$$

$$\Rightarrow \|w_2\|^2 = (w_2, w_2) = (-1) + (-1) + 1 + 1 + 0 = 2$$

$$\therefore \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(-1, 1, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{Let } w_3 = V_3 - \frac{(V_3, w_1)}{\|w_1\|^2} w_1 - \frac{(V_3, w_2)}{\|w_2\|^2} w_2 \quad \dots\dots\dots(2)$$

$$\text{We have } (V_3, w_1) = (V_3, V_1) = 1 + 1 + 1 = 3$$

$$(V_3, w_2) = 1 + (-1) + 1 + 1 = 2$$

Putting in eq.(ii) we get

$$w_3 = (1, 1, 1) - \frac{3}{2}(1, 1, 0) - \frac{2}{2}(-1, 1, 0) = (1, 1, 1) - (1, 1, 0) = (0, 0, 1)$$

$$\Rightarrow \|w_3\|^2 = (w_3, w_3) = 0 + 0 + 1 = 1$$

$$\frac{w_3}{\|w_3\|} = \frac{(0, 0, 1)}{1} = (0, 0, 1)$$

Hence an orthonormal basis for $\mathbb{R}^3$ is

$$\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0, 0, 1) \right\}$$

Q.22. Let V be a finite dimensional inner product space and $T \in V^*$. Prove that there exists a unique $y \in V$ such that $T(x) = \langle x, y \rangle \forall x \in V$. (2019)

Sol. Since T is a linear functional on V . So $T: V \rightarrow F$ is a homomorphism. We know that every finite dimensional inner product space has an orthogonal basis. Let $\{x_1, x_2, \dots, x_n\}$ be an orthogonal basis of V , then

$$\langle x_i, x_j \rangle = 0 \text{ for } i \neq j \text{ and } \langle x_i, x_i \rangle = 1 \text{ for each } i \quad \dots\dots\dots(i)$$

Any $x \in V$ can be written as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \quad \alpha_i \in F \quad \dots\dots\dots(ii)$$

$$\text{Let } T(x) = \beta_i \in F \quad \text{for } i = 1, 2, \dots, n \quad \dots\dots\dots(iii)$$

$$\text{Let } y = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n \in V$$

We have

$$\langle x, y \rangle = \left\langle \sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^n \beta_j x_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \langle x_i, x_j \rangle = \sum_{i=1}^n \alpha_i \beta_i \quad \text{by eq (i)}$$

$$\therefore \langle x, y \rangle = \sum_{i=1}^n \alpha_i \beta_i \quad \dots\dots\dots(iv)$$

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From eq.(ii)
 $T(x) = T(\alpha_1 x_1 + \dots + \alpha_n x_n) = \alpha_1 T(x_1) + \dots + \alpha_n T(x_n)$
 (since T is a homomorphism)

$$= \alpha_1 \beta_1 + \dots + \alpha_n \beta_n = \sum_{i=1}^n \alpha_i \beta_i \quad \dots (v)$$

From eq.(iv) and (v)

$$T(x) = \langle x, y \rangle \quad \forall x \in V \quad \dots (vi)$$

Now, we show that

$y \in V$ satisfying eq.(vi) is unique.

Let $y_1 \in V$ be such that $T(x) = \langle x, y_1 \rangle \quad \forall x \in V \quad \dots (vii)$

from eq.(vi) and (vii)

$$\langle x, y \rangle = \langle x, y_1 \rangle \quad \forall x \in V \quad \dots (viii)$$

We have $\langle x, y - y_1 \rangle = \langle x, y \rangle - \langle x, y_1 \rangle = 0$ by eq.(viii)

$$\therefore \langle x, y - y_1 \rangle = 0 \quad \forall x \in V$$

In particular, $\langle y - y_1, y - y_1 \rangle = 0$ (Take $x = y - y_1 \in V$)

$$\Rightarrow \langle y - y_1, y - y_1 \rangle = 0 \Rightarrow y = y_1$$

This proves the uniqueness of y .

Q.23. Define eigen values and eigen vectors of a square matrix. Prove that eigen values of a Hermitian matrix are real. (2018)

Sol. Eigen value and eigen vectors of a square matrix: -

An eigen value of square matrix is defined as eigen value of a transformation associated with matrix.

Let A be a square matrix of order $n \times n$ over field F .

A map $T_A: F^n \rightarrow F^n$ defined as

$$T_A(X) = AX,$$

$$\forall AX \in F^n$$

written as column vector i.e. $x =$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

eigen value of A def eigen value of T_A .

Let $\lambda \in F$ be an eigen value of A .

