

WINSOME SERIES

GEOMETRY

(PAPER-1)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो। BSc-1**



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- Q9. If a chord PQ a conic whose eccentricity and the length of semi-latus rectm are e and l respectively. Subtends a right angle at the focus S, prove that $\left(\frac{1}{SP} - \frac{1}{l}\right)^2 + \left(\frac{1}{SQ} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}$. (2017) -11

- Q10. Find the equation of the chord joining two points whose vectorial angles are α and β , on the conic -12

$$\frac{l}{r} = 1 + e \cos (\theta - \gamma) \quad (2017)$$

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$$\tan \frac{\beta}{2} = \frac{1 + 2e \cos \frac{2\alpha}{2} + e^2}{1 - 2e \sin \frac{2\alpha}{2} + e^2} \cot \frac{\alpha}{2} \quad (2019)$$

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 Find also the coordination of the foot & length of the perpendicular and the equation of the plane in which the perpendicular & the given straight line lie. (2016)
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- Q8. Find the image of the point (1, 2, 3) in the plane $x + 2y + 4z =$ - 31
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 between the planes
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- Q11. Using vector technique, find the equation of plane passing - 34
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- Q13. Define the shortest distance between any two line and find the shortest between the lines
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 $\vec{r} \cdot \vec{n}_3 + d_3 = 0 = \vec{r} \cdot \vec{n}_4 + d_4$ (2018) - 36 Q9.
- Q14. Using vector technique obtain the non-symmetrical form of equation of the orthogonal projection of the line
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- Q2. Find the limiting points of the co-axial system of sphere determined by
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- Q3. Find the enveloping Cylinder of the sphere
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 & $x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$ (2016) - 42 Q1.

- Q 8. Find the coordinates of centre & radius of the circle
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- Q 9. Show that the centres of all sections of sphere $x^2 - y^2 + z^2 = a^2$ by the planes through the point (α, β, γ) lie on a sphere. (2017) - 43
- Q 10. Find the equation of tangent line to the circle $3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z = 22$, $3x + 4y + 5z = 26$ at point $(1, 2, 3)$. (2017) - 44
- Q 11. Find the equation of sphere circumscribing the tetrahedron whose faces are-
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 $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (2017) - 46
- Q 12. Show that the equation of the sphere through the four points $(0, 0, 0), (-a, b, c), (a, -b, c), (a, b, -c)$ is
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- Q 13. A points moves so that the sum of the squares of its distance from the six faves of a cube is constant, show that its locus in a sphere. (2018) - 47
- Q 14. Find the equation of sphere having the circle $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 6$ as a great circle determine its centre & radius. (2018) - 48
- Q 15. Find the radical axis of the following spheres -
 $(x - 2)^2 + y^2 + z^2 = 1$
 $x^2 + (y - 3)^2 + z^2 = 6$
 $(x + 1)^2 + (y - 1)^2 + (z - 2)^2 = 6$. (2019) - 49
- Q 16. Prove that the centres of sphere which touches the line $y = mx, z = c$ & $y = -mx, z = c$, lies on the conicoid.
 $mxy + c(1 + m^2)z = 0$. (2015) - 51

Chapter-4

Cones, Cylinders & Generating lines

- Q 1. Find the equation of the cone whose centre is $(1, 2, 3)$ & guiding curve the circle $x^2 + y^2 + z^2 = 4, x + y + z = 1$ (2014) - 51

- Q2. Find the equation of the generating line of the hyperboloid - 53

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$
 which passing through the point (2, 3, -4). (2014, 2016)
- Q3. Find the equation of the cylinder whose generator are paral- - 54
 lel to the line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{3}$ and whose guiding curve is $x + 2y^2 = 1, z = 3$. (2015)
- Q4. Show that there are two systems of generators of a hyperbo- - 55
 loid of one sheet such that any generator of one system intersects any generator of the other system. Find the parametric equation of the hyperboloid of one sheet. (2015, 2017)
- Q5. Find two mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$. Which are also perpendicular to the line - 56

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$
 (2015, 2017, 2019)
- Q6. Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the ellipse - 57

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$
 (2016)
- Q7. Find the equation of the circular cylinder which passes - 58
 through the circle, $x^2 + y^2 + z^2 = 9, x - y + z = 3$ (2016)
- Q8. Find the equation of the cylinder which intersects the curve - 58
 $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$ & whose generators are parallel to the axis of z . (2017)
- Q9. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinates axes in A, B, - 5
 C. Prove that the equation to the cone generated by the lines drawn from O to meet the circle ABC is

$$yz \left[\frac{b}{c} + \frac{c}{b} \right] + zx \left[\frac{c}{a} + \frac{a}{c} \right] + xy \left[\frac{a}{b} + \frac{b}{a} \right] = 0$$
 (2018)
- Q10. Show that the lines drawn through the point (α, β, γ) whose -
 direction numbers l, m, n satisfy the relation $al^2 + bm^2 + cn^2$

= 0, generate the cone.

$$a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = 0 \quad (2018)$$

Q11. Find the equation of the cone formed by rotating the line $2x + 3y = 6, z = 0$ about the y -axis. (2018) – 61

Q12. Find the equation of the right circular cone which passes through the three lines drawn from the origin with direction numbers $1, 2, 2$; $2, 1, -2$; $2, -2, 1$ (2018) – 61

Q13. Find the equation of enveloping cylinder of sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generator parallel to the line $x = y = z$ (2019) – 62

Q14. Show that in general three normal can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2ax$ but if the point lies on the surface. – 63

$$27a(x^2 + y^2) + 8(a - z)^2 = 0 \text{ two of them coincides.} \quad (2019)$$

Chapter-5

Central Conicoids & Paraboloids

Q1. Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid to the tangent planes is $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$. (2014, 2017) – 64

Q2. Prove that the sum of squares of the projections of three conjugate semi-diameter of an ellipsoid on a given plane is constant. (2014, 2015) – 64

Q3. Show the five normals in general, can be drawn to a paraboloid from a given point. (2014, 2017) – 64

Q4. Find the locus of a luminous point $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if the ellipsoid, casts a shadow on the plane $z = 0$. (2015, 2017) – 65

Q5. Prove that the plane $3x + 4y = 1$ is a diametral plane of paraboloid, $5x^2 + 6y^2 = 2z$. Find the equation to the chord through the point $(3, 4, 5)$, which bisects it. (2015) – 66

Q6. Find the equation to the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which passes through the line $3x - 3y + 6z - 5 = 0$ $x + 9y - 3z = 0$ (2016) – 67

Q7. Find the equation of polar plane of a point with respect to the paraboloid $ax^2 + by^2 = 2z$. (2016) – 67

Q8. Prove that the length of normal chord through $P(\alpha, \beta, \gamma)$ of – 68

the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is equal to $4PG_3$. (2016)

- Q9. Find the equations to the two planes which contain the line given by $7x + 10y - 30 = 0$ & $5y - 3z = 0$ & touches the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$. - 70

- Q10. Show that the normal form (α, β, γ) to the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ lie on the cone - 72

$$\frac{\alpha}{x - \alpha} - \frac{\beta}{y - \beta} + \frac{a^2 - b^2}{z - \gamma} = 0. \quad (2018)$$

- Q11. Show that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid $x^2 - 2y^2 = 3z$ & find the point of contact. (2018) - 72

- Q12. Find the locus of straight line drawn through a fixed point (f, g, h) whose polar lines with respect to the quadrics, $ax^2 + by^2 + cz^2 = 1$ & - 73

$$ax^2 + by^2 + \gamma z^2 = 1 \text{ are coplanar.} \quad (2018)$$

- Q13. Show that the locus of the poles of tangent planes of $ax^2 + by^2 + cz^2 = 1$ with respect to $ax^2 + by^2 + \gamma z^2 = 1$ is the surface - 74

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1 \quad (2019)$$

- Q14. A cone envelops the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and is cut by plane $z = 0$ in a parabola. Show that its vertex lie on the plane $z = \pm c$. (2019) - 75

- Q15. Prove that the centre of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. - 76

by the plane through the three extremities of the different axes is the centroid of the triangle formed by those extremities. (2019)

Q. 1. Find the equation of the cone with vertex (ρ, β) & axis (θ, ϕ) .

Ans. Let the cone be $(\rho, \beta) & r \cos(\theta - \phi)$

The cone is $e = \frac{r \cos(\theta - \phi)}{r}$

$$e = \sqrt{\frac{r^2 \cos^2(\theta - \phi)}{r^2}}$$

This may be written as $\{1 - e^2 \cos^2(\theta - \phi)\}$
 $\rho \cos(\theta - \phi) = e r$
 Therefore $\{1 - e^2 \cos^2(\theta - \phi)\} = 1$
 Let L be

of semi-latus rectum L from the directrix

$$l = e \frac{r}{\cos(\theta - \phi)} = 2(p - r \cos(\theta - \phi))$$

$$p = \frac{l}{1 - e \cos(\theta - \phi)}$$

Q 2. Find the equation of the cone with vertex (ρ, β) & axis (θ, ϕ) .

Ans. Let the cone be $(\rho, \beta) & r \cos(\theta - \phi)$
 then

Chapter - 1

Polar Coordinates

Q. 1. Find the equation of the conic whose eccentricity, focus and the equation of corresponding directrix are respectively e , (ρ, β) & $r \cos(\theta - \alpha) = p$ and also find to latus rectum. (2014)

Ans. Let the eccentricity, focus & equation of corresponding directrix be e , (ρ, β) & $r \cos(\theta - \alpha) = p$. Let (r_1, θ_1) be any conic.

$$e = \frac{\text{The distance of the point } (r_1, \theta_1) \text{ from the focus } (\rho, \beta)}{\text{The distance of the point } (r_1, \theta_1) \text{ from the directrix}}$$

$$e = \frac{\sqrt{r_1^2 + \rho^2 - 2r_1\rho \cos(\theta - \beta)}}{|\rho - r_1 \cos(\theta_1 - \alpha)|}$$

This may be written as

$$\{1 - e^2 \cos^2(\theta_1 - \alpha)\} r_1^2 + 2 \{e^2 p \cos(\theta_1 - \alpha) - \rho \cos(\theta_1 - \beta)\} r_1 + \rho^2 - e^2 p^2 = 0$$

Therefore the equation of conic is -

$$\{1 - e^2 \cos^2(\theta - \alpha)\} r^2 + 2 \{e^2 p \cos(\theta - \alpha) - \rho \cos(\theta - \beta)\} r + \rho^2 - e^2 p^2 = 0$$

Let L be one of the extremities of the latus rectum and l be the length of semi-latus rectum. Then the distance of L from the focus = e , the distance of L from the directrix. i.e.,

$$l = e \text{ the distance of the focus from the directrix.}$$

$$= 2(p - \rho \cos(\beta - \alpha)).$$

$$p = \frac{l}{e} + \rho \cos(\beta - \alpha)$$

Q 2. Find the equation of the chord joining points ' α ' & ' β ' on the

conic $\frac{l}{r} = 1 + e \cos(\theta - r).$ (2014)

Ans. Let the coordinates of the extremities of the chord be (r_1, α) and (r_2, β) , then

$$\frac{1}{r_1} = 1 + e \cos \alpha \quad \dots(1)$$

and $\frac{l}{r} = 1 + e \cos \beta$... (2) get

If the equation of the chord is

$$\frac{l}{r} = A \cos \theta + B \sin \theta$$

Then $\frac{l}{r} = A \cos \alpha + B \sin \alpha$ (3)

and $\frac{l}{r} = A \cos \beta + B \sin \beta$ (4)

from (1) and (3), we have

$$A \cos \alpha + B \sin \alpha = 1 + e \cos \alpha$$

or $(A - e) \cos \alpha + B \sin \alpha = 1$

and (2) and (4), we have

$$A \cos \beta + B \sin \beta = 1 + e \cos \beta$$

or $(A - e) A \cos \beta + B \sin \beta$

Multiplying (5) by $\sin \beta$ and (6) by $\sin \alpha$, and subtracting, we get

$$(A - e) (\cos \alpha \sin \beta - \cos \beta - \sin \alpha) = \sin \beta - \sin \alpha$$

or $A - e = \frac{\sin \beta - \sin \alpha}{\sin(\beta - \alpha)}$

$$= \frac{2 \cos \frac{\beta + \alpha}{2} \sin \frac{\beta - \alpha}{2}}{2 \sin \frac{\beta - \alpha}{2} \cos \frac{\beta - \alpha}{2}}$$

$$= \frac{\cos \frac{\beta + \alpha}{2}}{\cos \frac{\beta - \alpha}{2}}$$

$$\therefore A = \cos \left(\frac{\beta + \alpha}{2} \right) \sec \left(\frac{\beta - \alpha}{2} \right) + e$$

Again, multiplying (5) by $\cos \beta$ and (6) by $\cos \alpha$, and subtracting we

$$B (\sin \alpha \cos \beta - \sin \beta \cos \alpha) = \cos \beta - \cos \alpha$$

or $B = \frac{\cos \beta - \cos \alpha}{\sin(\alpha - \beta)} = \sin \left(\frac{\beta + \alpha}{2} \right) \sec \left(\frac{\beta - \alpha}{2} \right)$

putting the values of A and B in the equation

$$\frac{l}{r} = A \cos \theta + B \sin \theta, \text{ we get}$$

$$\frac{l}{r} = \left[\cos \left(\frac{\beta + \alpha}{2} \right) \sec \left(\frac{\beta + \alpha}{2} + e \right) \right] \cos \theta + \left[\sin \left(\frac{\beta + \alpha}{2} \right) \sec \left(\frac{\beta + \alpha}{2} \right) \right] \sin \theta$$

... (5) or $\frac{l}{r} = e \cos \theta + \sec \left(\frac{\beta - \alpha}{2} \right) \left[\cos \left(\frac{\beta + \alpha}{2} \right) \cos \theta + \sin \left(\frac{\beta + \alpha}{2} \right) \sin \theta \right]$

... (6) or $\frac{l}{r} = e \cos \theta + \sec \left(\frac{\beta - \alpha}{2} \right) \cos \left(\frac{\beta + \alpha}{2} - \theta \right)$

Which is the required equation of the chord.

Q 3. Find the pole of line $\frac{l}{r} = a \cos \theta + b \sin \theta$ with respect to the

$$\text{conic } \frac{l}{r} = 1 + e \cos(\theta - r)$$

(2014)

Ans. Let the pole of the given line with respect to the above conic be (r_1, θ_1) . The polar of (r_1, θ_1) with respect to the conic is given by -

$$\left[\frac{l}{r} - e \cos(\theta - \gamma) \right] \left[\frac{l}{r_1} - e \cos(\theta_1 - \gamma) \right] = \cos(\theta - \theta_1)$$

which can be written as

$$\frac{l}{r} = (e \cos \gamma + q \cos \theta_1) \cos \theta + (e \sin \gamma + q \sin \theta_1) \sin \theta$$

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$$\text{where } q = \begin{cases} l \\ l_1 - e \cos(\theta_1 - \gamma) \end{cases}$$

Since, the straight line (1) & (3) are the polar of the same point. we have,

$$a = e \cos \gamma + q \cos \theta_1 \quad \dots(4)$$

$$b = e \sin \gamma + q \sin \theta_1 \quad \dots(5)$$

From (4) & (5), we get

$$\tan \theta_1 = \frac{b - e \sin \gamma}{a - e \cos \gamma}$$

$$q = \sqrt{(a - e \cos \gamma)^2 + (b - e \sin \gamma)^2}$$

$$q = \sqrt{a^2 + b^2 - 2e(a \cos \gamma + b \sin \gamma) + e^2}$$

$$\text{But } q = \begin{cases} l \\ l_1 - e \cos(\theta_1 - \gamma) \end{cases}$$

$$\text{Hence, } \frac{l}{\eta} = e \cos(\theta_1 - \gamma) + \frac{1}{q}$$

$$\frac{l}{\eta} = e(\cos \theta_1 \cos \gamma + \sin \theta_1 \sin \gamma) + \frac{1}{q}$$

$$\frac{l}{\eta} = \frac{e(a \cos \gamma + b \sin \gamma) - e^2 + 1}{q}$$

$$\text{for } \cos \theta_1 = \frac{a - e \cos \gamma}{q}$$

$$\sin \theta_1 = \frac{b - e \sin \gamma}{q}$$

Therefore,

$$\eta = \frac{lq}{e(a \cos \gamma + b \sin \gamma) - (e^2 - 1)}$$

$$r_2 = \frac{l\sqrt{a^2 + b^2 + e^2 - 2e(a \cos \gamma + b \sin \gamma)}}{e(a \cos \gamma + b \sin \gamma) - (e^2 - 1)}$$

