

WINSOME SERIES

Elementry Analysis (paper-2)

**उद्देश्य प्राप्ति तक जिस चीज का त्याग करना पड़े
उसे त्याग दो।**



KEY POINT AU



KEY POINT AU

B.Sc. Ist Year

Mathematics : Paper - II

Elementary Analysis

[Syllabus]

Unit - I- Statements, Connectives (Conjunctions, Disjunction, Negation, Conditional and Biconditional Joins) Statement formulas, Tautologies, Implication and equivalences, Statement Functions, Quantifiers, Sets, Relations, Equivalence and Order relations, Partitions, Functions, Direct and Inverse images of subsets under maps.

Axiomatic introduction of IR as a complete ordered Field, existence of square roots of positive real numbers.

Unit - II- Properties of Integers and Sequences - Natural numbers, First principle of Induction, Well ordering property of $\mathbb{N}$, Second Principle of induction, Integers and rational, Archimedean Property, Rational and Irrational Density Theorems, Division and Euclid can Algorithm in $\mathbb{Z}$, Primes, Fundamental Theorem of Arithmetic, Irrationality of surds.

Intervals in IR, Real sequences and their algebra, Limit of a sequence Bounded, convergent, monotone, and Cauchy sequences Cauchy's general principle of convergence, Algebra of limits (passage to limits under additional multiplications, inversions and inequality), Divergence of sequences.

Unit -III- Limits and Continuity: Real valued Functions of one variable, their graphs and algebra, Neighbourhoods of a point and limits points of subsets of $\mathbb{R}$, Limits of a function at limit point of its domain, sequential characterization of limit of a function. Algebra of limits, One sided limits, Limit of a function as $x \rightarrow \infty$. Infinite limits.

Continuity, Local Boundedness and local maintenance of sign, Boundedness and intermediate value properties of continuous functions over closed intervals, Image of a closed interval under continuous maps.

Unit - IV- Differentiability - Differentiability of a functions a point and its geometrical interpretation, algebra of differentiable functions. Chain rule of differentiation, sign of derivatives and monotonicity of functions, Interior Extremeum. Theorem Rolle's Theorem, Lagrange's Mean value Theorem Cauchy's Mean value Theorem.

Unit-V- Application of Derivatives - Higher derivatives, Leibnitz theorem, Taylor's and Maclaurin's Theorem with Lagrange's and Cauchy's Forms of remainders, Maxima and minima, Local extremum points (necessary and sufficient conditions), Intermediate Form L' Hospital's rule. Convexity and concavity, Points of inflexion, curvature value property in explicit form $y = f(x)$, Darboux's Theorem on intermediate value property of derivatives.

Content

Chapter-1

Properties of Integers And Sequences

- | | | |
|------|---|-----|
| Q1. | Show that every Cauchy sequence in $\mathbb{R}$ converges. (2014) | - 1 |
| Q2. | If r is an real number, then show that there is a unique integer m for which $m \leq r < m + 1$. | - 1 |
| Q3. | Set of all positive real number does not have an upper bound. (2014) | - 1 |
| Q4. | Let $f: x \rightarrow y$ be a map. Let $A, B \subseteq X$. Then show that,
(i) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(ii) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. (2014) | - 2 |
| Q5. | Define lower bound & greatest lower bound of a set real numbers. Prove that the greatest number of a set, if it exists, is the supremum (l.u.b.) of the set. (2015) | - 2 |
| Q6. | Prove, if R and S are two equivalence relation on a set A , then $R \cap S$ is also an equivalence relation. (2014, 2017) | - 3 |
| Q7. | Let P be a compound statement which is a tautology. Show that $(P \vee Q)$ is also a tautology for any statement Q . (2014) | - 4 |
| Q8. | Define an equivalence relation on a set. (2015) | - 4 |
| Q9. | Let $f: X \rightarrow Y$ be map let A & B be subsets of Y . Prove : $F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$. (2015) | - 4 |
| Q10. | Show that every open interval (a, b) in $\mathbb{R}$ contains a rational numbers. | - 5 |
| Or | Between any two distinct real number there exists infinitely many rational number. (2015, 2018) | |
| Q11. | State & Prove the fundamental theorem of authentic. (2015, 2017, 2019) | - 5 |
| Q12. | Define partial order relation. The relation R on the set of natural number defined by $a R b$ if a divides b is partially ordered relation on $\mathbb{N}$. (2016, 2019) | - 6 |
| Q13. | Prove: If $f: A \rightarrow B$ and $g: B \rightarrow C$ are each surjective, then $g \circ f$ is also surjective. (2016) | - 7 |
| Q14. | State the field axioms of the real number system. Prove that for $a, b \in \mathbb{R}$, $ab = 0 \Rightarrow [a = 0 \text{ or } b = 0]$ (2016) | - 7 |
| Q15. | State the division algorithm in integers and prove it. (2016) | - 8 |
| Q16. | State well ordering property of $\mathbb{N}$. Prove, if a is non zero rational numbers & $\sqrt{b}$ is irrational, then $a\sqrt{b}$ is irrational. (2016) | - 9 |