$\Leftrightarrow \lambda$ is an eigen value of T_A .

$\Leftrightarrow \exists X (\neq 0) \in F^n$ such that $T_A(x) = \lambda x$ or $AX = \lambda x$

To Prove - All the eigen value of a Hermitian matrix are real number.

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Proof -

Let A^* be Hermitian matrix over $\mathbb{C}$ then $A^* = A$ (i)
 Let λ be an eigen value of A (Hermitian matrix)

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } Ax = \lambda x$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } (Ax)^* = (\lambda x)^*$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } x^* A^* = \bar{\lambda} x^* \quad \left\{ \begin{array}{l} \because (AB)^* = B^* A^* \\ \& (\lambda A)^* = \bar{\lambda} A^* \end{array} \right.$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } x^* A = \bar{\lambda} x^* \quad \left\{ \because A^* = A \right.$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } x^* Ax = \bar{\lambda} x^* x \quad \left\{ \text{multiply by } x \right.$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } x^* \lambda x = \bar{\lambda} x^* x \quad \left\{ \text{by eq. (ii)} \right.$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } (\lambda - \bar{\lambda})(x^* x) = 0 \quad \because \langle x, y \rangle = y^* x$$

$$\Rightarrow \exists x (\neq 0) \in \mathbb{C}^n \text{ such that } (\lambda - \bar{\lambda}) \langle x, x \rangle = 0 \quad \left(\langle x, x \rangle = x^* x \right)$$

$$\Rightarrow \lambda - \bar{\lambda} = 0 \quad \left\{ \because \langle x, x \rangle = \|x\|^2 \neq 0 \text{ as } x \neq 0 \right.$$

$$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is purely real number.}$$

Q.24. Let F be a field and $A \in M_n(F)$. Define the trace of A . If $B \in M_n(F)$ is similar to A , then prove that A and B have same trace. Is the converse true? Justify your answer. (2019)

Sol. Trace of a Matrix: -

Let A be a square matrix of order n over a field F . The sum of the elements of A lying along the principal diagonal is called the trace of A . i.e. if $A = [a_{ij}]_{n \times n}$. Then

$$\text{tr} A = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

if $A, B \in M_n(F)$

Now we prove that A and B have the same trace.

Suppose that A and B are two similar matrices. Then there exists an invertible matrix C such that

$$B = C^{-1}AC$$

Let $C^{-1}A = D$

$$\text{Then } \text{tr} B = \text{tr}(DC) = \text{tr}(CD) \quad (\because \text{tr}(AB) = \text{tr}(BA))$$

$$= \text{tr}(CC^{-1}A) = \text{tr}(IA) = \text{tr} A$$

Hence similar matrices have the same trace.

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Unit V

Q.1. Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. (2014)

Sol. The characteristic equation of A is
 $|A - xI| = 0$

$$\begin{vmatrix} 1-x & 2 \\ 0 & 1-x \end{vmatrix} = 0 \Rightarrow (1-x)^2 - 0 \times 2 = 0$$

$$\Rightarrow (1-x)^2 = 0$$

$$\Rightarrow x = 1, 1$$

Since roots are equal. Hence A is not diagonalizable.

Q.2. Prove that real quadratic form on a vector space V determines a unique symmetric bilinear form on V given by

$$f_s(x, y) = \frac{1}{2} [q(x+y) - q(x) - q(y)], \quad \forall x, y \in V. \quad (2014)$$

Sol. Let f be a bilinear form associated with quadratic form q on V.

$$q(x) = f(x, x) \quad \forall x \in V \quad \dots\dots\dots(i)$$

$$f = f_s + f_a \quad \dots\dots\dots(ii)$$

From (i) and (ii) we have

$$q(x) = (f_s + f_a)(x, x) \quad \forall x \in V$$

$$= f_s(x, x) + f_a(x, x) \quad \forall x \in V$$

$$q(x) = f_s(x, x) \quad \forall x \in V$$

$$q(x+y) = f_s(x+y, x+y)$$

$$q(x+y) = f_s(x, x) + f_s(x, y) + f_s(y, x) + f_s(y, y)$$

$$\Rightarrow q(x+y) = q(x) + 2f_s(x, y) + q(y)$$

$$f_s(x, y) = \frac{1}{2} [q(x+y) - q(x) - q(y)]$$

Q.3. Reduce the equation $2y^2 + 2zx + 4y + 4 = 0$ to standard form by suitable orthogonal change of variables and identify the conicoid represented by it. (2014, 19)

Sol. Matrix of quadratic form $2y^2 + 2zx$ relative to standard basis of $\mathbb{R}^3$ is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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characteristic equation is $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$$\lambda = 1, 2, -1$$

(i) eigen vectors corresponding to $\lambda = 1$. Let (ℓ, m, n) be an eigen vector corresponding to 1.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$R_3 \rightarrow R_3 + R_1$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$\begin{matrix} -\ell + n = 0 \\ m = 0 \end{matrix} \Rightarrow m = 0, \ell = n$$

$$(1, 0, 1) \rightarrow \text{Unit vector } \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

(ii) Eigen vector corresponding to $\lambda = 2$. Let (ℓ, m, n) be an eigen vector corresponding to $\lambda = 2$.