Thus, the pole of the line (1) with respect of conic (2) is -

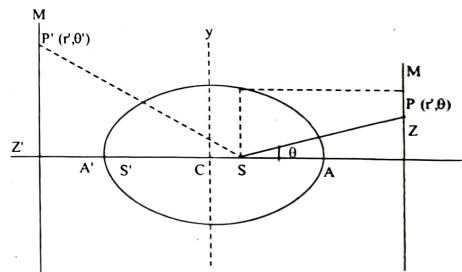
$$\frac{l\sqrt{a^2 + b^2 + e^2 - 2e(a \cos \gamma + b \sin \gamma)}}{e(a \cos \gamma + b \sin \gamma) - (e^2 - 1)}$$

$$\tan^{-1} = \frac{b - e \sin \gamma}{a - e \cos \gamma}$$

Q 4. Find the equation of the directrices of the ellipse

$$\frac{l}{r} = 1 + e \cos(\theta - \gamma). \quad (2015)$$

Ans. Let $\{P(r, \theta)\}$ be the point on the directrix ZM, and $P'(r', \theta')$ be any point on the directrix ZM' and SZ' the perpendicular from S, then $SP' = r'$, $\angle P'SZ' = \theta'$, then $\angle P'SZ' = \pi - \theta'$.



$$\therefore SZ' = r' \{\cos(\pi - \theta')\}$$

$$\text{But } SZ' = SC + CZ'$$

$$= ae + \frac{a}{e}$$

where a is the length of semimajor axis

$$= \frac{a(e^2 + 1)}{e}$$

$$\text{Also } l = \frac{b^2}{a} = \frac{a^2(1 - e^2)}{a} = a[1 - e^2]$$

$$\text{Hence } -r' = \cos \theta \cdot \frac{l(e^2 + 1)}{e(1 - e^2)}$$

$$\text{or } \frac{1}{r'} = \frac{-e(1 - e^2)}{(1 + e^2)} \cos \theta$$

This relation holds good for every point $P'(r', \theta')$ on the directrix $Z'M'$, Hence its equation is

$$\frac{1}{r} = \left(\frac{1 - e^2}{1 + e^2} \right) e \cos \theta$$

Q 5. Find the equation of the director circle of the circle.

$$\frac{l}{r} = 1 + e \cos \theta, e \neq 1$$

(2015)

Ans. The tangents at the point ' α ' & ' β ' of the conic $\frac{l}{r} = 1 + e \cos \theta$ are

$$\frac{l}{r} = e \cos \theta + \cos(\theta - \alpha) \quad \dots(1)$$

$$\& \quad \frac{l}{r} = e \cos \theta + \cos(\theta - \beta) \quad \dots(2)$$

Slope of (1) & (2) are m_1 & m_2 .

$$m_1 = \frac{-e + \cos \alpha}{\sin \alpha}, m_2 = \frac{(e + \cos \beta)}{\sin \beta}$$

So, if tangents will be perpendicular

$$m_1 \times m_2 = -1$$

$$\left[-\frac{e + \cos \alpha}{\sin \alpha} \right] \left[-\frac{e + \cos \beta}{\sin \beta} \right] = -1$$

$$e^2 + e(\cos \alpha + \cos \beta) + \cos(\alpha - \beta) = 0$$

$$e^2 + 2e \cos \left(\frac{\alpha + \beta}{2} \right) \cdot \cos \left(\frac{\alpha - \beta}{2} \right) + 2 \cos^2 \left(\frac{\alpha - \beta}{2} \right) - 1 = 0 \dots(3)$$

Again for point of intersection, we have to solve (1) & (2)

$$\cos(\theta - \alpha) = \cos(\theta - \beta)$$

$$\theta = \frac{\alpha + \beta}{2}$$

$$\text{Putting } \theta = \frac{\alpha + \beta}{2} \text{ in (1)}$$

$$\frac{l}{r} = e \cos \left(\frac{\alpha + \beta}{2} \right) + \cos \left[\left(\frac{\alpha + \beta}{2} \right) - \alpha \right]$$

$$\frac{l}{r} = e \cos \left(\frac{\alpha + \beta}{2} \right) + \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\therefore \cos \frac{\alpha - \beta}{2} = \frac{l}{r} - e \cos \theta$$

$$\text{Putting value of } \theta \text{ & } \cos \frac{\alpha - \beta}{2} \text{ in (3)}$$

$$e^2 + 2e \left(\frac{l}{r} - e \cos \theta \right) \cos \theta + 2 \left(\frac{l}{r} - e \cos \theta \right)^2 - 1 = 0$$

$$e^2 r^2 - 2e \cos \theta \cdot l + 2l^2 - r^2 = 0$$

$$r^2(1 - e^2) + 2el \cos \theta - 2l^2 = 0$$

This is the required equation of director circle.

Q. 6. Define the polar of a point with respect to a conic. Obtain the equation of polar of the point (r, θ) with respect to conic

$$\frac{l}{r} = 1 + e \cos \theta.$$

(2015, 2018)

Ans. The polar of a point (r_1, θ_1) w.r.t a conic of the chord of contact of the tangent drawn to the conic through the point (r, θ)

Let RP & RQ be the two tangents from the points $R(r, \theta)$ to the conic

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots(1)$$

Join PQ, PQ is chord obtained from joining the point of contact.

Equation of chord PQ is given by -

$$\frac{l}{r} = e \cos \theta + \sec \frac{\beta - \alpha}{2} \cos \left[\frac{\alpha + \beta}{2} - e \right] \quad \dots(2)$$

Now, the equation of tangent RP & RQ-

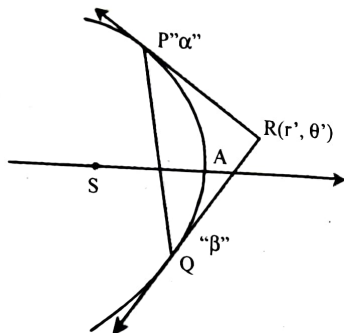
$$\frac{l}{r} = e \cos \theta + \cos(\theta - \alpha) \quad \dots(3)$$

$$\frac{l}{r} = e \cos \theta + \cos(\theta - \beta) \quad \dots(4)$$

As these tangents passes through (r_1, θ_1)

$$\frac{l}{r_1} = e \cos \theta_1 + \cos(\theta_1 - \alpha) \quad \dots(5)$$

$$\frac{l}{r_1} = e \cos \theta_1 + \cos(\theta_1 - \beta) \quad \dots(6)$$



Subtracting (6) from (5).

$$\cos(\theta_1 - \alpha) = \cos(\theta_1 - \beta)$$

$$\theta_1 - \alpha = \theta_1 - \beta.$$

$$\alpha = \beta \quad \dots(7)$$

$$\& \theta_1 = \frac{\alpha + \beta}{2} \quad \dots(8)$$

Putting values of θ_1 from (8) in (5)

$$\frac{l}{r_1} = e \cos \theta_1 + \cos \left[\frac{\alpha + \beta}{2} - \alpha \right]$$

$$\frac{l}{r_1} - e \cos \theta_1 = \cos \frac{\beta - \alpha}{2} \quad \dots(9)$$

Putting $\theta_1 = \frac{\alpha + \beta}{2}$ & value of $\cos = \frac{\beta - \alpha}{2}$ from (9) in (2).

$$\left(\frac{l}{r} - e \cos \theta \right) \left(\frac{l}{r_1} - e \cos \theta_1 \right) = \cos(\theta - \theta_1)$$

$$\frac{l}{r} = e \cos \theta + \left[\frac{l}{r_1} - e \cos \theta_1 \right] \cos(\theta - \theta_1)$$

Which is the required eqⁿ of polar or chord of contact.

Q. 7. If the line $\frac{l}{r} = A \cos \theta + B \sin \theta$ touches the conic

$$\frac{l}{r} = 1 + e \cos \theta. \text{ Then prove that } (A - e)^2 + B^2 = 1. \quad (2016)$$

Ans. Suppose the given line is a tangent to the conic at α . The equation of the tangent at α is -

$$\frac{l}{r} = e \cos \theta + \cos(\theta - \alpha)$$

$$\frac{l}{r} = (e + \cos \alpha) \cos \theta + \sin \alpha \sin \theta$$

The equation of the given line is

$$\frac{l}{r} = A \cos \theta + B \sin \theta$$

Since two equations represents the same line, we have

$$A = e + \cos \alpha \text{ \& } B = \sin \alpha$$

$$A - e = \cos \alpha, B = \sin \alpha.$$

Squaring & adding, we get -

$$(A - e)^2 + B^2 = 1.$$

Q. 8. Find the locus of the pole of a chord of the conic $\frac{l}{r} = 1 + e \cos \theta$, which sustends a constant angle 2β at the focus of conic. (2016)

Ans. Let the pole of the chord joining the point $\alpha + \beta$ & $\alpha - \beta$ on the conic be (r_1, θ_1) .

Equation of chord -

$$\frac{l}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha)$$

$$\frac{l}{r} = (e + \sec \beta \cos \alpha) \cos \theta + \sec \beta \sin \alpha \sin \theta. \quad \dots(1)$$

The equation of polar of the point (r_1, θ_1)

$$\left(\frac{l}{r} - e \cos \theta\right) \left(\frac{l}{r_1} - e \cos \theta_1\right) = \cos(\theta - \theta_1)$$

$$\frac{l}{r} = (e + q \cos \theta_1) \cos \theta + q \sin \theta_1 \sin \theta. \quad \dots(2)$$

where $\frac{l}{r_1} = e \cos \theta_1 = \frac{1}{q}$.

Since, the lines (1) & (2) are polar of same point (r_1, θ_1) , we have
 $e + q \cos \theta_1 = e + \sin \beta \cos \alpha \Rightarrow q \cos \theta_1 = \sec \beta \cos \alpha$
 $\& q \sin \theta_1 = \sec \beta \sin \alpha \Rightarrow q \sin \theta_1 = \sec \beta \sin \alpha$

Squaring & adding these we get,

$$q = \sec \beta.$$

$$\left(\frac{l}{r_1} - e \cos \theta_1\right) = \frac{1}{\sec \beta}$$

$$\frac{l \sec \beta}{r_1} = 1 + e \sec \beta \cos \theta_1$$

Therefore, the locus of point (r_1, θ_1) is the conic

$$\frac{l \sec \beta}{r} = 1 + e \sec \beta \cos \theta$$

Q 9. If a chord PQ of a conic whose eccentricity and the length of semi-latus rectum are e and l respectively. Subtends a right angle at the focus S, prove that (2017)

$$\left(\frac{1}{SP} - \frac{1}{l}\right)^2 + \left(\frac{1}{SQ} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}.$$

Ans. Let the equations of the conic be

$$\frac{l}{r} = 1 + e \cos \theta$$

and the vectorial angle of P be α .

Since the chord PQ subtends a right angle at the focus, the vectorial

angle of Q is $\frac{\pi}{2} + \alpha$. Therefore

$$\frac{l}{SP} = 1 + e \cos \alpha \text{ \& } \frac{l}{SQ} = 1 + e \cos \theta \left(\frac{\pi}{2} + \alpha\right)$$

$$\frac{1}{SP} - \frac{1}{l} = \frac{e}{l} \cos \alpha \quad \dots(1)$$

and $\frac{l}{SQ} - \frac{1}{l} = \frac{-e \sin \alpha}{l}$

Squaring & adding the corresponding sides of these conditions, we

get-

$$\left(\frac{1}{SP} - \frac{1}{l}\right)^2 = \left(\frac{e}{l} \cos \alpha\right)^2$$

$$\left(\frac{1}{SP} - \frac{1}{l}\right)^2 = \left(\frac{-e}{l} \sin \alpha\right)^2$$

$$\Rightarrow \left(\frac{1}{SP} - \frac{1}{l}\right)^2 + \left(\frac{1}{SQ} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}$$

Q 10. Find the equation of the chord joining two points whose vectorial angles are α and β , on the conic

$$\frac{l}{r} = 1 + e \cos(\theta - \gamma) \quad (2017)$$

Ans. The polar coordinates of P & Q are

$$\left[\frac{l}{1 + e \cos(\alpha - \gamma)}, \alpha\right] \text{ \& \& } \left[\frac{l}{1 + e \cos(\beta - \gamma)}, \beta\right]$$

respectively. Let the equation of the chord joining the points P & Q

be

$$\frac{C}{r} = a \cos \theta + b \sin \theta, \text{ i.e.,}$$

$$\frac{l}{r} = A \cos \theta + B \sin \theta$$

...(1)

$$\text{where } A = \frac{al}{C} \text{ \& \& } B = \frac{bl}{C}$$

Since, point P & Q lie on the chord (1), we have

$$1 + e \cos(\alpha - \gamma) = A \cos \alpha + B \sin \alpha$$

...(2)

$$\& 1 + e \cos(\beta - \gamma) = A \cos \beta + B \sin \beta$$

...(3)

Solving (2) & (3), we get

$$A = e \cos \gamma + \frac{\sin \beta - \sin \alpha}{\sin(\beta - \alpha)}$$

$$A = e \cos \gamma + \sec \left[\frac{\beta - \alpha}{2} \right] \cdot \cos \left[\frac{\alpha + \beta}{2} \right] \quad \dots(4)$$

$$\& B = e \sin \gamma + \sec \left[\frac{\beta - \alpha}{2} \right] \sin \left[\frac{\alpha + \beta}{2} \right] \quad \dots(5)$$

Using (4) & (5) in equation (1), we have

$$\frac{l}{r} = e (\cos \theta \cos \gamma + \sin \theta \sin \gamma)$$

$$+ \sec \left(\frac{\beta - \alpha}{2} \right) \left[\cos \theta \cos \frac{\alpha + \beta}{2} \sin \theta \sin \frac{\alpha + \beta}{2} \right]$$

$$\frac{l}{r} = e \cos(\theta - \gamma) + \sec \left(\frac{\beta - \alpha}{2} \right) \cos \left[\theta - \frac{\alpha + \beta}{2} \right]$$

Q 11. Find the equation of the polar of the point (r_1, θ_1) with respect to the conic $\frac{l}{r} = 1 + e \cos(\theta - \gamma)$. (2017)

Ans. Let the equation of the conic be

$$\frac{l}{r} = 1 + e \cos(\theta - \gamma) \quad \dots(1)$$

& P (r_1, θ_1) be the given point. Suppose a line through P meets the conic at the points B & C, and the tangents at B & C meet at the point Q (ρ, α) . Then the chord BC is the chord of contact of the point Q with respect to the conic. Therefore, its equation is

$$\left[\frac{l}{r} - e \cos(\theta - \gamma) \right] \left[\frac{l}{\rho} - e \cos(\alpha - \gamma) \right] = \cos(\theta - \alpha)$$

Since, this line passes through the point P (r_1, θ_1) , we have

$$\left[\frac{l}{r_1} - e \cos(\theta_1 - \gamma) \right] \left[\frac{l}{\rho} - e \cos(\alpha - \gamma) \right] = \cos(\theta_1 - \alpha)$$

WINSOME SERIES

GEOMETRY (PAPER-1)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो।**



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Hence the locus of Q (ρ, α) is

$$\left[\frac{l}{r} - e \cos(\theta - \gamma) \right] \left[\frac{l}{r_1} - e \cos(\theta_1 - \gamma) \right] = \cos(\theta - \theta_1)$$

This is the equation of the polar of the point (r_1, θ_1) with respect to the conic (1)

This equation may be written as-

$$\frac{l}{r} = e \cos(\theta - \gamma) + q \cos(\theta - \theta_1) \quad \dots(2)$$

Where $q = \frac{l}{\left\{ \frac{l}{r_1} - e \cos(\theta_1 - \gamma) \right\}}$

The equation (2) may be written as

$$\frac{l}{r} (e \cos \gamma + q \cos \theta_1) \cos \theta + (e \sin \gamma + q \sin \theta_1) \sin \theta$$

or $\frac{l}{r} = a \cos \theta + b \sin \theta$

where $a = e \cos \gamma + q \cos \theta_1$

& $b = e \sin \gamma + q \sin \theta_1$

This shows that the polar of a point with respect to a conic is a straight line.

Q 12. If the normal at one of the extremities of the latus rectum of the conic $\frac{l}{r} = 1 + e \cos \theta$ meets the curve again at P, Show

that $SP = \frac{l(1 + 3e^2 + e^4)}{(1 + e^2 + e^4)}$, where S is the focus of the conic.

(2017)

Ans. The equation of conic is -

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots(1) \text{ \& } SP = r$$

The coordinate of one extremity P is $\left[l, \frac{\pi}{2} \right]$

The equation of the normal at $(l, \pi/2)$ is

$$\frac{l}{r} \left[\frac{e \sin \pi/2}{\left(1 + e \cos \frac{\pi}{2} \right)} \right] = e \sin \theta + \sin \left[\theta - \frac{\pi}{2} \right]$$

$$\frac{le}{r} = e \sin \theta - \cos \theta$$

$$\frac{le}{r} + \cos \theta = e \sin \theta \quad \dots(2)$$

from (1), we have,

$$\cos \theta = \frac{1}{e} \left[\frac{l}{r} - 1 \right] = \left[\frac{l-r}{r} \right] \frac{1}{e}$$

Putting the value of $\cos \theta$ in (2), we get

$$\frac{le}{r} + \left[\frac{l-r}{r} \right] \cdot \frac{1}{e} = e \sqrt{1 - \left(\frac{l-r}{r} \right)^2} \cdot \frac{1}{e^2}$$

$$le^2 + (l-r) = e \sqrt{e^2 r^2 - (l-r)^2}$$

Squaring both sides, we get

$$Pe^4 + 2le^2(l-r) + (l-r)^2 = e^4 r^2 - e^2(l-r)^2$$

$$(l-r) [(1+e^2)(l-r) + 2le^2 + e^4(l+r)] = 0$$

$$(1+e^2)(l-r) + 2le^2 + e^4[l+r] = 0$$

$$r[1+e^2-e^4] = l + le^2 + 2le^2 + e^4 l$$

$$r = \frac{l(1 + 3e^2 + e^4)}{(1 + e^2 - e^4)}$$

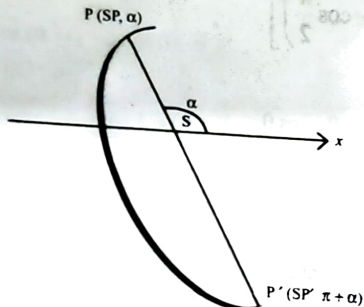
$$SP = r = \frac{l(1 + 3e^2 + e^4)}{(1 + e^2 - e^4)}$$

- Q 13. Prove that the sum of the reciprocals of the segments of any focal chord of a conic is constant. (2018).

Ans. Let the equation of the conic be

$$\frac{l}{r} = 1 + e \cos \theta$$

Let PSP' be any focal chord at an angle α to the axis



$$\therefore \frac{l}{SP} = 1 + e \cos \alpha \quad (\because r = SP, \theta = \alpha) \quad \dots(1)$$

$$\& \frac{l}{SP'} = 1 + e \cos (\pi + \alpha)$$

$$\text{or } \frac{l}{SP'} = 1 - e \cos \alpha \quad \dots(2)$$

Adding (1) & (2), we get

$$\frac{l}{SP} + \frac{l}{SP'} = 2$$

$$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$$

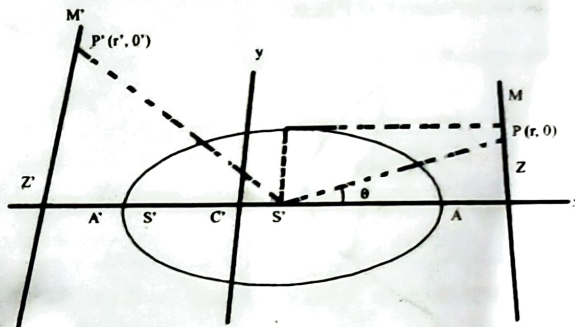
which is a constant as l is constant.

- Q 14. Find the equation of directrix of a conic section. (2018)

Ans. Let $P(r, \theta)$ be the point on the directrix ZM. & $P'(r', \theta')$ be any point on the directrix Z'M' & SZ' the perpendicular from S, then

$$SP' = r' \quad \angle P' SZ' = \theta'$$

$$\text{then } \angle P' SZ' = \pi - \theta'.$$



$$\therefore SZ' = r' [\cos (\pi - \theta')]$$

$$\text{But } SZ' = SC + CZ' = ae + \frac{a}{e}$$

$$\text{where } a \text{ is the length of semimajor axis} = \frac{a(e^2 + 1)}{e}$$

$$\text{Also, } l = \frac{b^2}{a} = \frac{a^2(1 - e^2)}{a} = a(1 - e^2)$$

$$\text{Here, } \frac{l}{r'} = \frac{-e(1 - e^2)}{(1 + e^2)} \cos \theta'$$

This relation hold good for every point $P'(r', \theta')$ on the directrix Z'M'. Hence its equation is -

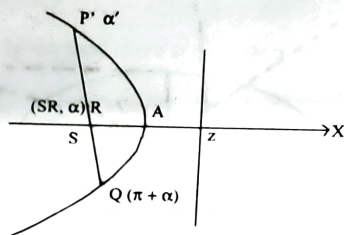
$$\frac{l}{r} = - \left[\frac{1 - e^2}{1 + e^2} \right] e \cos \theta$$

- Q 15. Prove that the locus of the middle point of focal chord of a conic section is a conic section. (2018)

Ans. Let the given conic be

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots(1)$$

Let PSQ be the focal chord of the conic (1) and R be the mid point of PSQ. Let the vectorial angle of P be α .



$\therefore$ Vectorial angle of Q is $(\pi + \alpha)$ and that of R is ' α '

$\therefore$ Polar coordinates P and Q are (SP, α) and $(SQ, \pi + \alpha)$ respectively.

Let the coordinates of R be (SR, α)

$\therefore$ P and Q lie on (1),

$$\therefore \frac{l}{SP} = 1 + e \cos \alpha \quad \dots(2)$$

$$\text{and } \frac{l}{SQ} = 1 - e \cos \alpha \quad \dots(3)$$

$$\text{Now } SR = SP - PR = SP - \frac{1}{2}(SP + SQ) = \frac{SP - SQ}{2}$$

$$\text{or } 2SR = SP - SQ$$

$$= \frac{l}{1 + e \cos \alpha} - \frac{l}{1 - e \cos \alpha} \quad [\text{using (2) and (3)}]$$

$$= \frac{-2el \cos \alpha}{1 - e^2 \cos^2 \alpha} \quad \text{or} \quad SR = \frac{-el \cos \alpha}{1 - e^2 \cos^2 \alpha}$$

$$\text{Hence locus of R (SR, } \alpha) \text{ is } r = \frac{-el \cos \theta}{1 - e^2 \cos^2 \theta}$$

$$\text{or } r(1 - e^2 \cos^2 \theta) = -el \cos \theta$$

$$\text{or } r^2(1 - e^2 \cos^2 \theta) + elr \cos \theta = 0$$

Changing the above equation in Cartesian form, we get

$$x^2 + y^2 - e^2 x^2 + elx = 0$$

$$\text{or } (1 - e^2)x^2 + y^2 + elx = 0$$

This is second degree equation in x and y, so it represents a conic section.

It will represent an ellipse, a parabola or a hyperbola according as h^2

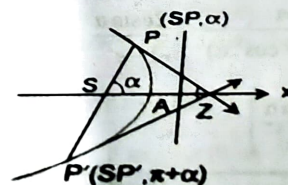
$-ab < 0$ i.e. according as $0 - (1 - e) < 0$ i.e. as $e > 1$ which is also condition that the original conic is an ellipse, a parabola or a hyperbola.

Q 16. If PSP' is a focal chord of a conic show that the angle b/w the tangents at the point P & P' is $\tan^{-1} \left[\frac{2e \sin \alpha}{1 - e^2} \right]$ where α is the angle b/w the axis & chord. (2019)

Ans. The equation of conic is

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots(1)$$

Let PSP' be a focal chord of (1) which makes angle α with the major axis.



So, the vectorial angle of P & P' are α and $\pi + \alpha$, respectively

Now equation of tangent at α is

$$\frac{l}{r} = \cos(\theta - \alpha) + e \cos \theta$$

$$\text{or } \frac{l}{r} = \cos \theta \cos \alpha + \sin \theta \sin \alpha + e \cos \theta$$

$$\Rightarrow l = r \cos \theta (e + \cos \alpha) + r \sin \theta \sin \alpha$$

Changing into cartesian form, we get

$$l = (e + \cos \alpha)x + y \sin \alpha$$

$\therefore$ Slope of the tangent at P = m_1

$$= -\frac{e + \cos \alpha}{\sin \alpha}$$

Replacing α by $\pi + \alpha$ in (3), we get

Slope of tangent at P', say m_2

$$\therefore m_2 = -\frac{e + \cos(\pi + \alpha)}{\sin(\pi + \alpha)}$$

$$= \frac{e - \cos \alpha}{\sin \alpha}$$

Let ϕ be the angle b/w tangential at P and P', then

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$= \frac{\frac{e - \cos \alpha}{\sin \alpha} + \frac{e + \cos \alpha}{\sin \alpha}}{1 - \left[\frac{e - \cos \alpha}{\sin \alpha} \right] \left[\frac{e + \cos \alpha}{\sin \alpha} \right]}$$

$$= \frac{2e \sin \alpha}{\sin^2 \alpha - (e^2 - \cos^2 \alpha)} = \frac{2e \sin \alpha}{1 - e^2}$$

$$\phi = \tan^{-1} \left[\frac{2e \sin \alpha}{1 - e^2} \right]$$

- Q 17.** Show that polar equation of the circle circumscribing the triangle formed by the tangents to the parabola $\frac{l}{r} = 1 + \cos \theta$ at points whose vectorial angles are $2\alpha, 2\beta$ & 2γ is $2r \cos \alpha \cos \beta \cos \gamma = l \cos(\theta - \alpha - \beta - \gamma)$. (2019)

Ans. The point of intersection of tangents at the points whose vectorial

... (2) angles, are $2\alpha, 2\beta$ and 2γ are

$$P \left(\frac{l}{2} \sec \beta, \sec \beta, \beta + \gamma \right), Q \left(\frac{l}{2} \sec \alpha, \sec \alpha, \alpha + \gamma \right)$$

$$\dots (3) \text{ and } R \left(\frac{l}{2} \sec \alpha, \sec \beta, \alpha + \beta \right)$$

Let the equation of the circle circumscribing triangle PQR be

$$r = d \cos(\theta - \lambda) \quad \dots (1)$$

where d is diameter of the circle

If P, Q, R, lies on (1), then

$$\frac{1}{2} \sec \beta \sec \gamma = d \cos(\beta + \gamma - \lambda) \quad \dots (2)$$

$$\frac{1}{2} \sec \gamma \sec \alpha = d \cos(\gamma + \alpha - \lambda) \quad \dots (3)$$

$$\text{and } \frac{1}{2} \sec \alpha \sec \beta = d \cos(\alpha + \beta - \lambda) \quad \dots (4)$$

Dividing (2) by (3), we get

$$\frac{\sec \beta}{\sec \alpha} = \frac{\cos \alpha}{\cos \beta} = \frac{\cos(\beta + \gamma + \lambda)}{\cos(\gamma + \alpha - \lambda)}$$

It is possible when $\lambda = \alpha + \beta + \gamma$

Putting value of λ in (2) we get

$$\frac{l}{2} \sec \beta \sec \gamma = d \cos \alpha$$

$$\text{or } d = \frac{l}{2} \sec \alpha \sec \beta \sec \gamma$$

Again, putting the values of λ and d from above in (1), we get

$$r = \frac{l}{2} \sec \alpha \sec \beta \sec \gamma \cos(\theta - \alpha - \beta - \gamma)$$

$$\text{or } 2r = l \sec \alpha \sec \beta \sec \gamma \cos(\theta - \alpha - \beta - \gamma)$$

$$\text{or } 2r \cos \alpha \cos \beta \cos \gamma = l \cos(\theta - \alpha - \beta - \gamma)$$

- Q 18.** If the normal to the conic $\frac{l}{r} = 1 + e \cos \theta$ at the point α again meets the curve at the point β then prove that -

$$\tan \frac{\beta}{2} = \frac{1 + 2e \cos^2 \frac{\alpha}{2} + e^2}{1 - 2e \sin^2 \frac{\alpha}{2} + e^2} \cot \frac{\alpha}{2}.$$

(2019)

Ans. The equation of the normal β to the conic

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots(1)$$

$$\frac{le \sin \alpha}{(1 + e \cos \alpha)r} = e \sin \theta + \sin(\theta - \alpha) \quad \dots(2)$$

β . The conic (1) & the line (2) meet at the point whose vectorial angle is

Therefore the point $\left[\frac{l}{1 + e \cos \beta}, \beta \right]$ lies on the line (2), i.e.,

$$\frac{e \sin \alpha (1 + e \cos \beta)}{1 + e \cos \alpha} = e \sin \beta + \sin(\beta - \alpha)$$

$$\Rightarrow e(\sin \alpha - \sin \beta) + e^2(\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin(\beta - \alpha)(1 + e \cos \alpha)$$

$$\Rightarrow 2e \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} = -\sin(\alpha - \beta)(1 + e \cos \alpha + e^2)$$

$$\Rightarrow 2e \cos \frac{\alpha + \beta}{2} = 2 \cos \frac{\alpha - \beta}{2} (1 + e \cos \alpha + e^2)$$

for $\sin \frac{\alpha - \beta}{2} \neq 0$ as $\alpha \neq \beta$.

$$\Rightarrow e \left[\cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right]$$

$$= - \left[\cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right] (1 + e \cos \alpha + e^2)$$

$$\Rightarrow e \left[1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right] = - \left[1 + \tan \frac{\alpha}{2} \tan \beta \right] [1 + e \cos \alpha + e^2]$$

$$\Rightarrow \tan \frac{\alpha}{2} \tan \frac{\beta}{2} (1 + e \cos \alpha + e^2 - e) = - (1 + e \cos \alpha + e^2)$$

$$\Rightarrow \tan \frac{\beta}{2} = - \frac{1 + e(1 + \cos \alpha) + e^2}{1 + e(1 + \cos \alpha) + e^2} \cot \frac{\alpha}{2}$$

$$\Rightarrow \tan \frac{\beta}{2} = - \frac{1 + 2e \cos^2 \frac{\alpha}{2} + e^2}{1 - 2e \sin^2 \frac{\alpha}{2} + e^2} \cot \frac{\alpha}{2}$$

Chapter - 2
Straight Lines & Planes

Q 1. Find the equation of the plane passing through the points (5, 2, -3) and perpendicular to the planes $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 9$ & $\vec{r} \cdot (\hat{i} + 3\hat{j} - 5\hat{k}) = -3$. (2014, 2016)

Ans. As the required plane passes through the point $\vec{a} = (5\hat{i} + 2\hat{j} - 3\hat{k})$

& perpendicular to the given plane

$$\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 9 \quad \dots(1)$$

$$\& \vec{r} \cdot (\hat{i} + 3\hat{j} - 5\hat{k}) = -3 \quad \dots(2)$$

Therefore required plane passes through $\vec{a}$ & parallel to the normal vector of planes (1) & (2).

Hence, equation of required plane is -

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{b}) = 0$$

Now,

$$\vec{b} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -5 \end{vmatrix}$$

$$= \hat{i}(2) + \hat{j}(11) + \hat{k}(7).$$

Vector equation of the plane is

$$[\vec{r} - (5\hat{i} + 2\hat{j} - 3\hat{k})] \cdot [2\hat{i} + 11\hat{j} + 7\hat{k}] = 0$$

$$\vec{r} \cdot (2\hat{i} + 11\hat{j} + 7\hat{k}) - 11 = 0$$

$$\vec{r} \cdot (2\hat{i} + 11\hat{j} + 7\hat{k}) = 11$$

Q 2. Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ on the plane $3x + 4y + 5z = 0$. (2015)

Ans. The given line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \dots(1)$$

Any point on this line A is

$$(1+2r, 2+3r, 3+4r)$$

If A lies on the given plane

$$3x + 4y + 5z = 0, \text{ we have}$$

$$3(1+2r) + 4(2+3r) + 5(3+4r) = 0$$

$$3 + 6r + 8 + 12r + 15 + 20r = 0$$

$$r = -1$$

from equation (2), the point

$$A \text{ is } (-1, -1, -1)$$

Now Let B be the point of the perpendicular from C on the given plane

i.e., it is normal to the given plane and as such the direction ratio are 3, 4, 5

The equation of BC of line passing through C(1, 2, 3) & having (3, 4, 5) as direction ratios are

$$\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$$

Any point on this line BC is

$$(1+3r, 2+4r, 3+5r). \text{ If point B i.e., if the point lies on the given plane,}$$

then we have

$$3(1+3r) + 4(2+4r) + 5(3+5r) = 0$$

$$3 + 9r + 8 + 16r + 15 + 25r = 0$$

$$r = -2$$

$$\text{The point B } (-7, -6, -7)$$

The direction ratio of the projection AB of given line are

$$(-7+1, -6+2, -7+3) = (-6, -4, -4)$$

The required equations of projection AB are

$$\frac{x+1}{-6} = \frac{y+1}{-4} = \frac{z+1}{-4}$$

Q 3. Find the symmetrical form, the equation of line $x + y + z + 1 = 0, 4x + y - 2z + 2 = 0$ (2015)

Ans. Let l, m, n be the d.c's of the line.

Then as this line lies on both the given planes, so it is perpendicular

to the normal to these planes.