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$R_3 \rightarrow R_3 + 2R_1 \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow \begin{matrix} \ell - 2n = 0 \\ -3n = 0 \end{matrix} \Rightarrow n = 0$$

$$\text{then } \ell = 0$$

$$(\ell, m, n) = (0, 1, 0)$$

(iii) eigen vectors corresponding to $\lambda = -1$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$R_3 \rightarrow R_3 - R_1 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow \ell + n = 0, \ell = -n$$

$$3m = 0 \Rightarrow m = 0$$

$(-1, 0, 0)$ or $(1, 0, -1)$

Let orthogonal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotating the axis without changing the origin:-

Now axis are x', y', z' where x, y, z and x', y', z' are related by transformation -

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z', y = y', z = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z'$$

from these transformation the given eq.(i) reduced to

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$$

eq.(i) becomes

$$x'^2 + 2y'^2 - z'^2 + 4y' = -4$$

$$x'^2 + 2y'^2 + 4y' - z'^2 = -4$$

$$(x')^2 + 2(y'^2 + 2y') - z'^2 = -4$$

$$(x')^2 + 2(y' + 1)^2 - (z')^2 = -4 + 2$$

$$-1(x')^2 - 2y'^2 + (z')^2 = 2$$

Nature - This is hyperboloid of two sheets.

Here $y = y' + 1$

New transformation x', y, z' .

Final transformation are -

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$$x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z'$$

$$y = y - 1$$

$$z = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z'$$

Q.4. Reduce the quadratic form

$q(x, y, z) = x^2 - 2yz + xz$ on $\mathbb{R}^3(\mathbb{R})$ to conical form $\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$ by congruent reduction and find the transformation relation between (x', y', z') and (x, y, z) . Also find the index and signature of the quadratic form. (2014)

Sol. q on $\mathbb{R}^3$, $q(x, y, z) = x^2 - 2yz + xz$ matrix of q relative to standard basis of $\mathbb{R}^3$ is

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 1/2 & -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 1/2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{1}{2}R_1 \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 0 & -1 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \rightarrow C_3 - \frac{1}{2}C_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1/4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \leftrightarrow C_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 4R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$C_3 \rightarrow C_3 - 4R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} \wedge \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$\therefore$ Normal form of q is

$$x'^2 - \frac{1}{4}y'^2 + 4z'^2$$

when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

Rank=3,

Index=2

Singneture=1

Q.5. Find the matrix of the quadratic form -

$q(x_1, x_2, x_3) = 2x_1x_2 + 4x_1x_3 + 2x_2x_3$ on $\mathbb{R}^3(\mathbb{R})$ relative to the ordered basis $\{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$ of $\mathbb{R}^3(\mathbb{R})$.

Sol. 'Given that

$$q(x_1, x_2, x_3) = 2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

The polar form of the symmetric bilinear form f associated with q is given by

$$f(x, y) = \frac{1}{2}[q(x+y) - q(x) - q(y)]$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$

$$(x+y) = (x_1+y_1, x_2+y_2, x_3+y_3)$$

$$f((x_1, x_2, x_3)(y_1, y_2, y_3)) = \frac{1}{2} [2(x_1+y_1)(x_2+y_2) + 4(x_1+y_1)(x_3+y_3) + 2(x_2+y_2)(x_3+y_3) - (2x_1x_2 + 4x_1x_3 + 2x_2x_3) - (2y_1y_2 + 4y_1y_3 + 2y_2y_3)]$$

$$= x_1y_2 + y_1x_2 + 2(x_1y_3 + y_1x_3) + x_2y_3 + y_2x_3$$

Basis $S = \{e_1 = (1, 1, 0), e_2 = (1, 0, 1), e_3 = (0, 0, 1)\}$

Now

$$f(e_1, e_1) = f((1, 1, 0), (1, 1, 0)) = 2, f(e_2, e_1) = 4, f(e_3, e_1) = 3$$

$$f(e_1, e_2) = f((1, 1, 0), (1, 0, 1)) = 4, f(e_3, e_2) = 4, f(e_2, e_2) = 2$$

$$f(e_1, e_3) = f((1, 1, 0), (0, 0, 1)) = 3, f(e_3, e_3) = 2, f(e_2, e_3) = 0$$

Hence matrix of f relative to the basis S is

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

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Q.6. Reduce the following equation in standard form by suitable orthogonal change of variables

$$4y^2 + 4z^2 + 4yz - 2x - 14y - 22z + 33 = 0$$

and tell the nature of the surface represented by it.