∴ We have

$$l + m + n = 0, 4l + m - 2n = 0$$

Solving these we get,

$$\frac{l}{1+2} = \frac{m}{+2-4} = \frac{n}{4-1}$$

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

Putting $z = 0$, in the given eqⁿ we have

$$x + y = 1, 4x + y = -2$$

$$\text{Solving, } x = \frac{-1}{3}, y = \frac{-2}{3}$$

∴ The required line meets the plane $z = 0$

$$\text{at } \left[\frac{-1}{3}, \frac{-2}{3}, 0 \right]$$

∴ from (1) & (2), the required equation in symmetric form are-

$$\frac{x + \frac{1}{3}}{1} = \frac{y + \frac{2}{3}}{-2} = \frac{z - 0}{1}$$

$$\text{or } \frac{3x + 1}{3} = \frac{3y + 2}{-6} = \frac{z}{1}$$

& its direction ratio are 3, -6, 1.

Q 4. Find the length & the equations of the s.d. between the lines

$$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-6}{1}$$

$$\& \frac{x+3}{7} = \frac{y+3}{-6} = \frac{z+3}{1}$$

Ans. The given lines are

$$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-6}{1}$$

$$\& \frac{x+3}{7} = \frac{y+3}{-6} = \frac{z+3}{1} \quad \dots(2)$$

Let l, m, n be the directions cosines of the line of S.D. to the two given

lines

Then we have

$$-4l + 3m + 2n = 0,$$

$$-4l + m + n = 0$$

Solving, we get

$$\frac{l}{3-2} = \frac{m}{-8+4} = \frac{n}{-4+12}$$

$$\frac{l}{1} = \frac{m}{-4} = \frac{n}{8} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1^2 + (-4)^2 + 8^2}} = \frac{1}{9}$$

∴ The d.c.'s of the S.D. are $(1/9), -(4/9), (8/9)$

& the d.r.'s of S.D. are 1, -4, 8

Also A (-3, 6, 0) is a point on the line (1)

and B (-2, 0, 7) is a point on the line (2)

The length of SD = projection of join of A & B on the line whose d.c.'s are given by (3)

$$= \frac{1}{9} [(-3) - (-2)] - \frac{4}{9} [6 - 0] + \frac{8}{9} [0 - 7]$$

$$= \frac{-1}{9} - \frac{24}{9} - \frac{56}{9} = -9$$

Equation of SD-

The equation of the plane through the line (2) & S.D. is

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ 1 & -4 & 8 \end{vmatrix} = 0$$

$$\Rightarrow 32x + 34y + 13z = 108$$

And the equation of the plane through the line (2) & S.D. is

...(4)

$$\begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -4 & 8 \end{vmatrix} = 0$$

$$\Rightarrow 4x + 11y + 5z = 27$$

$\therefore$ from equation (4) & (5) the equations of the S.D. are

$$32x + 34y + 13z = 108$$

$$4x + 11y + 5z = 27.$$

Q 5. Find the equations of the perpendicular drawn from the point

$$(5, 7, 3) \text{ to straight line } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{5}$$

Find also the coordination of the foot & length of the perpendicular and the equation of the plane in which the perpendicular & the given straight line lie. (2016)

Ans. The given point P (5, 7, 3) & the line is

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{5}$$

Any point on it is N (15 + 3r, 29 + 8r, 5 + 5r)

$\therefore$ The d.r.'s of line PN are

$$(15 + 3r) - 5, (29 + 8r) - 7, (5 + 5r) - 3$$

$$\text{i.e., } 10 + 3r, 22 + 8r, 2 + 5r$$

Let N be the foot of the perpendicular from P to (1), then PN is perpendicular to (1), and so we have

$$3(10 + 3r) + 8(22 + 8r) - 5(2 + 5r) = 0$$

$$r = -2$$

$\therefore$ from (2), the foot N of the perpendicular PN is

$$(15 - 6, 29 - 16, 5 + 10) \text{ or } (9, 13, 15)$$

And from (3), the d.r.'s of the perpendicular PN are 4, 6, 12. The required equation of perpendicular PN are given by -

$$\frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12}$$

$$\therefore \text{PN} = \text{distance b/w P (5, 7, 3) \& N (9, 13, 15)}$$

$$= \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = 14$$

Again the equation of the plane containing the given line (1), is

$$A(x - 15) + B(y - 29) + C(z - 5) = 0 \quad \dots(4)$$

$$\text{where, } A(3) + B(-8) + C(-5) = 0 \quad \dots(5)$$

If the plane (4) passes through

$$P(5, 7, 5) = 0$$

If the plane (4) passes through P(5, 7, 3), also then

$$A(5 - 15) + B(7 - 29) + C(3 - 5) = 0$$

$$5A + 11B + C = 0 \quad \dots(6)$$

Solving (5) & (6), we have

$$\frac{A}{8 + 55} = \frac{B}{-25 - 3} = \frac{C}{33 - 40}$$

$$\text{or } \frac{A}{9} = \frac{B}{-4} = \frac{C}{1}$$

$\therefore$ From (4) the required equation of the plane through the line (1) and the perpendicular PN is

$$9(x - 15) - 4(y - 29) - (z - 5) = 0$$

$$9x - 4y - z = 14.$$

Q 6. The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$. Find the equation of the plane in its new position. (2017)

Ans. Here, we are to find the equation of the plane through the line of intersection of the plane $x - 2y + 3z = 0$ and $2x + 3y - 4z - 5 = 0$ and at right angle to the plane $x - 2y + 3z = 0$.

Now the equation of the plane through the line of intersection of the planes.

$$x - 2y + 3z = 0 \& 2x + 3y - 4z = 5$$

$$\text{is } (x - 2y + 3z) + \lambda(2x + 3y - 4z - 5) = 0$$

$$(1 + 2\lambda)x + (3\lambda - 2)y + z(3 - 4\lambda) = 5\lambda \quad \dots(1)$$

If this plane is perpendicular to the plane $x - 2y + 3z = 0$, then we have

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$1(1 + 2\lambda) - 2(3\lambda - 2) + 3(3 - 4\lambda) = 0$$

$$\text{or } \lambda = 7/8$$

Substituting the value of λ in (1), the required equation is

$$\left[1 + \frac{7}{4}\right]x + \left[\frac{21}{8} - 2\right]y + \left[3 - \frac{7}{2}\right]z = \frac{35}{8}$$

$$22x + 5y - 4z = 35.$$

Q 7. Find the equation of the plane through three non-collinear point $\vec{a}, \vec{b}, \vec{c}$ & find its distance from point $\vec{d}$. (2017)

Ans. Let $\vec{a}, \vec{b}$ & $\vec{c}$ are the position vectors of the three non-collinear points.

A, B & C respectively then the line segment AB, BC & AC lie on the plane on which A, B & C lie.

$\therefore$ A vector perpendicular to plane

$$= \vec{AB} \times \vec{AC}$$

$$\therefore \vec{n} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})$$

Equation of plane

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

becomes

$$\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

...(1)

Hence, equation (1) represent the plane passing through three given points.

Perpendicular distance of a point from the plane, perpendicular distance P, of the point A (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is given by

$$P = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Q 8. Find the image of the point (1, 2, 3) in the plane $x + 2y + 4z = 38$. (2017)

Ans. We have, $x + 2y + 4z = 38$

...(1)

Let image of the point P (1, 2, 3) in the line (1) be Q (α, β, γ). Then PM

is perpendicular to (1) & M is the mid point of PQ.

Therefore, direction cosines of PM are perpendicular of (1, 2, 4) which are d.r.'s of the normal to (1)

Thus equation of PM is

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{4} \quad \dots(2)$$

At point on (2) is M ($k+1, 2k+2, 4k+3$)

M lies on (1), hence we get

$$(k+1) + 2(2k+2) + 4(4k+3) = 38$$

$$\Rightarrow k = 1$$

Therefore, M (2, 4, 7)

$$\text{Hence, } 2 = \frac{1+\alpha}{2} \Rightarrow \alpha = 3$$

$$4 = \frac{2+\beta}{2} \Rightarrow \beta = 6$$

$$7 = \frac{3+\gamma}{2} \Rightarrow \gamma = 11$$

Therefore, co-ordinates of Q are (3, 6, 11)

Q 9. Find the vector equations of the bisector planes of the angle between the planes

$$3x - 2y + 6z + 8 = 0 \text{ \& }$$

$$2x - y + 2z + 3 = 0$$

(2018)

Ans. Vector equations of the given planes are

$$\vec{r} \cdot \vec{n}_1 = -8 \text{ where } \vec{n}_1 = (3\hat{i} - 2\hat{j} + 6\hat{k}), d_1 = -8$$

$$\text{and } \vec{r} \cdot \vec{n}_2 = -3 \text{ where } \vec{n}_2 = (2\hat{i} - \hat{j} + 2\hat{k}), d_2 = -3$$

The equations of the two bisector planes are

$$\vec{r} \cdot (\hat{n}_1 - \hat{n}_2) = \frac{d_1}{|\hat{n}_1|} - \frac{d_2}{|\hat{n}_2|}$$

$$\text{and } \vec{r} \cdot (\hat{n}_1 + \hat{n}_2) = \frac{d_1}{|\hat{n}_1|} + \frac{d_2}{|\hat{n}_2|}$$

$$\text{or } \vec{r} \cdot \left[\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{3^2 + (-2)^2 + 6^2}} - \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (2)^2}} \right]$$

$$= \frac{-8}{\sqrt{9+4+36}} - \frac{3}{\sqrt{4+1+4}}$$

and

$$\vec{r} \cdot \left[\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{9+4+36}} - \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} \right] = \frac{-8}{\sqrt{9+4+36}} - \frac{3}{\sqrt{4+1+4}}$$

$$\text{or } \vec{r} \cdot \left[\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{7} - \frac{2\hat{i} - \hat{j} + 2\hat{k}}{3} \right] = -\frac{8}{7} - \frac{3}{3}$$

$$\text{and } \vec{r} \cdot \left[\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{7} + \frac{2\hat{i} - \hat{j} + 2\hat{k}}{3} \right] = -\frac{8}{7} + \frac{3}{3}$$

$$\text{or } \vec{r} \cdot [9\hat{i} - 6\hat{j} + 18\hat{k} - 14\hat{i} + 7\hat{j} + 14\hat{k}] = -24 + 21$$

$$\text{and } \vec{r} \cdot [9\hat{i} - 6\hat{j} + 18\hat{k} + 14\hat{i} - 7\hat{j} + 14\hat{k}] = -24 - 21$$

$$\text{or } \vec{r} \cdot (-5\hat{i} + \hat{j} + 4\hat{k}) = -3$$

$$\text{and } \vec{r} \cdot [23\hat{i} - 13\hat{j} + 32\hat{k}] = -45$$

Q 10. Using vector technique, show that the lines,

$$\vec{r} = (2\hat{j} - 3\hat{k}) + \lambda(i + 2j + 3\hat{k}) \text{ \& }$$

$$\vec{r} = (2i + 6j + 3\hat{k}) + \mu(2i + 3j + 4\hat{k}) \text{ are coplaner. (2019)}$$

$$\text{Ans. } \vec{r} = (2j - 3\hat{k}) + \lambda(i + 2j + 3\hat{k}).$$

$$\vec{a} = (2j - 3\hat{k})$$

$$\vec{b} = (i + 2j + 3k)$$

$$\vec{r} = (2i + 6j + 3\hat{k}) + \mu(2i + 3j - 4k)$$

$$\vec{a}' = (2i + 6j + 3\hat{k})$$

$$\vec{b}' = (2i + 3j - 4k)$$

The twolines will be coplaner iff -

$$(\vec{a} - \vec{a}') \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

$$\text{Now, } \vec{a} - \vec{a}' = (2\hat{i} + 4\hat{j} + 6\hat{k})$$

$$\vec{b} \times \vec{b}' = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 3 & -4 \end{vmatrix}$$

$$= i(-8-9) - j(-4-6) + k(3-4)$$

$$= -17\hat{i} + 10\hat{j} - k$$

$$\therefore (\vec{a} - \vec{a}') \cdot (\vec{b}_2 \times \vec{b}_2)$$

$$= (2\hat{i} + 4\hat{j} + 6\hat{k}) \cdot (-17\hat{i} + 10\hat{j} - k)$$

$$= -34 + 40 - 6 = 0$$

Hence, the given lines are coplanar.

Q 11. Using vector technique, find the equation of plane passing through the line of intersection of planes $x + 2y + 2z - 4 = 0$ & $2x + y - z + 5 = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. (2019)

Ans. Vector equation of plane are -

$$r \cdot \vec{b} = 4, \text{ where, } \vec{b} = x + zy + xz \quad \dots(1)$$

$$r \cdot \vec{b} = -5, \text{ where, } \vec{b}' = 2x + y - z \quad \dots(2)$$

As the required plane passes through the point. $\vec{a} = 4x + 5y - 3z$ & perpendicular to given planes,

Hence, equation of required plane is -

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{b}') = 0$$

$$\text{Now, } \vec{b} \times \vec{b}' = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= i(-2-2) - j(-1-4) + k(2-4) = -4i + 5j - 2k$$

∴ vector equation of the plane is -

$$[r - (4i + 5j - 3k)] \cdot [-4i + 5j - 2k] = 0$$

$$r \cdot (-4i + 5j - 2k) = (4i + 5j - 3k) \cdot (-4i + 5j - 2k)$$

$$r \cdot (-4i + 5j - 2k) = -16 + 25 + 6$$

$$r \cdot (-4i + 5j - 2k) - 3 = 0$$

Q 12. Using vector technique find the equation of the plane passing through the points $3i + j + \hat{k}$, $i - 2j + 3\hat{k}$ and parallel to x-axis. (2018)

Ans. The required plane is passing through point $3i + j + \hat{k}$ & $i + 2j + 3\hat{k}$ and is parallel to x-axis, so that it is perpendicular to yz-plane i.e.,
 $r \cdot N = 0$, $N = (1, 0, 0)$ (1)

where $r = xi + yj + z\hat{k}$ is the position vector. Let the equation of plane passing through $3i + j + \hat{k}$ be

$$[r - (3i + j + \hat{k})] \cdot N_1 = 0 \quad \dots(2)$$

Since, the plane (2), passes through $i - 2j + 3\hat{k}$, & is perpendicular to plane (1), we get

$$[(i - 2j + 3\hat{k}) - (3i + j + \hat{k})] \cdot N_1 = 0$$

$$\& N_1 \times N = 0$$

$$\text{or } (-2i - 3j + 2\hat{k}) \cdot N_1 = 0 \& N_1 \cdot N = 0$$

Equation (3), suggest that if we taken

$$N_1 = \lambda(-2i - 3j + 2\hat{k}) \times N$$

$$= \lambda(-2i - 3j + 2\hat{k}) \times (i - 0j + 0\hat{k})$$

$$= \lambda(0i + 2j + 3\hat{k}) = \lambda(2j + 3\hat{k})$$

Putting the value of N_1 in equation (2). we get required equation as-

$$[(xi + yj + z\hat{k}) - (3i + j + \hat{k})] \cdot \lambda(2j + 3\hat{k}) = 0$$

$$\text{or } [(x-3)\hat{i} + (y-1)\hat{j} + (z-1)\hat{k}] \cdot \lambda(2\hat{j} + 3\hat{k}) = 0$$

$$\text{or } (x-3)0 + (y-1)2 + (z-1)3 = 0$$

$$\text{or } 2y + 3z - 5 = 0$$

Q 13. Define the shortest distance between any two line and find the shortest between the lines

$$\vec{r} \cdot \vec{n}_1 + d_1 = 0 = \vec{r} \cdot \vec{n}_2 + d_2$$

$$\vec{r} \cdot \vec{n}_3 + d_3 = 0 = \vec{r} \cdot \vec{n}_4 + d_4$$

(2018)

Ans. A straight line perpendicular to both the given non intersecting lines (skew lines), is called the line of shortest distance and the portion of the line, intercepted between the non-intersecting lines, called length of the shortest distance between the lines.

Let the equation of the line be

$$r \cdot n_1 + d_1 = 0 = r \cdot n_2 + d_2$$

$$\& r \cdot n_3 + d_3 = 0 = r \cdot n_4 + d_4$$

The given line are parallel to the vector $N_1 \times N_2$ & $N_3 \times N_4$ respectively.

Let the line be passing through the point 'a' & 'c' respectively, then,

$$a \cdot N_1 + d_1 = 0, \quad a \cdot N_2 + d_2 = 0$$

$$c \cdot N_3 + d_3 = 0, \quad c \cdot N_4 + d_4 = 0 \quad \dots(1)$$

& the equation of the line are -

$$r = a + t(N_1 \times N_2)$$

$$\& r = c + s(N_3 \times N_4)$$

Therefore, the shortest distance (SD) between the lines is -

$$SD = \frac{\pm(a-c) \cdot \{(N_1 \times N_2) \times (N_3 \times N_4)\}}{|(N_1 \times N_2) \times (N_3 \times N_4)|} \quad \dots(2)$$

$$\text{But } (a-c) \cdot \{(N_1 \times N_2) \times (N_3 \times N_4)\}.$$

$$= a \cdot \{(N_1 \times N_2) \times (N_3 \times N_4)\} -$$

$$c \cdot \{(N_1 \times N_2) \times (N_3 \times N_4)\}.$$

$$= a \cdot \{[N_1 N_3 N_4] N_2 - [N_2 N_3 N_4] N_1\} -$$

$$c \cdot \{[N_1 N_2 N_4] N_3 - [N_2 N_2 N_3] N_4\}$$

$$= -d_2 [N_3 N_4 N_1] + d_1 [N_2 N_3 N_4] + d_3 [N_4 N_1 N_2] - d [N_1 N_2 N_3]$$

....(3)

Putting the value of equation (3) into equation (2),

$$\pm d_1[N_2N_3N_4] - d_2[N_3N_4N_1] + d_3[N_4N_1N_2]$$

$$SD = \frac{(N_1 \times N_2) \times (N_3 \times N_4)}{-d_4[N_1N_2N_3]}$$

Q 14. Using vector technique obtain the non-symmetrical form of equation of the orthogonal projection of the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ on the plane $3x + 4y + 5z = 0$. (2018)

Ans. Equations of the given lines & plane are

$$r = a + tb, \quad a = (1, 2, 3), \quad b = (2, 3, 4) \quad \dots(1)$$

$$\& r \cdot N = 0, \quad N = (3, 4, 5) \quad \dots(2)$$

The required projection of line (1) on the plane (2) and the line of intersection of the plane (2) and the plane passing through the line (1) & perpendicular to the plane (2).

Let the equation of the plane passing through the line (1) is

$$(r - a) \cdot N_1 = 0 \quad \dots(3)$$

$$\text{where, } N_1 \cdot b = 0$$

Since, the plane (3) is perpendicular to the plane (2),

$$N_1 \cdot N = 0 \quad \dots(4)$$

Now (4) & (5) imply that N_1 is collinear with $b \times N$

$$N_1 = \lambda (b \times N).$$

Hence, the equation of the plane passing through the line (1) and perpendicular to the plane (2) is

$$(r - a) \cdot (b \times N) = 0$$

Thus the non symmetrical form of the equations of the projection of the line (1) on the plane (2) are -

$$r \cdot N = 0 \quad \text{or} \quad 3x + 4y + 5z = 0$$

$$(r - a) \cdot (b \times N) = 0 \quad \text{or} \quad x - 2y + z = 0$$

Which is equivalent to symmetrical form

$$\frac{x}{7} = \frac{y}{1} = \frac{z}{-5}$$

Chapter - 3

Sphere

Q 1. Find the centre & radius of the circle passing through the points (2,0,0), (0,3,0) & (0,0,4). (2014)

Ans. The general equation of the sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Since, it passes through (2, 0, 0), (0, 3, 0) & (0, 0, 4), then

$$4 + 0 + 0 + 4u + 0 + 0 = 0$$

$$4 + 4u = 0$$

$$u = -1 \quad \dots(1)$$

$$0 + 9 + 0 + 0 + 6v + 0 = 0$$

$$v = -\frac{3}{2} \quad \dots(2)$$

$$0 + 0 + 16 + 0 + 0 + 8w = 0$$

$$w = -2 \quad \dots(3)$$

Centre is $(-u, -v, -w)$

$$C: \left(1, \frac{3}{2}, 2\right)$$

$$\text{Radius is } \sqrt{u^2 + v^2 + w^2}$$

$$R = \sqrt{1^2 + \left(\frac{3}{2}\right)^2 + (2)^2}$$

$$R = \sqrt{1 + \frac{9}{4} + 4}$$

$$R = \frac{\sqrt{29}}{2} \text{ unit}$$

Q 2. Find the limiting points of the co-axial system of sphere determined by $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$ & $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$ (2014)

Ans. The equation of co-axial system of sphere is

WINSOME SERIES

GEOMETRY

(PAPER-1)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो।**



KEY POINT AU



KEY POINT AU

$$(x^2 + y^2 + z^2 + 3x - 3y + 6) + \lambda(x^2 + y^2 + z^2 - 6y - 6z + 6) = 0$$

$$(x^2 + y^2 + z^2) + \frac{(3 - 0\lambda)x}{(1 + \lambda)} + \frac{(-3 - 6\lambda)y}{(1 + \lambda)} + \frac{(0 - 6\lambda)z}{(1 + \lambda)} + 6 = 0$$

$$\text{its centre is } \left[\frac{-3}{2(1 + \lambda)}, \frac{3 + 6\lambda}{2(1 + \lambda)}, \frac{6\lambda}{2(1 + \lambda)} \right]$$

& radius is equal to zero, gives -1

$$\frac{9}{4(1 + \lambda)^2} + \frac{9(1 + 2\lambda)^2}{4(1 + \lambda)^2} + \frac{9\lambda^2}{(1 + \lambda)^2} - 6 = 0$$

$$9 + 9(1 + 2\lambda)^2 + 36\lambda^2 = 6(1 + \lambda)^2$$

$$9(1 + 1 + 4\lambda^2 + 4\lambda + 4\lambda^2) = 6(1 + \lambda)^2$$

$$\lambda = \frac{1}{2}, -1$$

Putting these values in (1), we get limiting point as -
(-1, 2, 1) & (-2, 1, -1).

Q 3. Show that the polar plane of any point on the line $\frac{x}{2} = \frac{y-1}{3} = \frac{z+3}{4}$ with respect to the sphere $x^2 + y^2 + z^2 = 1$, passes through the line $\frac{2x+3}{13} = \frac{y-1}{-3} = \frac{z}{-1}$. (2019, 2015)

Ans. Any point on the line -

$$\frac{x}{2} = \frac{y-1}{3} = \frac{z+3}{4} = r$$

$$P(2r, 3r+1, 4r-3)$$

The polar plane of the point P with respect to the sphere $x^2 + y^2 + z^2 = 1$ is

$$x \cdot 2r + y \cdot (3r+1) + z \cdot (4r-3) = 1$$

If this plane equation (1), is to pass through the given line. ... (1)

$$\frac{2x+3}{13} = \frac{y-1}{-3} = \frac{z}{-1}$$

$$\text{or } \frac{x+3/2}{13} = \frac{y-1}{-6} = \frac{z-0}{-2} \quad \dots (2)$$

then the point $(-3/2, 1, 0)$ on this line must lie on the plane (1) & so we must have

$$(-3/2)2r + 1 \cdot (3r+1) + 0 \cdot (4r-3) = 1$$

which is true for all values of r

$\therefore$ the point $(-3/2, 1, 0)$ lies on the plane (1).

Also normal to plane (1) must be perpendicular to the line (2) with d.r.'s 13, -6, -2 i.e., we must have

$$2r(13) + (1+3r) \cdot (-6) + (-3+4r) \cdot (-2) = 0$$

which is also true for all values of r

$\therefore$ Polar plane (1), passes through the given line (2).

Q 4. Find the equation of sphere having the circle $x^2 + y^2 = a^2$, $z = 0$ for a great circle. Determine its centre & radius.

(2015, 2019)

Ans. The equation of sphere through the given circle is

$$(x^2 + y^2 - a^2) + \lambda(z) = 0$$

$$x^2 + y^2 + z^2 + 0x + 0y + \lambda z - a^2 = 0$$

& its centre is - $[0, 0, -\lambda]$

If the given circle is the great circle, sphere (1) then the centre of the circle (1) must lie on the plane of circle.

$$0 + 0 + (-\lambda) = 0$$

$$\lambda = 0$$

Equation of required sphere is -

$$(x^2 + y^2 + z^2 = a^2)$$

& its centre is $(0, 0, 0)$.

$$\therefore \text{radius } \sqrt{0^2 + 0^2 + 0^2 + a^2} = a.$$

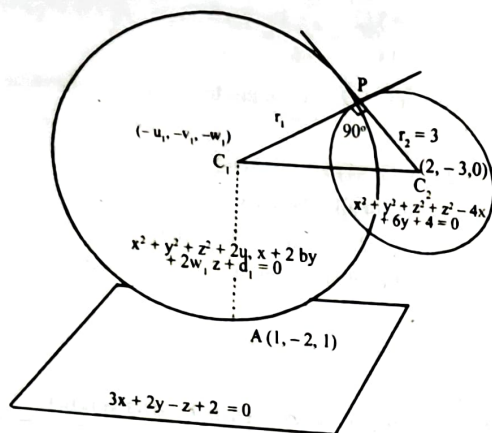
Q 5. Find the equation of sphere which touches the plane $3x + 2y - z + 2 = 0$ at point $(1, -2, 1)$ & cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$. (2016)

Ans. The given sphere is

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$$

and the plane is

... (1)



$$3x + 2y - z + 2 = 0$$

Centre of the sphere (1) is (2, -3, 0) and radius = 3

Suppose the sphere meets the plane at A.

Then dir's of AC_1 are (3, 2, -1)

Now equation of AC_1 is

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} \quad (\text{say})$$

any point on it, is $1 + 3r, -2 + 2r(1-r)$, say $c_1(-u_1, v_1, -w_1)$

But AC_1 = radius of the sphere, = r_1

$$\therefore r_1^2 = (1 + 3r - 1)^2 + (-2 + 2r + 2)^2 + (1 - r - 1)^2$$

$$r_1^2 = 9r^2 + 4r^2 + r^2$$

$$\text{or } r_1^2 = 14r^2$$

$$\text{or } r_1 = r\sqrt{14}$$

Since the spheres cut orthogonally, therefore

$$(PC_1)^2 + (PC_2)^2 = (C_1C_2)^2$$

$$(1 + 3r - 2)^2 + (-2 + 2r + 3)^2 + (1 - r - 0)^2 = 12r^2 + 9$$

$$\text{or } 9r^2 + 1 - 6r + 4r^2 + 4r + 1 + r^2 + 1 - 2r = 14r^2 + 9$$

$$\text{or } 14r^2 - 4r + 3 = 14r^2 + 9$$

$$\text{or } r = -3/2$$

Hence, centre c_1 of the required sphere is

$$\left[1 + 3\left(-\frac{3}{2}\right), -2 + 2\left(-\frac{3}{2}\right), 1 - \left(-\frac{3}{2}\right) \right]$$

$$\text{or } \left(-\frac{7}{2}, -5, +5/2\right) \text{ and radius} = \sqrt{14}r = \sqrt{14} \times \frac{3}{2} = \frac{3}{2}\sqrt{14}$$

Hence, required equation of the sphere by using,

$$(x-u_1)^2 + (y-v_1)^2 + (z-w_1)^2 = r^2 \text{ is}$$

$$\left(x + \frac{7}{2}\right)^2 + (y+5)^2 + \left(z - \frac{5}{2}\right)^2 = \frac{9}{4} \times 14$$

$$\text{or } x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$$

Q 6. Find the limiting points of the coaxial system of sphere determined by the spheres -

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$$

$$\& x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$$

(2016)

Ans. The equation of the coaxial system sphere is

$$(x^2 + y^2 + z^2 + 3x - 3y + 6) + \lambda(x^2 + y^2 + z^2 + 4x - 2y + 2z + 6) = 0$$

$$x^2 + y^2 + z^2 + \left(\frac{3+4\lambda}{1+\lambda}\right)x - \left(\frac{3+2\lambda}{1+\lambda}\right)y + \left[\frac{2\lambda}{1+\lambda}\right]z + 6 = 0$$

$$\text{Its centre is } \left[\frac{3+4\lambda}{1+\lambda}, \frac{-3+2\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right]$$

...(1)

Equating, its radius to zero, we get

$$\left(\frac{3+4\lambda}{1+\lambda}\right)^2 + \left(\frac{3+2\lambda}{1+\lambda}\right)^2 + \frac{(2\lambda)^2}{(1+\lambda)^2} - 6 = 0$$

$$18\lambda^2 + 24\lambda + 12 = 0$$

$$\lambda = \frac{-24 \pm \sqrt{24^2 - 4 \times 18 \times 12}}{2 \times 18} = \frac{-24 \pm 16.9i}{36}$$

$$\lambda = -0.66 \pm 0.47i$$

Substituting these values of λ in the coordinates of above centre, the required limiting points are-

$$\left[\frac{3+4(-0.66 \pm 0.47i)}{(1-0.66 \pm 0.47i)}, \frac{-3+2(-0.66 \pm 0.47i)}{1-0.66 \pm 0.47i}, \frac{2(-0.66 \pm 0.47i)}{1-0.66 \pm 0.47i} \right]$$

$$= (2.98 + 1.39i, 3 - 1.39i, 2.64 - 0.89i)$$

$$\& (2.98 - 1.39i, 3 + 1.39i, 2.64 + 0.89i)$$

Q 7. Find the coordinates of centre & radius of the circle

$$x + 2y + 2z = 15,$$

$$x^2 + y^2 + z^2 - 2y - 4z = 11.$$

(2016)

Ans. Given circles are-
 $x + 2y + 2z = 15$

$$\Rightarrow 0 \cdot x^2 + 0 \cdot y^2 + 0 \cdot z^2 + 2 \cdot \frac{1}{2}x + 2 \cdot 1 \cdot y + 2 \cdot 1 \cdot z - 15 = 0$$

So, the centre of circle is-

$$(-u, -v, -w) = \left(-\frac{1}{2}, -1, -1 \right)$$

$$\& \text{radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{\frac{1}{4} + 1 + 1 - 15}$$

$$= \sqrt{-ve}$$

Which is imaginary hence it is not an equation of circle.
 Now, the equation of other circle is

$$x^2 + y^2 + z^2 - 2y - 4z = 11$$

$$u = 0, v = 1, z = 2 \& d = -11$$

then the centre of circle is-

$$(-u, -v, -w) = (0, -1, -2)$$

$$\& \text{radius} = \sqrt{0^2 + (-1)^2 + (-2)^2 - (-11)}$$

$$= 4$$

Q 8. Show that the centres of all sections of sphere $x^2 + y^2 + z^2 = a^2$ by the planes through the point (α, β, γ) lie on a sphere.

(2017)

Any plane through (α, β, γ) is

$$l(x - \alpha) + m(y - \beta) + n(z - \gamma) = 0 \quad \dots(1)$$

& the line perpendicular to it form the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \dots(2)$$

The foot of the perpendicular is the point of intersection of eq (1) &

(2).

Thus to find the locus of the foot of perpendicular, we should eliminate l, m, n between eq (1) & (2).

$$x(x - \alpha) + y(y - \beta) + z(z - \gamma) = 0$$

$$\Rightarrow x^2 - x\alpha + y^2 - y\beta + z^2 - z\gamma = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0.$$

Q 9. Find the equation of tangent line to the circle $3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z = 22, 3x + 4y + 5z = 26$ at point $(1, 2, 3)$. (2017)

Ans. Equation of circle is given by sphere

$$3x^2 + 2y^2 + 3z^2 - 2x - 3y - 4z - 22 = 0 \quad \dots(1)$$

$$\& \text{the plane } 3x + 4y + 5z - 26 = 0 \quad \dots(2)$$

Now, equation of tangent plane at

$(1, 2, 3)$ to sphere (1) is -

$$\Rightarrow x(1) + y(2) + z(3) + \left(\frac{-1}{3} \right)(x+1)$$

$$+ \left(\frac{-1}{2} \right)(y+2) + \left(\frac{-2}{3} \right)(z+3) - 22 = 0$$

$$\Rightarrow \frac{2x}{3} + \frac{3y}{2} + \frac{7z}{3} - \frac{1}{3} - 1 - 2 - 22 = 0$$

$$\Rightarrow 4x + 9y + 14z - 25 \times 6 = 0$$

$$\Rightarrow 4x + 9y + 14z - 152 = 0. \quad \dots(3)$$

The required tangent line is the intersection of plane (2) & (3) if l, m, n are d.r.'s of tangent line, then,

$$3l + 4m + 5n = 0 \quad \dots(4)$$

$$4l + 9m + 14n = 0 \quad \dots(5)$$

Solving (4) & (5), we have

$$\frac{l}{56-45} = \frac{m}{20-52} = \frac{n}{27-16}$$

$$\frac{l}{11} = \frac{m}{-32} = \frac{n}{11}$$

∴ The required tangent line at (1, 2, 3)

$$\frac{x-1}{11} = \frac{y-2}{32} = \frac{z-3}{11}$$

Q 10. Find the equation of sphere circumscribing the tetrahedron whose faces are-

$$\frac{y}{b} + \frac{z}{c} = 0, \frac{z}{c} + \frac{x}{a} = 0, \frac{x}{a} + \frac{y}{b} = 0$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Ans. Given faces of tetrahedron are

$$\frac{y}{b} + \frac{z}{c} = 0 \quad \dots(1)$$

$$\frac{z}{c} + \frac{x}{a} = 0 \quad \dots(2)$$

$$\frac{x}{a} + \frac{y}{b} = 0 \quad \dots(3)$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(4)$$

Adding (1), (2) and (3), we get

$$2\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\text{or } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \quad (\%)$$

Subtracting (1), (2) and (3) from (5) respectively, we get

$$\frac{x}{a} = 0 \quad \text{or} \quad x = 0$$

$$\frac{y}{b} = 0 \quad \text{or} \quad y = 0$$

$$\frac{z}{c} = 0 \quad \text{or} \quad z = 0$$

Hence (1), (2), (3) meet at (0,0,0)

Adding (2) and (3), we get

$$\frac{x}{a} + \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0 \quad \dots(6)$$

$$\text{or } \frac{x}{a} + 1 = 0$$

or $x = -a$

putting $x = -a$ in (2) and (3), we get

$y = b$ and $z = c$

Hence faces (2), (3) and (4) meet at $(-a, b, c)$

Similarly other points of intersection are $(a, -b, c)$ and $(a, b, -c)$

Let the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots(7)$$

$$\therefore \text{Equation (7) passes through } (0,0,0), \text{ so } d = 0 \quad \dots(8)$$

Again, when (7) passes through $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$ then,

we have

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0 \quad \dots(9)$$

$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc = 0 \quad \dots(10)$$

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0 \quad \dots(11)$$

Adding (9) and (10), we get

$$2(a^2 + b^2 + c^2) + 4wc = 0$$

$$\text{or } 2w = -\frac{a^2 + b^2 + c^2}{c}$$

Similarly, we get

$$2u = -\frac{a^2 + b^2 + c^2}{a}, 2v = -\frac{a^2 + b^2 + c^2}{b}$$

Putting these values in (7), we get

$$x^2 + y^2 + z^2 - \frac{a^2 + b^2 + c^2}{a}x - \frac{a^2 + b^2 + c^2}{b}y - \frac{a^2 + b^2 + c^2}{c}z = 0$$

$$\text{or } x^2 + y^2 + z^2 - (a^2 + b^2 + c^2) \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right) = 0$$

Which is the required equation of the sphere.