Sol. Given that

$$4y^2 + 4z^2 + 4yz - 2x - 14y - 22z + 33 = 0$$

(2012)

.....(i)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

Characteristic Equation of A is

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 2, 6, 0$$

Eigen vectors corresponding to $\lambda = 2$ is $(0, -1, 1)$

Eigen vectors corresponding to $\lambda = 6$ is $(0, 1, 1)$

Eigen vectors corresponding to $\lambda = 0$ is $(1, 0, 0)$

orthogonal matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Let us rotate the co-ordinate axis to new axis (x, y, z) and (x', y', z') are related by transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$x = z'$$

$$y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$$

$$z = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$$

Using these transformation eq.(i) becomes

$$2x'^2 + 6y'^2 + 0z'^2 + \left(\frac{14}{\sqrt{2}} - \frac{22}{\sqrt{2}}\right)x' + \left(-\frac{14}{\sqrt{2}} - \frac{22}{\sqrt{2}}\right)y' + (-2)z' = -33$$

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$$\text{or } \left(2x'^2 - \frac{8}{\sqrt{2}}x'\right) + \left(6y'^2 - \frac{36}{\sqrt{2}}y'\right) - 2z' = -33$$

$$\text{or } 2\left(x'^2 - \frac{4}{\sqrt{2}}x'\right) + 6\left(y'^2 - \frac{6}{\sqrt{2}}y'\right) - 2z' = -33$$

$$\text{or } 2\left(x' - \frac{2}{\sqrt{2}}\right)^2 + 6\left(y' - \frac{3}{\sqrt{2}}\right)^2 - 2z' = -33 + 4 + 27 = -2$$

$$\text{or } 2X^2 + 6Y^2 = 2Z' - 2$$

$$\text{or } 2X^2 + 6Y^2 = 2(Z' - 1)$$

$$\text{or } 2X^2 + 6Y^2 = 2Z$$

$$\text{or } X^2 + 3Y^2 = Z$$

Elliptical Paraboloid

$$X = x' - \sqrt{2}$$

$$Y = y' - \frac{3}{\sqrt{2}}$$

$$Z = z' - 1$$

$$x = z + 1$$

$$y = -\frac{1}{\sqrt{2}}(x + \sqrt{2}) + \frac{1}{\sqrt{2}}\left(y + \frac{3}{\sqrt{2}}\right)$$

$$z = \frac{1}{\sqrt{2}}(x + \sqrt{2}) + \frac{1}{\sqrt{2}}\left(y + \frac{3}{\sqrt{2}}\right)$$

Q.7. Is matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ diagonalizable? If yes, find a matrix P

such that $P^{-1}AP$ is a diagonal matrix.

(2013)

Sol. Given that $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristics equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

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$$\text{or } \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore \lambda = -3, -3, 5$$

(i) Eigen vectors corresponding to $\lambda = -3$.

Let $X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = -3$.

$$\therefore [\lambda + 3I]X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

On solving using Echlon form we get homogeneous equation having solution $(-2, 1, 0)$ and $(3, 0, 1)$.

(ii) Eigen vectors corresponding to $\lambda = 5$.

Let $X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = 5$. Then

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{aligned} R_3 &\rightarrow R_3 - 7R_1 \\ R_2 &\rightarrow R_2 + 2R_1 \end{aligned} \quad \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{aligned} R_1 &\rightarrow -R_1 \\ R_3 &\rightarrow R_3 + 2R_2 \end{aligned} \quad \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \rightarrow -\frac{R_2}{8} \quad \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

On further solving Echlon form whose equation is equivalent to

$$x_1 + 2x_2 + 5x_3 = 0$$

$$x_2 + 2x_3 = 0$$

We get one linear independent solution $(1, 2, -1)$.

Thus there are three linear independent eigen vectors $(-2, 1, 0)$, $(3, 0, 1)$ and $(1, 2, -1)$. Hence given matrix A is diagonalizable.