Q 11. Show that the equation of the sphere through the four points $(0, 0, 0)$, $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$ is

$$x^2 + y^2 + z^2 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$$

Ans. The equation of the sphere passing through the origin is

This passes through the three points

$$(0, 0, 0), (a, -b, c), (a, b, -c) \text{ if } \dots(1)$$

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0$$

$$a^2 + b^2 + c^2 - 2ua - 2vb + 2wc = 0 \dots(2)$$

$$a^2 + b^2 + c^2 - 2ua + 2vb - 2wc = 0 \dots(3)$$

Subtracting these in pair, we get

$$ua = vb = wc$$

$$\therefore u = \frac{k}{a}, v = \frac{k}{b}, w = \frac{k}{c}$$

Putting the value of u, v, w in any one equation,

$$a^2 + b^2 + c^2 - \frac{2ka}{a} + \frac{2kb}{b} + \frac{2kc}{c} = 0$$

$$2k = -(a^2 + b^2 + c^2)$$

$$k = -\frac{1}{2}(a^2 + b^2 + c^2)$$

Hence, equation of sphere is

$$x^2 + y^2 + z^2 - (a^2 + b^2 + c^2) \left[\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right] = 0$$

The radius of sphere is $\sqrt{u^2 + v^2 + w^2}$

$$\frac{1}{a}(a^2 + b^2 + c^2) \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

Q 12. A point moves so that the sum of the squares of its distance from the six faces of a cube is constant, show that its locus is a sphere. (2018)

Ans. Let six faces of a cube is given by

$$x = \pm a, y = \pm b, z = \pm c$$

Let the point $P(x, y, z)$ moves in the cube such that its square of distance from six faces are constant. Then using distance formula we have

$$(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = k^2$$

$$\text{or } x_1^2 + y_1^2 + z_1^2 - 2ax_1 - 2by_1 - 2cz_1 + a^2 + b^2 + c^2 = k^2$$

Now replace (x_1, y_1, z_1) by x, y, z , we get

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - k^2 = 0$$

which is a sphere.

Q 13. Find the equation of sphere having the circle $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ as a great circle determine its centre & radius. (2018)

Ans. Equation of circle is -

$$x^2 + y^2 + z^2 = 9, x - 2y + 2z = 5. \dots(1)$$

Equation of sphere passing through circle (1) is

$$S + \lambda u = 0$$

$$x^2 + y^2 + z^2 - 9 + \lambda(x - 2y + 2z - 5) = 0$$

$$x^2 + y^2 + z^2 - 9 - \lambda x - 2\lambda y + 2\lambda z - 5\lambda = 0$$

$$x^2 + y^2 + z^2 + \lambda - 2\lambda y + 2\lambda z - 5\lambda - 9 = 0 \dots(2)$$

Centre of sphere (2) is $\left[\frac{-\lambda}{2}, \lambda, -\lambda \right]$

Since, circle (1) is great circle for sphere centre & radius of sphere is equal to centre & radius of circle. Centre of sphere (2) lies on plane

$$x - 2y + 2z - 5 = 0$$

$$\therefore \frac{-\lambda}{2} - 2\lambda - 2\lambda - 5 = 0$$

$$-9\lambda = 10$$

$$\lambda = \frac{-10}{9}$$

so centre of circle is

$$\left[\frac{5}{9}, \frac{-10}{9}, \frac{10}{9} \right]$$

Putting the value of λ in eq (2), we get equation of sphere -

$$x^2 + y^2 + z^2 - \frac{10x}{9} + \frac{20y}{9} - \frac{20z}{9} + \frac{50}{9} - 9 = 0$$

$$9x^2 + 9y^2 + 9z^2 - 10x + 20y - 20z - 31 = 0$$

Radius of sphere be

$$= \sqrt{\frac{25}{81} + \frac{100}{81} + \frac{100}{81} - \frac{31}{9}}$$

$$= \frac{2}{9} \sqrt{14}$$

Q 14. Find the radical axis of the following spheres -

$$(x-2)^2 + y^2 + z^2 = 1$$

$$x^2 + (y-3)^2 + z^2 = 6$$

$$(x+1)^2 + (y-1)^2 + (z-2)^2 = 6.$$

(2019)

Ans. $S_1 = (x-2)^2 + y^2 + z^2 - 1 = 0$

$$S_2 = x^2 + (y-3)^2 + z^2 - 6 = 0$$

$$S_3 = (x+1)^2 + (y+1)^2 + (z-2)^2 - 6 = 0$$

Radical plane (1) and (2) is

$$S_1 - S_2 = 0$$

i.e., $4x - 6y = 0$

or $2x - 3y + 0, z = 0$

...(4)

and, radical plane of (1) and (3) is

$$8x + 2y - 4z = 0$$

or $4x + y - 2z = 0$

...(5)

Planes (4) and (5) don't have constant term, so it is clear that the

radical axis is passing through origin. i.e., (0, 0, 0)

If l, m, n dc's of radical axis, then we have

$$2l - 3m + 0, n = 0 \text{ and } 4l + m - 2n = 0$$

solving, we get

$$\frac{l}{6} = \frac{m}{4} = \frac{n}{14} \quad \text{or} \quad \frac{l}{3} = \frac{m}{2} = \frac{n}{7}$$

$\therefore$ Hence, equation of radical axis is

$$\frac{x-0}{3} = \frac{y-0}{2} = \frac{z-0}{7} \quad \text{or} \quad \frac{x}{3} = \frac{y}{2} = \frac{z}{7}$$

Q 15. Prove that the centres of sphere which touches the line $y = mx, z = c$ & $y = -mx, z = -c$, lies on the conicoid. $mxy + c(1 + m^2)z = 0$. (2015)

Ans. Let the equation required spheres be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots(1)$$

This sphere meets the plane $z = c$ in a circle given by

$$x^2 + y^2 + c^2 + 2ux + 2vy + 2wc + d = 0, z = c$$

$$\text{or } x^2 + y^2 + 2ux + 2vy + 2wc + d + c^2 = 0, z = c \quad \dots(2)$$

This line $y = mx, z = c$ meets to sphere is point which this line meets the circle (2). Thus putting $y = mx$ in (2), we get

$$x^2(1 + m^2) + 2x(\alpha + mv) + (2wc + c^2 + d) = 0 \quad \dots(3)$$

This equation being quadratic in x , gives two values of x . If the line $y = mx, z = c$ touches the given sphere then the roots of the equation (3), must be equal and for this the discriminant of the equation (3) must be zero.

$$\text{i.e., } [2(u + mv)]^2 - 4(1 + m^2)(2wc + c^2 + d) = 0 \quad \dots(4)$$

Similarly the line $y = -mx, z = -c$ will be a tangent to the given sphere if

$$[2(u - mv)]^2 - 4(1 + m^2)(-2wo + c^2 + d) = 0 \quad \dots(5)$$

Subtracting equation (5) from equation (2), we get

$$\Rightarrow 4(u^2 + m^2v^2 + 2muv) - 4(2wc + c^2 + d + 2wcm^2 + c^2m^2 + dm^2) - 4(u^2 + m^2v^2 - 2muv) + 4(-2wc + c^2 + d - 2wcm^2 + c^2m^2 + dm^2) = 0$$

$$\Rightarrow 4muv - 4wc - 4wcm^2 = 0$$

$$\Rightarrow muv = wc(1 + m^2)$$

This can be written as

$$m(-u)(-v) + (1 + m^2)(-w)c = 0$$

locus of the centre $(-u, -v, -w)$ is $mx + (1 + m^2)z = 0$.

$\Rightarrow$ condition for orthogonality

Chapter - 4

Cones, Cylinders & Generating lines

Q 1. Find the equation of the cone whose centre is $(1, 2, 3)$ & guiding curve the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$ (2014)

Ans. The equation of the guiding curve are given by

$$x^2 + y^2 + z^2 = 4, \quad \dots(1)$$

$$\text{and } x + y + z = 1 \quad \dots(2)$$

The equation of any line through $(1, 2, 3)$ are

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{x+y+z-6}{l+m+n} \quad \dots(3)$$

The line meets the plane $x + y + z = 1$ at the point given by

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{1-6}{l+m+n} = \frac{-5}{l+m+n}$$

$$\text{i.e. at the point } \left(1 - \frac{5l}{l+m+n}, 2 - \frac{5m}{l+m+n}, 3 - \frac{5n}{l+m+n} \right)$$

$$\text{or } \left(\frac{m+n+4l}{l+m+n}, \frac{2l+2n-3m}{l+m+n}, \frac{3l+3m-2n}{l+m+n} \right)$$

If this point lies on (1), then we get

$$(m+n+4l)^2 - (2l-3m+2n)^2 + (3l+3m-2n)^2 = 4(l+m+n)^2 \quad \dots(4)$$

Eliminating (l, m, n) from (3) and (4), we get

$$[(y-2) + (z-3) - 4(x-1)]^2 + [2(x-1) - 3(y-3) + 2(z-2)]^2 + [3(x-1) + 3(y-2) - 2(z-3)]^2 = 4[(x-1) + (y-2) + (z-3)]^2$$

$$\text{or } (y+z-4x-1)^2 + (2x-3y+2z-2)^2 + (3x+3y-2z-3)^2 = 4(x+y+z)^2$$

$$\text{or } 5x^2 + 3y^2 + z^2 - 6yz - 4zx - 2xy + 6x + 8y + 10z - 20 = 0$$

Which is the required equation of the cone.

Q 2. Find the equation of the generating line of the hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

which passing through the point $(2, 3, -4)$. (2014, 2016)

Ans. Let the equation of the generating line through (2, 3, -4) be

$$\frac{x-2}{l} = \frac{y-3}{m} = \frac{z-4}{n} = r \text{ (say)}$$

Then co-ordinates of any point on the line (1) is

$$(2+lr, 3+mr, -4+nr)$$

As (1) is the generator of the hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

So, the above point will lie on (2), therefore, we have

$$\frac{(l+2r)^2}{4} + \frac{(3+mr)^2}{9} - \frac{(-4+nr)^2}{16} = 1$$

$$r^2 \left(\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} \right) - 2r \left(\frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16} \right) + \left(\frac{4}{4} + \frac{16}{16} + \frac{9}{9} - 1 \right) = 0$$

∴ This is true for all value of r,

$$\therefore \frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0$$

$$\text{and } \frac{l}{2} + \frac{m}{3} + \frac{n}{4} = 0$$

Eliminating n from (3) and (4), we get

$$\frac{l^2}{4} + \frac{m^2}{9} - \left(\frac{6l+4m}{3} \right) \cdot \frac{1}{16} = 0 \text{ this gives,}$$

when $l=0$, then from (4), we have

$$4m+3n=0 \Rightarrow \frac{m}{3} = \frac{n}{-4}$$

Hence dr's of the first generating are

$$\frac{l}{0} = \frac{m}{3} = \frac{n}{-4}$$

and

...(5)

When $m=0$, then (4) gives, second generator with dr's as,

$$\frac{l}{1} = \frac{m}{0} = \frac{n}{-2}$$

...(6)

Hence equations of required generators will be

$$\frac{x-2}{0} = \frac{y-3}{3} = \frac{z+4}{-4}$$

$$\text{and } \frac{x-2}{1} = \frac{y-3}{0} = \frac{z+4}{-2}$$

Q 3. Find the equation of the cylinder whose generator are parallel

to the line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{3}$ and whose guiding curve is $x+2y^2=$

$$1, z=3.$$

(2015)

Ans. Let (α, β, γ) be any point on the surface of cylinder so that the equation of its generator through this point are,

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$$

$$\left(\alpha + \frac{3-\gamma}{3}, \beta + \frac{2\gamma-6}{3}, 3 \right)$$

This point will lie on the surface $x^2+2y^2=1$.

$$\left[\alpha + \frac{3-\gamma}{3} \right]^2 + 2 \left[\beta + \frac{2\gamma-6}{3} \right]^2 = 1$$

$$(3\alpha+3-\gamma)^2 + 2(3\beta+2\gamma-6)^2 = 9$$

Hence, locus of point (α, β, γ) will be

$$\Rightarrow (3x-z+3)^2 + 2(3y+2z-6)^2 = 9$$

$$\Rightarrow 9x^2 + z^2 + 9 - 6xz - 6z + 18x + 2(9y^2 + 4z^2 + 36 + 12yz - 24z - 36y) = 9$$

$$\Rightarrow 9x^2 + 18y^2 + 9z^2 + 24yz - 6xz + 18x - 72y - 54z - 72 = 0$$

This is the required equations of cylinder.

- Q 4.** Show that there are two systems of generators of a hyperboloid of one sheet such that any generator of one system intersects any generator of the other system. Find the parametric equation of the hyperboloid of one sheet.
(2015, 2017)

Ans. Consider the generators of two different systems

$$\frac{x}{a} + \frac{z}{c} = \lambda \left[1 + \frac{y}{b} \right] \quad \dots(1)$$

$$\frac{x}{a} - \frac{z}{c} = \lambda \left[1 - \frac{y}{b} \right] \quad \dots(2)$$

$$\frac{x}{a} + \frac{z}{c} = \mu \left[1 - \frac{y}{b} \right] \quad \dots(3)$$

$$\frac{x}{a} - \frac{z}{c} = \mu \left[1 + \frac{y}{b} \right] \quad \dots(4)$$

These are four equations in three unknown x, y, z . We show that they are consistent, i.e., the values of x, y, z obtained from any three equations satisfy the remaining one.

Now, eqs (1) & (3)

$$\left(1 + \frac{y}{b} \right) = \mu \left[1 - \frac{y}{b} \right]$$

$$\text{or } \frac{y}{b} = \frac{\mu - \lambda}{\mu + \lambda}$$

Then eq (1) & (2)

$$\frac{x}{a} + \frac{z}{c} = \frac{2\lambda\mu}{\mu + \lambda}$$

$$\& \frac{x}{a} - \frac{z}{c} = \frac{2}{\mu + \lambda}$$

Adding & subtracting these, we get

$$\frac{x}{a} = \frac{\mu\lambda + 1}{\mu + \lambda}$$

$$\frac{z}{c} = \frac{\mu\lambda - 1}{\mu + \lambda}$$

Substituting these values of $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ in eq (4), we obtain

$$\frac{\mu\lambda + 1}{\mu + \lambda} - \frac{\mu\lambda - 1}{\mu - \lambda} = \frac{1}{\mu} \left[1 + \frac{\mu - 1}{\mu + \lambda} \right]$$

$$\frac{2}{\mu + \lambda} = \frac{2}{\mu + \lambda}$$

The four equations (1) to (4) are consistent for the above values of x, y, z . Hence, any member of λ system intersects any member of μ system in the point

$$\left[a \frac{(\lambda\mu + 1)}{(\lambda + \mu)}, b \frac{(-\lambda + \mu)}{(\lambda + \mu)}, c \frac{(\lambda\mu - 1)}{(\lambda + \mu)} \right]$$

These parametric equations of the hyperboloid of one sheet are

$$x = a \frac{(\lambda\mu + 1)}{(\lambda + \mu)}$$

$$y = b \frac{(-\lambda + \mu)}{(\lambda + \mu)}$$

$$z = c \frac{(\lambda\mu - 1)}{(\lambda + \mu)}$$

where, λ & μ are parameters.