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Let $P = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Then $P^{-1} = \frac{1}{8} \begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$

Therefore $P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Q.8. Reduce the quadratic form $q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4zx$ on $\mathbb{R}^3(\mathbb{R})$ to canonical form by $\alpha x'^2 + \beta y'^2 + \gamma z'^2$ by congruent reduction and find the transformation relation between (x, y, z) and (x', y', z') and tell the index and signature of the quadratic form. (2013)

Sol. Given that

$$q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4zx$$

$$q(x, y, z) = x^2 + xy - 2xz + xy + 5y^2 + 0yz - 2zx + 0zy - 4z^2$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 5 & 0 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 5 & 0 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 4 & 2 \\ -2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \rightarrow C_2 - C_1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 2 \\ -2 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$R_3 \rightarrow 2R_1 + R_3$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \rightarrow 2C_1 + C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2.2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \rightarrow C_3 - 2C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^t \quad Q$$

Normal form of q is

$$X'^2 + 2Y'^2 - 6Z'^2$$

Where $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

$$x = x' - y' + 4z'$$

$$y = y' - 2z'$$

$$z = z'$$

Rank = 3, index = 2, signature = 1.

Q.9. Prove that every bilinear form on a real vector space can be uniquely expressed as the sum of a symmetric and skewsymmetric bilinear forms. (2013, 19)

Sol. Let f be a bilinear form on $V(\mathbb{R})$ consider the following bilinear form
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on $V(\mathbb{R})$.

$$f_s(x, y) = \frac{1}{2}[f(x, y) + f(y, x)] \quad \dots\dots\dots(i) \quad \forall x, y \in \mathbb{R}$$

$$\text{and } f_{ss}(x, y) = \frac{1}{2}[f(x, y) - f(y, x)] \quad \dots\dots\dots(ii)$$

Claim: - f_s is symmetric and f_{ss} is skew symmetric.

$$\begin{aligned} \text{Since we have } f_s(y, x) &= \frac{1}{2}[f(y, x) + f(x, y)] = \frac{1}{2}[f(x, y) + f(y, x)] \\ &= f_s(x, y) \end{aligned}$$

$$\text{Therefore } f_s(x, y) = f_s(y, x) \quad \forall x, y \in V$$

 $\Rightarrow f_s$ is symmetric.

$$\begin{aligned} \text{Since } f_{ss}(x, y) &= \frac{1}{2}[f(y, x) - f(x, y)] \\ &= -\frac{1}{2}[f(x, y) - f(y, x)] = -f_{ss}(x, y) \quad \forall x, y \in V \end{aligned}$$

$$\therefore f_{ss}(x, y) = -f_{ss}(x, y) \quad \forall x, y \in V$$

Adding eq.(i) and (ii) we get

$$f(x, y) = f_s(x, y) + f_{ss}(x, y) \quad \dots\dots\dots(iii)$$

$$\text{or } f(x, y) = (f_s + f_{ss})(x, y) \quad \forall x, y \in V$$

Uniqueness:-

Let if possible f have another representation viz

$$f = g_s + g_{ss} \quad \dots\dots\dots(iv)$$

where g_s is symmetric and g_{ss} is skew-symmetric.

$$\begin{aligned} \therefore f(x, y) + f(y, x) &= (g_s + g_{ss})(x, y) + (g_s + g_{ss})(y, x) \\ &= g_s(x, y) + g_{ss}(x, y) + g_s(y, x) + g_{ss}(y, x) \\ &= g_s(x, y) + g_{ss}(x, y) + g_s(x, y) - g_{ss}(x, y) \\ &= 2g_s(x, y) \end{aligned}$$

$$\therefore g_s(x, y) = \frac{1}{2}[f(x, y) + f(y, x)] \quad \forall x, y \in V$$

$$= f_s(x, y) \quad \forall x, y \in V \quad \text{from eq.(i)}$$

$$\Rightarrow g_s = f_s$$

from eq.(iv) and (v) we get

$$f_s + f_{ss} = g_s + g_{ss}$$

$$f_{ss} = g_{ss}$$

Hence representation is unique.

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Q.10. Reduce the equation

$$22x^2 - 12xy + 17y^2 - 112x + 92y + 178 = 0$$

into standard form by suitable orthogonal change of variables and tell the nature of the conic section represented by it. (2013)

Ans. Given that

$$22x^2 - 12xy + 17y^2 - 112x + 92y + 178 = 0$$

$$\text{Hence } A = \begin{bmatrix} 22 & -6 \\ -6 & 17 \end{bmatrix}$$

Its characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 22-\lambda & -6 \\ -6 & 17-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 39\lambda + 338 = 0$$

Solving we get $\lambda = 13, 26$ Eigen vector corresponding to $\lambda = 13$

$$(A - 13I)X = 0$$

$$\begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$9x - 6y = 0 \quad \Rightarrow \quad -6x + 4y = 0$$