- Q 5.** Find two mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$. Which are also perpendicular to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

(2015, 2017, 2019)

Ans. Equation of the cone is

$$5xy - 8xz - 3yz = 0$$

and the equation of the given line is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

Let a line perpendicular to the line (2) be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

$$\therefore l + 2m + 3n = 0$$

If the line (3) is generator of the cone (1), then we have

$$5mn - 8nl - 3lm = 0$$

Eliminating n from (4) and (5), we get

$$(5m - 8l) \left(-\frac{l + 2m}{3} \right) - 3lm = 0$$

$$\text{or } 4l^2 + lm - 5m^2 = 0$$

$$\text{or } (l - m)(4l + 5m) = 0$$

when $l - m = 0$, then we have $n = -l$

$$\text{i.e. } \frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$$

$$\text{Again, when } 4l + 5m = 0, \text{ then; } l = -\frac{5}{4}m, n = -\frac{1}{4}m$$

$$\text{i.e. } \frac{l}{5} = \frac{m}{-4} = \frac{n}{1}$$

Hence, the other two mutually perpendicular generator are

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \text{ and } \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

$$\text{or } x = y = -z \text{ and } 4x - 5y = 20z$$

Q 6. Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$

(2016)

Ans. Any line through (α, β, γ) is

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \dots(1)$$

It meets the plane $z = 0$ at

$$\left[\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0 \right]$$

& if it lie on the given conic we have

$$\frac{1}{a^2} \left[\alpha - \frac{l\gamma}{n} \right]^2 + \frac{1}{b^2} \left[\beta - \frac{m\gamma}{n} \right]^2 = 1 \quad \dots(2)$$

Eliminating l, m, n between (1) & (2)

We get the equation of the required cone as -

$$\frac{1}{a^2} \left[\alpha - \left(\frac{x - \alpha}{z - \gamma} \right) \gamma \right]^2 + \frac{1}{b^2} \left[\beta - \left(\frac{y - \beta}{z - \gamma} \right) \gamma \right]^2 = 1$$

$$\text{or } b^2(\alpha z - \gamma x)^2 + a^2(\beta z - \gamma y)^2 = a^2 b^2 (z - \gamma)^2$$

Q 7. Find the equation of the circular cylinder which passes through the circle, $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ (2016)

Ans. Let us consider distance from the centre $(0, 0, 0)$ of the sphere $x^2 + y^2 + z^2 = 9$ to the plane $x - y + z = 3$ be p then

$$p = \frac{3}{\sqrt{1+1+1}} = \frac{3}{\sqrt{3}}$$

$$p = \sqrt{3}$$

If the radius of the sphere be R then $R = 3$

Hence the radius of the circle is given by

$$r^2 = R^2 - p^2$$

$$r^2 = 9 - 3$$

$$r = \sqrt{6}$$

Hence, the axis of right circular cylinder is the normal to the plane through the centre, i.e.,

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}, \quad \text{& its radius equal to } \sqrt{6}.$$

Therefore the equation of the cylinder is

$$[x(-1) - y(1)]^2 + [y(1) - z(-1)]^2 + [z(1) - x(1)]^2 = (1+1+1)$$

$$(-x-y)^2 + (y+z)^2 + (z-x)^2 = 18$$

$$x^2 + y^2 + 2xy + y^2 + z^2 + 2yz + z^2 + x^2 - 2xz = 18$$

$$x^2 + y^2 + z^2 + xy + yz - zx = 9$$

Q 8. Find the equation of the cylinder which intersects the curve $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$ & whose generators are parallel to the axis of z . (2017)

Ans. The equation of required cylinder is obtained by eliminating z between the equations

$$ax^2 + by^2 + cz^2 = 1$$

$$\& lx + my + nz = p$$

For this, putting the value of $z = \frac{p - lx - my}{n}$ in the other equation.

We get,

$$ax^2 + by^2 + c \left[\frac{p - lx - my}{n} \right]^2 = 1$$

$$\Rightarrow \frac{ax^2n^2 + by^2n^2 + c(p^2 + l^2x^2 + m^2y^2 - 2plx + 2lmy - 2my^2)}{n^2} = 1$$

$$\Rightarrow ax^2n^2 + by^2n^2 + cp^2 + cl^2x^2 + cm^2y^2 - 2clpx + 2clmy - 2cm^2y^2 = n^2$$

$$\Rightarrow x^2(a^2n^2 + cl^2) + y^2(b^2n^2 + cm^2) + 2clmy - 2clpx - 2cm^2y^2 = n^2 = 0$$

This is the equation of required cylinder.

Q 9. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinates axes in A, B, C. Prove that the equation to the cone generated by the lines drawn from O to meet the circle ABC is

$$yz \left[\frac{b}{c} + \frac{c}{b} \right] + zx \left[\frac{c}{a} + \frac{a}{c} \right] + xy \left[\frac{a}{b} + \frac{b}{a} \right] = 0 \quad (2018)$$

Ans. The circle ABC is determined by the plane ABC whose equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(1)$$

& the sphere passing through the point A, B, C. Since one more point is required to find the equation of sphere, we take the origin as the fourth convenient point the equation of the sphere OABC is now seen to be

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \quad \dots(2)$$

& the circle ABC is given by the equation (1) & (2).

To obtain the equation of the cone whose vertex is the origin and which passes through the circle ABC. It is sufficient to make the equation (2) homogeneous with the help of (1). For such an equation will represent a surface through the intersection of the surfaces given by (1) & (2) & being a homogeneous equation it will also represent a cone.

Both the equation (1) & (2) should be made homogeneous equation it will be also represent a cone.

Now eliminating t ,

$$x^2 + y^2 + z^2 - (ax + by + cz) \left[\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right] = 0$$

$$\text{is } yz \left[\frac{b}{c} + \frac{c}{b} \right] + xz \left[\frac{c}{a} + \frac{a}{c} \right] + xy \left[\frac{a}{b} + \frac{b}{a} \right] = 0$$

Q 10. Show that the lines drawn through the point (a, b, g) whose direction numbers l, m, n satisfy the relation $al^2 + bm^2 + cn^2 = 0$ generate the cone.

$$a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = 0 \quad (2018)$$

Ans. Any line through (α, β, γ) whose direction cosine are l, m, n is

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = k$$

$$\Rightarrow l = \frac{x - \alpha}{k}, m = \frac{y - \beta}{k}, n = \frac{z - \gamma}{k}$$

Also it is given that l, m, n satisfies the relation $al^2 + bm^2 + cn^2 = 0$
Therefore,

$$a\left[\frac{x-\alpha}{k}\right]^2 + b\left[\frac{y-\beta}{k}\right]^2 + c\left[\frac{z-\gamma}{k}\right]^2 = 0$$

$$\frac{1}{k^2} [a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2] = 0$$

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

Ex. 12. Find the equation of the cone formed by rotating the line $2x - 3y = 6, z = 0$ about the y -axis. (2018)

Ans. Given the line are

$$2x - 3y = 6, z = 0$$

The equation of the given line can be written as -

$$\frac{x}{3} = \frac{y+2}{-2} = \frac{z}{0}$$

where $(0, -2, 0)$ is the points of intersection with the y -axis is -

$$\cos^{-1} \frac{(-2)}{\sqrt{3^2 + (-2)^2}} \text{ is } \cos^{-1} \frac{(-2)}{\sqrt{13}}$$

If the equation of the generator in any other position be -

$$\frac{x}{l} = \frac{y+2}{m} = \frac{z}{n}$$

the angle which it makes with the y -axis is

$$\cos^{-1} \frac{m}{\sqrt{l^2 + m^2 + n^2}}$$

$$\therefore \cos^{-1} \frac{-2}{\sqrt{13}} = \cos^{-1} \frac{m}{\sqrt{l^2 + m^2 + n^2}}$$

$$\Rightarrow 4(l^2 + m^2 + n^2) = 13m^2$$

$$4l^2 - 9m^2 + 4n^2 = 0$$

Hence, the equation of the cone generated by revolving line is

$$4x^2 - 9y^2 + 4z^2 = 0$$

$$4x^2 - 9y^2 + 4z^2 = 0$$

$$4x^2 - 9y^2 + 4z^2 = 0$$

12. Find the equation of the right circular cone which passes through the three lines drawn from the origin with direction numbers

$$1, 2, 2; 2, 1, -2; 2, -2, 1$$

(2018)

Ans. Let the axis of the cone be the line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$$

Then since the axis is equally inclined to the generators of the cone,

we have

$$\lambda + 2\mu + 2\nu = \pm (2\lambda + \mu - 2\nu) = \pm (2\lambda - 2\mu + \nu) \quad \dots(1)$$

Taking both the plus sign on the right sides, we write two equation

$$\lambda - \mu - 4\nu = 0, \lambda - 4\mu - \nu = 0$$

which give

$$\frac{\lambda}{5} = \frac{\mu}{1} = \frac{\nu}{1}$$

if the equation of a generator be now

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ we have}$$

$$\frac{l\lambda + m\mu + n\nu}{\sqrt{l^2 + m^2 + n^2}} = \frac{\lambda + 2\mu + 2\nu}{\sqrt{1 + 4 + 4}}$$

$$\text{is } (5l + m + n)^2 = 9(l^2 + m^2 + n^2)$$

Therefore, the equation of the cone is -

$$(5x + y + z)^2 = 9(x^2 + y^2 + z^2)$$

$$8x^2 - 4y^2 - 4z^2 + yz + 5zx + 5xy = 0.$$

Q 13. Find the equation of enveloping cylinder of sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generator parallel to the line $x = y = z$. (2019)

Ans. Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r \text{ (say)}$$

Any point on this generator is

WINSOME SERIES

GEOMETRY (PAPER-1)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो।**



KEY POINT AU



KEY POINT AU

$$(\alpha + r, \beta + r, \gamma + r)$$

If this point lies on the given sphere, then we get

$$(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - 2(\alpha + r) + 4(\beta + r) = 1$$

$$\text{or } 3r^2 + 2r(\alpha + \beta + \gamma + 1) + (\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1)$$

Since this generator is a tangent to the given sphere, so the value of r obtained from eq (1) are equal and the conditions for the same is

$$[2(\alpha + \beta + \gamma + 1)]^2 = 4 \cdot 3(\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1)$$

$$\text{or } \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta - 4\alpha + 5\beta - \gamma - 2 = 0$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ i.e., the required equation of the enveloping cylinders is

$$x^2 + y^2 + z^2 - yz - zx - xy - 4x + 5y - z = 2 = 0$$

Q 14. Show that in general three normal can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2az$ but if the point lies on the surface.

27a ($x^2 + y^2$) + 8(a - z)^2 = 0 two of them coincides. (2019)

Ans. Let the equation to any secant APQ through $A(\alpha, \beta, \gamma)$ be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \text{ (say)} \quad \dots(1)$$

Then r_1 and r_2 the measures of AP and AQ are the roots of

$$a(\alpha + lr)^2 + b(\beta + mr)^2 = 2(\gamma + nr)$$

$$\text{or } (a\alpha^2 + b\beta^2) + 2r(a\alpha l + b\beta m - n) + (a\alpha^2 + b\beta^2 - 2\gamma) = 0 \quad \dots(2)$$

Again if ρ is the measure of AR, we have

$$AR = \rho = \frac{r_1 r_2}{r_1 + r_2}, \text{ Since AR is the harmonic mean between AP and AQ}$$

$$= -\frac{a\alpha^2 + b\beta^2 - 2\gamma}{-(a\alpha l + b\beta m - n)} \quad \dots(3)$$

And if (X, Y, Z) is the point R which lies on (1) for $r = \rho$ we have

$$X - \alpha = l\rho, Y - \beta = m\rho, Z - \gamma = n\rho$$

Where ρ is the distance of R from (α, β, γ)

Eliminating ρ between (3) and (4), we have

$$a\alpha(X - \alpha) + b\beta(Y - \beta) - (Z - \gamma) + a\alpha^2 + b\beta^2 - 2\gamma = 0$$

$$\text{i.e. } a\alpha X + b\beta Y - Z - \gamma = 0$$

Hence the locus of R is $a\alpha x + b\beta y = z + \gamma$

Which is polar plane of (α, β, γ) with respect to paraboloid $ax^2 + by^2 = 2z$.

Q 15. Find the enveloping Cylinder of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. (2014)

Ans. Let $P(\alpha, \beta, \gamma)$ be a point of the cylinder. Then the equation of the generator of the cylinder through P drawn parallel to the given lines are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \quad \dots(1)$$

Any point on this generator is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point lies on the given sphere, we have

$$\begin{aligned} &(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + \\ &2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0 \\ &\text{or } r^2(l^2 + m^2 + n^2) + 2r(l\alpha + m\beta + n\gamma + ul + vm + wn) \\ &+ (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \end{aligned} \quad \dots(2)$$

Since the generator (1) is tangent line to the given sphere, so the two values of r given by (2) must be equal & the condition for the same is

$$B^2 = 4AC$$

$$\begin{aligned} &[2(l\alpha + m\beta + n\gamma + ul + vm + wn)]^2 \\ &= 4(l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) \end{aligned}$$

$\therefore$ The equation of enveloping cylinder or the locus of $P(\alpha, \beta, \gamma)$ is

$$\begin{aligned} &(lx + my + nz + ul + vm + wn)^2 = \\ &(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) \\ &\Rightarrow [l(x + u) + m(y + v) + n(z + w)]^2 = \\ &(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) \end{aligned}$$

Either $\alpha = 0$ or $\beta = 0$

...(3)

Case I: If $\alpha = 0$,

$$\text{then } \left\{ \frac{1}{a^2} \right\} \left\{ \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right\} = \frac{1}{b^2} \left[\frac{\gamma^2}{c^2} - 1 \right]$$

$\therefore$ The locus of (α, β, γ) in this case will be

$$x = 0, \frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2} = 1$$

Case II. If $\beta = 0$

$$\text{then } \frac{1}{a^2} \left\{ \frac{\gamma^2}{c^2} - 1 \right\} = \frac{1}{b^2} \left\{ \frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1 \right\}$$

$\therefore$ locus of (α, β, γ) is

$$y = 0, \frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1$$

Q 5. Prove that the plane $3x + 4y = 1$ is a diametral plane of paraboloid, $5x^2 + 6y^2 = 2z$. Find the equation to the chord through the point $(3, 4, 5)$, which bisects it. (2015)

Ans. Let the plane $3x + 4y = 1$ bisects the line parallel to

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \dots(1)$$

Then locus of the mid points of the chords parallel to (1) is

$$5lx + 6my - n = 0$$

This should be identical to given plane $3x + 4y = 1$

Comparing (2) and (3), we get

$$\frac{5l}{3} = \frac{6m}{4} = \frac{n}{1}$$

$$\text{or } \frac{1}{9} = \frac{m}{10} = \frac{n}{15}$$

Hence the required chord is $\frac{x-3}{9} = \frac{y-4}{10} = \frac{z-5}{15}$

6. Find the equation to the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which passes through the line $3x - 3y + 6z - 5 = 0$ $x + 9y - 3z = 0$ (2016)

Ans. Any plane through the given line is

$$(x + 9y - 3z) + \lambda(3x - 3y + 6z - 5) = 0$$

$$(1 + 3\lambda)x + (9 - 3\lambda)y + (6\lambda - 3)z = 5\lambda \quad \dots(1)$$

If this plane touches the given hyperboloid

$$2x^2 - 6y^2 + 3z^2 = 5$$

$$\text{or } \frac{x^2}{(5/2)} + \frac{y^2}{(-5/6)} + \frac{z^2}{(5/3)} = 1$$

$$\text{then } a^2P^2 + b^2Q^2 + c^2R^2 = P^2$$

$$\Rightarrow \left(\frac{5}{2} \right) (1 + 3\lambda)^2 + (-5/6) (9 - 3\lambda)^2 - (5/3) (6\lambda - 3)^2 = (5\lambda)^2$$

$$\Rightarrow 15(9\lambda^2 + 6\lambda + 1) - 5(9\lambda^2 - 54\lambda + 81) + 10(36\lambda^2 - 36\lambda + 9) = 150\lambda^2$$

$$\Rightarrow 300\lambda^2 - 300 = 0$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

$\therefore$ From equation (1), the required tangent planes are -

$$(1 \pm 3)x + (9 \mp 3)y + (\pm 6 - 3)z = \pm 5$$

$$4x + 6y + 3z = 5$$

$$\& 2x - 12y + 9z = 5.$$

Q 7. Find the equation of polar plane of a point with respect to the paraboloid $ax^2 + by^2 = 2z$. (2016)

Ans. The normal at $(x_1, y_1, z_1) = 0$ to the paraboloid $x^2 + y^2 = 2az$ is

$$\frac{a(x - x_1)}{x_1} = \frac{a(y - y_1)}{y_1} = \frac{z - z_1}{-1}$$

If this passes through a point (α, β, γ) , then we have

$$\frac{a(\alpha - x_1)}{x_1} = \frac{a(\beta - y_1)}{y_1} = \frac{\gamma - z_1}{-1} = \lambda \text{ (say)}$$

$$\therefore x_1 = \frac{a\alpha}{a+\lambda}, y_1 = \frac{a\beta}{a+\lambda}, z_1 = \gamma + \lambda$$

But (x_1, y_1, z_1) lies on the paraboloid.

$$\therefore \frac{a^2\alpha^2}{(a+\lambda)^2} + \frac{a^2\beta^2}{(a+\lambda)^2} = 2a(\gamma + \lambda) \quad \dots(3)$$

Which is cubic in λ , giving three values of λ showing that three normals can be drawn from a given point to the paraboloid.