Solving we get

Choose $y = k$ then $x = 2k/3$

$$\text{then } X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k/3 \\ k \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\text{or } X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Similarly eigen vector corresponding to $\lambda = 26$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\text{Let } P = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$$

$$\text{Then } x = \frac{2}{\sqrt{13}}x' - \frac{3}{\sqrt{13}}y'$$

$$y = \frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'$$

Therefore eq.(i) becomes

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$$22x'^2 - 12 \left[\frac{2}{\sqrt{13}} x' - \frac{3}{\sqrt{13}} y' \right] \left[\frac{3}{\sqrt{13}} x' + \frac{2}{\sqrt{13}} y' \right] + 17y'^2$$

$$- 112 \left[\frac{2}{\sqrt{13}} x' - \frac{3}{\sqrt{13}} y' \right] + 92 \left[\frac{3}{\sqrt{13}} x' + \frac{2}{\sqrt{13}} y' \right] + 178 = 0$$

Solving we get

$$\frac{X^2}{2} + \frac{Y^2}{1} = 1$$

Where $X = x' + \frac{2}{\sqrt{13}}$, $Y = y' - \frac{14}{\sqrt{13}}$

Nature is ellipse.

Q.11. Prove that an $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigen vectors. (2013)

Ans. Suppose A is diagonalizable. Then A is similar to a diagonal matrix $D = \text{dia}[\lambda_1, \lambda_2, \dots, \lambda_n]$. Therefore there exists an invertible matrix

$$P = [x_1, x_2, \dots, x_n] \text{ such that } P^{-1}AP = D$$

i.e. $AP = PD$

$$A[x_1, x_2, \dots, x_n] = [x_1, x_2, \dots, x_n] \text{dia}[\lambda_1, \lambda_2, \dots, \lambda_n]$$

i.e. $[Ax_1, Ax_2, \dots, Ax_n] = [\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n]$

i.e. $Ax_1 = \lambda_1 x_1, Ax_2 = \lambda_2 x_2, \dots, Ax_n = \lambda_n x_n$

Therefore $x_1, x_2, \dots, x_n$ are eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. Since the matrix P is non-singular, therefore its column vectors $x_1, x_2, \dots, x_n$ are linearly independent. Hence A possesses n linearly independent eigen vectors.

Conversely, suppose that A possesses n linearly independent eigen vectors $x_1, x_2, \dots, x_n$ and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigen values. Then

$$Ax_1 = \lambda_1 x_1, Ax_2 = \lambda_2 x_2, \dots, Ax_n = \lambda_n x_n$$

Let $AP = [x_1, x_2, \dots, x_n] = [Ax_1, Ax_2, \dots, Ax_n]$

$$= [\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n] = [x_1, x_2, \dots, x_n] \text{dia}[\lambda_1, \lambda_2, \dots, \lambda_n]$$

$$= PD$$

Since the column vectors $x_1, x_2, \dots, x_n$ of the matrix P are linearly independent, therefore P is invertible and P^{-1} exists.

Therefore $AP = PD \Rightarrow P^{-1}AP = P^{-1}PD$

$$\Rightarrow P^{-1}AP = D$$

$\Rightarrow A$ is similar to a diagonal matrix D .

$\Rightarrow A$ is diagonalizable.

Q.12. Let f be the bilinear form on $\mathbb{R}^3$ defined by

$$f((x_1, x_2, x_3), (y_1, y_2, y_3)) = 2x_1y_1 - 3x_2y_2 + 4x_3y_3 + x_1y_2 + 2x_2y_1$$

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Find

(i) Symmetric and skew symmetric parts of f ;

(ii) The matrix of f with respect to ordered basis

$$\{(1, 1, 0), (1, -1, 0), (1, 1, 1)\} \text{ of } \mathbb{R}^3.$$

Sol.

Given that

(2017)

$$f\left(\underbrace{(x_1, x_2, x_3)}_x, \underbrace{(y_1, y_2, y_3)}_y\right) = 2x_1y_1 - 3x_2y_2 + 4x_3y_3 + x_1y_2 + 2x_2y_1$$