Writing (3) as

$$f(\lambda) = 2(\gamma + \lambda)(a + \lambda)^2 - a(\alpha^2 + \beta^2) = 0 \quad \dots(4)$$

$$\Rightarrow f'(\lambda) = 2(\alpha + \lambda)^2 + 4(\gamma + \lambda)(a + \lambda) = 0 \quad \dots(5)$$

If two normals coincide, then two roots of (4) must be equal, which shows that (4) and (5) must have a common linear factor.

From (5), we have

$$\begin{aligned} 2(a + \lambda)[(a + \lambda) + 2(\gamma + \lambda)] &= 0 \\ \Rightarrow (a + \lambda) + 2(\gamma + \lambda) &= 0 \quad [\because a + \lambda \neq 0] \\ \Rightarrow a + 3\lambda + 2\gamma &= 0 \\ \Rightarrow \lambda &= -\frac{a + 2\gamma}{3} \end{aligned}$$

putting the value of λ in (4), we get

$$2 \left[\left(\gamma - \frac{a + 2\gamma}{3} \right) \left(a - \frac{a + 2\gamma}{3} \right)^2 \right] = a(\alpha^2 + \beta^2)$$

$$\text{or } 2 \left(\frac{\gamma - a}{3} \right) \left(\frac{2a - 2\gamma}{3} \right) = a(\alpha^2 + \beta^2)$$

$$\text{or } 27a(\alpha^2 + \beta^2) = -8(a - \gamma)^3$$

$$\text{or } 27a(\alpha^2 + \beta^2) + 8(a - \gamma)^3 = 0$$

Locus of (α, β, γ) is

$$27a(x^2 + y^2) + 8(a - z)^3 = 0$$

Q 8. Prove that the length of normal chord through $P(\alpha, \beta, \gamma)$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is equal to $4PG_3$ (2016)

Ans. Let the coordinate of P be (x_1, y_1, z_1) , then equation of the normal to the given ellipsoid at P (x_1, y_1, z_1) is

$$\frac{x - x_1}{\frac{px_1}{a^2}} = \frac{-y_1}{\frac{py_1}{b^2}} = \frac{z - z_1}{\frac{pz_1}{c^2}} = r \quad (\text{say})$$

$$\text{where } \frac{1}{p^2} = \frac{a^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} \quad \dots(1)$$

Any point on (1) is

$$\left(x_1 + \frac{rpx_1}{a^2}, y_1 + \frac{rpy_1}{b^2}, z_1 + \frac{rpz_1}{c^2} \right)$$

where $r = PQ$

If Q lies on ellipsoid, then

$$\frac{1}{a^2} \left[\left(x_1 + \frac{rpx_1}{a^2} \right)^2 \right] + \frac{1}{b^2} \left[\left(y_1 + \frac{rpy_1}{b^2} \right)^2 \right] + \frac{1}{c^2} \left[\left(z_1 + \frac{rpz_1}{c^2} \right)^2 \right] = 1$$

$$\text{or } r^2 p^2 \left[\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right] + 2rp \left[\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} \right] = 0$$

as (x_1, y_1, z_1) lies on the ellipsoid.

$$\text{or } rp \left[\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right] + 2 \cdot \frac{1}{p^2} \quad \text{from (1)}$$

$$\therefore r = -\frac{2}{p^3 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)}$$

Again, normal meets $z = 0$ in G_3

$$\therefore PG_3 = \frac{-c^2}{p} \text{ in magnitude}$$

Now, if length of Normal = 4 PG,

$$\therefore \frac{2}{p^3} = \frac{4c^2}{P} \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)$$

$$\text{or } \frac{1}{p^2} = 2C^2 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)$$

$$\text{or } \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} = 2c^2 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)$$

$$\text{or } \frac{x_1^2}{a^6} (2c^2 - a^2) + \frac{y_1^2}{b^6} (2c^2 - b^2) + \frac{z_1^2}{c^4} = 0.$$

Now, locus of (x_1, y_1, z_1) is

$$\frac{x^2}{a^6} (ac^2 - a^2) + \frac{y^2}{b^6} (2c^2 - b^2) + \frac{z^2}{c^4} = 0.$$

Q 9. Find the equations to the two planes which contain the line given by $7x + 10y - 30 = 0$ & $5y - 3z = 0$ & touches the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$. (2018)

Ans. Any plane through $7x + 10y - 30 = 0$

$$5y - 3z = 0 \text{ is}$$

$$7x + 10y - 30 + \lambda(5y - 3z) = 0$$

$$7x + 5y(2 + \lambda) - 3\lambda z = 30$$

Let the equation of the plane be

$$lx + my + nz = P$$

The equation of the planes (1) & (2) is same so comparing the equation (1). & (2), we get

$$l = 7, m = 5(2 + \lambda)$$

$$n = -3\lambda, p = 30$$

$$\text{If it touches } 7x^2 + 5y^2 + 3z^2 = 60$$

$$\text{or } \frac{7}{60}x^2 + \frac{5}{60}y^2 + \frac{3}{60}z^2 = 1 \quad \dots(3)$$

General equation of conicoid is

$$ax^2 + by^2 + cz^2 = 1$$

Comparing the equation(3) & (4), we get

$$a = \frac{7}{60}, b = \frac{5}{60}, c = \frac{3}{60}$$

but we know that the condition tangency

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

Hence putting the value of l, m, n, a, b, c and P . We get

$$7^2 \times \frac{60}{7} + 5^2(2 + \lambda)^2 \times \frac{60}{5} + 3^2\lambda^2 \times \frac{60}{3} = 30^2$$

$$7 \times 60 + 300(2 + \lambda)^2 + 180\lambda^2 = 900$$

$$2\lambda^2 + 3\lambda + 2\lambda + 3 = 0$$

$$2\lambda^2 + 5\lambda + 3 = 0$$

$$(2\lambda + 3)(\lambda + 3) = 0$$

$$\lambda = \frac{-3}{2}, -1.$$

substituting $\lambda = -1$, in eq (1), we get

$$7x + 10y - 30 - 1(5y - 3z) = 0$$

$$10x - 5y - 30 = 0 \quad \dots(4)$$

By substituting $\lambda = \frac{-3}{2}$ in eq (2), we get

$$7x + 5y \left(2 - \frac{3}{2} \right) - 3 \left(\frac{-3}{2} \right) z = 30$$

$$7x + 10y - \frac{15y}{2} + \frac{9z}{2} = 30$$

$$14x + 5y + 9z = 60 \quad \dots(5)$$

Eqⁿ (4) & (5) gives required plane.

Q 10. Show that the normal form (α, β, γ) to the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ lies on the cone $\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$.

(2018)

Ans. Equation to the polar plane of (α, β, γ) w.r.t the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$$

$$\frac{x}{a^2} + \frac{y}{b^2} = (z + \gamma)$$

...(1)

Any line through (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

...(2)

If (2) is perpendicular to (1), then

$$\frac{\alpha}{l} = \frac{\beta}{m} = -\frac{1}{n} = K \text{ (say)}$$

$$\therefore \frac{\alpha}{l} = a^2 K, \frac{\beta}{m} = b^2 K, K = -\frac{1}{n}$$

$$\therefore \frac{\alpha}{l} - \frac{\beta}{m} = (a^2 - b^2)K = \frac{-(a^2 - b^2)}{n}$$

...(3)

Eliminating l, m, n from (2) and (3), we get

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$$

Q 11. Show that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid $x^2 - 2y^2 = 3z$ & find the point of contact. (2018)

Ans. The equation of paraboloid can be written as,

$$\frac{2x^2}{3} - \frac{4y^2}{3} = 2z$$

...(1)

This equation of the tangent plane at any points (α, β, γ) of the given paraboloid (1) is -

$$a\alpha x + b\beta y = z + \gamma$$

...(2)

where, $a = \frac{2}{3}, b = \frac{-4}{3}$

equation (2), of the tangent plane can be written as -

$$\frac{2}{3}a\alpha x - \frac{4}{3}\beta y = z + \gamma$$

$$\text{i.e., } 2\alpha x - 4\beta y - 3z - 3\gamma = 0$$

...(3)

If the given plane touches the paraboloid (1) then the given equation of the plane & equation (3) represents the same plane. Therefore comparing the coefficients we have,

$$\frac{2\alpha}{2} = \frac{-4\beta}{-4} = \frac{-3}{-1} = \frac{-3\gamma}{3}$$

$$\text{Thus, } \alpha = \beta = -\gamma = 3$$

Hence the plane touches the paraboloid at point $(3, 3, -3)$.

Q 12. Find the locus of straight line drawn through a fixed point (f, g, h) whose polar lines with respect to the quadrics, $ax^2 + by^2 + cz^2 = 1$ & $ax^2 + by^2 + gz^2 = 1$ are coplanar. (2018)

Ans. Let $\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n}$ be a line through the given points, whose

polar lines with respect to the conicoids are given by

$$afx + bgy + chz - 1 = 0 = a'x + b'my + c'nz$$

$$a'fx + \beta gy + \gamma hz - 1 = 0 = \alpha'x + \beta'my + \gamma'nz$$

These lines are coplanar if -

$$\begin{vmatrix} af & bg & ch & 1 \\ \alpha f & \beta g & \gamma h & 1 \\ al & bm & cn & 0 \\ \alpha l & \beta m & \gamma n & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (\alpha-a)f & (\beta-b)g & (\gamma-c)h \\ al & bm & cn \\ \alpha l & \beta m & \gamma n \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (\alpha-a)\frac{f}{l} & (\beta-b)\frac{g}{m} & (\gamma-c)\frac{h}{n} \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = 0$$

Hence locus of the required line is the cone.

$$\frac{(\alpha-a)f}{(x-f)}(by-c\beta) + \frac{(\beta-b)g}{(y-g)}(ca-\alpha\gamma) + \frac{(\gamma-c)h}{(z-h)}(a\beta-b\alpha) = 0.$$

Q 13. Show that the locus of the poles of tangent planes of $ax^2 + by^2 + cz^2 = 1$ with respect to $ax^2 + by^2 + gz^2 = 1$ is the surface

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1 \quad (2019)$$

Ans. Any tangent plane of conicoid $ax^2 + by^2 + cz^2 = 1$ is
 $lx + my + mz = p$... (1)

$$\text{where } p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \quad \dots (2)$$

Let (x_1, y_1, z_1) be the pole of (1) with respect to $ax^2 + by^2 + cz^2 = 1$, then polar plane of (x_1, y_1, z_1) is
 $\alpha x x_1 + \beta y y_1 + \gamma z z_1 = 1$... (3)

Equation (3) is identical with (1), so on comparing in x, y, z we get

$$\frac{\alpha x_1}{l} = \frac{\beta y_1}{m} = \frac{\gamma z_1}{n} = \frac{1}{p}$$

$$x_1 = \frac{lp}{\alpha}, y_1 = \frac{mp}{\beta}, z_1 = \frac{np}{\gamma}$$

or $l = \alpha x_1 p, m = \beta y_1 p, n = \gamma z_1 p$
 Putting these values in (2), we get

$$\frac{(\alpha x_1 p)^2}{a} + \frac{(\beta y_1 p)^2}{b} + \frac{(\gamma z_1 p)^2}{c} = p^2$$

$$\text{or } \frac{\alpha^2 x_1^2}{a} + \frac{\beta^2 y_1^2}{b} + \frac{\gamma^2 z_1^2}{c} = 1$$

$\therefore$ Locus of (x_1, y_1, z_1) is

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1$$

Q 14. A cone envelops the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and is cut by plane $z = 0$ in a paraboloid. Show that its vertex lie on the plane $z = \pm c$. (2019)

Ans. Let the vertex of cone be (α, β, γ) . Equation of the surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots (1)$$

Then equation of the enveloping cone of conicoid (1) with vertex (α, β, γ) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right) = \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} - 1 \right)^2$$

Now, section of (1) by the plane $z = 0$, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right) = \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1 \right)^2; z = 0$$

$$\text{or } \frac{x^2}{a^2} \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right) + \frac{y^2}{b^2} \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right) - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 \right)$$

$$\Rightarrow \frac{\alpha^2 x^2}{a^4} + \frac{\beta^2 y^2}{b^4} + 1 + \frac{2\alpha\beta xy}{a^2 b^2} - \frac{2\beta y}{b^2} - \frac{2\alpha x}{a^2}; z = 0$$

This will represent a parabola, if the product of coefficient of x^2 and y^2

is equal to the square of the coefficient of $\frac{xy}{2}$, i.e., by using $h^2 = ab$.

$$\text{i.e. If } \frac{1}{a^2} \left[\frac{\beta^2}{b^2} + \left(\frac{\gamma^2}{c^2} - 1 \right) \right] \cdot \frac{1}{b^2} \left[\frac{\alpha^2}{a^2} + \left(\frac{\gamma^2}{c^2} - 1 \right) \right] = \left(\frac{\alpha\beta}{a^2 b^2} \right)^2$$

$$\text{or If } \frac{\beta^2}{b^2} \left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1 \right) + \left(\frac{\gamma^2}{c^2} - 1 \right) \left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1 \right) = \frac{\alpha^2 \beta^2}{a^2 b^2}$$

or If

$$\frac{\beta^2 \alpha^2}{a^2 b^2} + \frac{\beta^2}{b^2} \left(\frac{\gamma^2}{c^2} - 1 \right) + \left(\frac{\gamma^2}{c^2} - 1 \right) \left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1 \right) = \frac{\alpha^2 \beta^2}{a^2 b^2}$$

$$\text{or If } \left(\frac{\gamma^2}{c^2} - 1 \right) \left[\frac{\beta^2}{b^2} + \frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1 \right] = 0$$

$$\text{or If } \frac{\gamma^2}{c^2} - 1 = 0$$

or If $\gamma^2 = c^2$

Hence locus of (a, b, g) is

$$z^2 = c^2$$

or $z = \pm c$.

[$\because (\alpha, \beta, \gamma)$ does not lie on the conicoid]

Q 15. Prove that the centre of the section of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

by the plane through the three extremities of the different axes is the centroid of the triangle formed by those extremities.

(2019)

Ans.

Let the coordinates of extremities be $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$

The equation of the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(1)$$

Let the centre of section be (α, β, γ)

$\therefore$ Equation of the section is

$$\frac{\alpha}{a^2}(x - \alpha) + \frac{\beta}{b^2}(y - \beta) + \frac{\gamma}{c^2}(z - \gamma) = 0$$

$$\text{or } \frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \quad \dots(2)$$

Planes (1) and (2) are identical, so on comparing the coefficient of x, y and z , we get

$$\frac{\alpha}{1} = \frac{\beta}{1} = \frac{\gamma}{1} = \frac{\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}}{1} = \lambda \text{ (say)} \quad \dots(3)$$

$$\therefore \alpha = a\lambda, \beta = b\lambda, \gamma = c\lambda$$

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = (\lambda^2 + \lambda^2 + \lambda^2) = 3\lambda^2$$

$$\lambda = 3\lambda^2$$

[using (3)]

$$\text{or } \lambda = \frac{1}{3}$$

$$\therefore \alpha = \frac{a}{3}, \beta = \frac{b}{3}, \gamma = \frac{c}{3}$$

$$\text{Hence centre of section is } \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right)$$

Which is clearly the centroid of $\triangle ABC$.

WINSOME SERIES

GEOMETRY

(PAPER-1)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो।**



KEY POINT AU



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