(i) Symmetric

$$f_s = \frac{1}{2} [f(y, x) + f(x, y)]$$

$$= \frac{1}{2} [2y_1x_1 - 3y_2x_2 + 4y_3x_3 + y_1x_2 + 2y_2x_1$$

$$+ 2y_1x_1 - 3y_2x_2 + 4x_3y_3 + y_2x_1 + 2x_2y_1]$$

$$= \frac{1}{2} [4y_1x_1 - 6y_2x_2 + 8x_3y_3 + 3y_2x_2 + 3x_2y_1]$$

Skew symmetric

$$f_{ss} = \frac{1}{2} [f(y, x) - f(x, y)]$$

$$= \frac{1}{2} [2y_1x_1 - 3y_2x_2 + 4x_3y_3 + y_1x_2 + 2y_2x_1$$

$$- 2y_1x_1 + 3y_2x_2 - 4x_3y_3 - y_2x_1 - 2x_2y_1]$$

$$f_{ss} = \frac{1}{2} [x_1y_2 - x_2y_1]$$

(ii)

$$a_{11} : f((1, 1, 0), (2, 2, 0)) = 2 - 3 + 1 + 2 = 2$$

$$a_{12} : f((1, 1, 0), (1, -1, 0)) = 2 + 3 - 1 + 2 = 6$$

$$a_{13} : f((1, 1, 0), (1, 1, 1)) = 2 - 3 + 1 + 2 = 2$$

$$a_{21} : f((1, 1, 0), (1, 1, 0)) = 2 + 3 + 1 - 2 = 4$$

$$a_{22} : f((2, -1, 0), (1, -1, 0)) = 2 - 3 - 1 - 2 = -4$$

$$a_{23} : f((1, -1, 0), (1, 1, 1)) = 2 + 3 + 1 - 2 = 4$$

$$a_{31} : f((1, -1, 0), (1, 1, 0)) = 2 - 3 + 1 + 2 = 2$$

$$a_{32} : f((1, 1, 1), (1, -1, 0)) = 2 + 3 - 1 + 2 = 6$$

$$a_{33} : f((1, 1, 1), (1, 2, 1)) = 2 - 3 + 4 + 1 + 2 = 6$$

The matrix of f with respect to the ordered basis $\{(1, 1, 0), (1, -1, 0), (1, 1, 1)\}$ is

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$$A = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 4 & 4 \\ 2 & 6 & 2 \end{bmatrix}$$

Q.13. Prove that every quadratic form q on a real vector space V determines a unique symmetric bilinear form f_q on V given by:

$$f_q(x, y) = \frac{1}{2} [q(x+y) - q(x) - q(y)], \quad \forall x, y \in V \quad (2018)$$

Sol. Let f be bilinear form on V such that

$$q(x) = f(x, x), \quad \forall x \in V \quad \text{.....(i)}$$

Also f can be expressed uniquely as

$$f = f_s + f_a \quad \text{.....(ii)}$$

where $f_s(x, y) = \frac{1}{2}(f(x, y) + f(y, x))$

is a symmetrical bilinear form and

$$f_a(x, y) = \frac{1}{2}(f(x, y) - f(y, x))$$

is a skew symmetrical bilinear form.

Hence $q(x) = f(x, x)$

$$= f_s(x, x) + f_a(x, x) \quad \text{from eq.(i)}$$

$$= f_s(x, x) \quad (\because f_a(x, x) = 0) \quad \text{from eq.(ii)}$$

Then q determines the unique symmetrical bilinear form f_s .

Now $q(x+y) = f_s(x+y, x+y)$

$$= f_s(x, x) + f_s(x, y) + f_s(y, x) + f_s(y, y)$$

$$= f_s(x, x) + 2f_s(x, y) + f_s(y, y) \quad (\because f_s(x, y) = f_s(y, x))$$

$$= q(x) + 2f_s(x, y) + q(y)$$

$$\text{Hence } 2f_s(x, y) = q(x+y) - q(x) - q(y)$$

$$f_s(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$$

Q.14. Let V be a vector space of dimension $n \in \mathbb{N}$ over a field F . Let $T: V \rightarrow V$ be a linear transformation. If T has n distinct eigen values, then prove that T is diagonalizable. Is the converse true? Justify your answer. (2019)

Sol. Suppose that T has m distinct eigen values say, $C_1, C_2, \dots, C_m$. Let $V_1, V_2, \dots, V_m$ be the corresponding eigen vectors.

Claim - $S = \{V_1, V_2, \dots, V_m\}$ are linearly independent over F .
By induction on m .

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if $m = 1$ Then $S = \{V_1\}$ is linearly independent. (Since $V_1 \neq 0$)

Suppose that the set $\{V_1, V_2, \dots, V_K\}$ is linearly independent where $K < m$. We shall prove that the set $\{V_1, V_2, \dots, V_K, V_{K+1}\}$ is linearly independent.

$$\text{Let } \alpha_1 V_1 + \dots + \alpha_{K+1} V_{K+1} = 0 \quad \text{where } \alpha_i \in F \quad \text{.....(i)}$$

$$\Rightarrow T(\alpha_1 V_1 + \dots + \alpha_{K+1} V_{K+1}) = T(0) = 0$$

$$\Rightarrow \alpha_1 T(V_1) + \dots + \alpha_{K+1} T(V_{K+1}) = 0 \quad (\because T(V_i) = C_i V_i)$$

$$\alpha_1 (C_1 V_1) + \dots + \alpha_{K+1} (C_{K+1} V_{K+1}) = 0 \quad \text{.....(ii)}$$

Multiplying eq.(i) by the scalar C_{K+1} and subtracting from eq.(ii), we get

$$\alpha_1 (C_1 - C_{K+1}) V_1 + \dots + \alpha_K (C_K - C_{K+1}) V_K = 0$$

$$\Rightarrow \alpha_1 (C_1 - C_{K+1}) = 0, \dots, \alpha_K (C_K - C_{K+1}) = 0$$

Since $\{V_1, V_2, \dots, V_K\}$ is linearly independent.

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_K = 0, \text{ since } C_1, C_2, \dots, C_{K+1} \text{ are all distinct.}$$

Putting these values in eq.(i) we get

$$\alpha_{K+1} V_{K+1} = 0 \quad \Rightarrow \quad \alpha_{K+1} = 0 \quad (\because V_{K+1} \neq 0)$$

Thus $\alpha_1 = 0, \dots, \alpha_K = 0, \alpha_{K+1} = 0$ implies that the set $\{V_1, V_2, \dots, V_K, V_{K+1}\}$ is linearly independent. This complete the induction.

Hence S is linearly independent.

Since $\dim V = m$, $\beta = \{V_1, V_2, \dots, V_m\}$ is an ordered basis of V which consist of eigen vectors of T . Hence T is diagonalizable. Since a linear operator T on a finite dimensional vector space V is diagonalizable if and only if there exists a basis of V consisting of eigen vectors of T .

Q.15. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Prove that there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ such that $P^T A P$ is a diagonal matrix. (2019)

Sol. Let $X_1, X_2, \dots, X_n$ be eigen vectors of A corresponding to its eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively.

$\therefore X_1, X_2, \dots, X_n$ are linearly independent and orthogonal to each other.

Normalizing the vectors

$$\text{Let } Y_1 = \frac{X_1}{\|X_1\|}, Y_2 = \frac{X_2}{\|X_2\|}, \dots, Y_n = \frac{X_n}{\|X_n\|}$$

Clearly $Y_1, Y_2, \dots, Y_n$ are also eigen vectors of A corresponding to eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively.

$$\therefore AY_1 = \lambda_1 Y_1, AY_2 = \lambda_2 Y_2, \dots, AY_n = \lambda_n Y_n \quad \text{.....(i)}$$

Let $P = (Y_1, Y_2, \dots, Y_n)$

clearly P is orthogonal matrix.

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$$P^{-1} = \begin{bmatrix} Y_1^t \\ Y_2^t \\ \vdots \\ Y_n^t \end{bmatrix}_{n \times 1} \quad A = [A]$$

$$P^{-1}A = \begin{bmatrix} Y_1^t A \\ Y_2^t A \\ \vdots \\ Y_n^t A \end{bmatrix} = \begin{bmatrix} Y_1^t A^t \\ Y_2^t A^t \\ \vdots \\ Y_n^t A^t \end{bmatrix}$$

($\because A$ is symmetric i.e. $A^t = A$)

$$= \begin{bmatrix} (AY_1)^t \\ (AY_2)^t \\ \vdots \\ (AY_n)^t \end{bmatrix} = \begin{bmatrix} (\lambda_1 Y_1)^t \\ (\lambda_2 Y_2)^t \\ \vdots \\ (\lambda_n Y_n)^t \end{bmatrix}$$

from eq.(i)

$$= \begin{bmatrix} \lambda_1 Y_1^t \\ \lambda_2 Y_2^t \\ \vdots \\ \lambda_n Y_n^t \end{bmatrix}$$

($\because (\lambda A)^t = \lambda A^t$)

$$= \begin{bmatrix} \lambda_1 & 0_F & \dots & 0_F \\ 0_F & \lambda_2 & \dots & 0_F \\ \vdots & \vdots & \ddots & \vdots \\ 0_F & 0_F & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} Y_1^t \\ Y_2^t \\ \vdots \\ Y_n^t \end{bmatrix}$$

$$= \text{Diag}[\lambda_1, \lambda_2, \dots, \lambda_n] P^{-1}$$

$$\therefore P^{-1}AP = \text{Diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$$