- Q17. Let $\langle u_n \rangle$ & $\langle v_n \rangle$ are two convergent sequences such that $u_n \geq v_n \forall n$ and $\lim_{n \rightarrow \infty} u_n = a$ & $\lim_{n \rightarrow \infty} v_n = b$, then prove $a \leq b$.
 Or Let $\langle a_n \rangle$ & $\langle b_n \rangle$ be convergent sequences on $\mathbb{R}$ with limit a & b respectively. If $a_n < b_n$ for all n , prove that $a \leq b$. (2016)
- Q18. Define a Cauchy sequence. Prove that every Cauchy sequence in $\mathbb{R}$ is bounded. (2017, 2019)
- Q19. If $f: X \rightarrow Y$ be any arbitrary mapping & $B \subseteq Y$, then $f(f^{-1}(B)) = B$, if f is bijective. (2017, 2019)
- Q20. Prove that every bounded monotonic sequence in $\mathbb{R}$ is a convergent sequence. (2017, 2018)
- Q21. Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \quad n \in \mathbb{N}$$
 is monotonic increasing or convergent. (2017)
- Q22. Prove that every non-empty subset of the set of positive integer has at least an element. (2018)
- Q23. Construct a truth table for
 $(P \Rightarrow Q) \Leftrightarrow P \wedge (\neg Q)$ where P & Q are mathematical statements. (2019)
- Q24. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where A, B, C are sets. (2019)
- Q25. For $x, y \in \mathbb{R}$. Prove that
 $|x + y| \leq |x| + |y|$. (2019)
- Q26. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that
 $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$, where $|l| < 1$, then prove $\lim_{n \rightarrow \infty} u_n = 0$. (2019)
- Q27. Let R be a relation in the set of integers $\mathbb{Z}$ given by
 $R = \{(x, y) : x = y \text{ or } x - y = 1\}$. Is R on equivalence relation? Support. (2012)
- Q28. Give an example of symmetric, transitive relation on the set $\{0, 1, 2\}$, which is not reflexive. (2013)
- Q29. Let $X = \{a, b, c\}$, then show that $\Delta = \{(x, x), (y, y), (z, z)\}$ is an equivalence relation. (2013)
- Q30. A relation R in the set of real number $\mathbb{R}$ is given by $x R y$ if $x - y + \sqrt{3}$ is an irrational number. Is R an equivalence relation?

- tion? (2012)
- Q31. Find the negative of $(p \vee q) \wedge (p \Rightarrow q)$. (2012) 16
- Q32. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a map defined by $f(x) = \sin x$, $x \in \mathbb{R}$, find $f^{-1}(0)$. (2013) -16
- Q33. Let $f: X \rightarrow Y$ be an arbitrary mapping and $A \subseteq X$, then
 (i) $A \subseteq f^{-1}(f(A))$
 (ii) $A = f^{-1}(f(A))$, if f is injective. (2012) -16
- Chapter- 2**
Limits & Continuity
- Q1. If a function f is continuous on a closed interval (a, b) then show that it attains its bound at least once in (a, b) . (2014) -17
- Q2. Show that $\lim_{x \rightarrow 0} \frac{1}{x} \sin \frac{1}{x}$, $x \neq 0$ does not exist. (2014) -18
- Q3. Show that the function f defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
 is continuous only at $x = 0$. (2014, 2019) -18
- Q4. If $\langle u_n \rangle$ and $\langle v_n \rangle$ are two convergent sequences such that
 $\lim_{n \rightarrow \infty} u_n = l_1$, & $\lim_{n \rightarrow \infty} v_n = l_2$, then $\lim_{n \rightarrow \infty} (u_n \cdot v_n) = l_1 \cdot l_2$. (2014, 2016) -18
- Q5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a map show that if $\lim_{x \rightarrow a} f(x) = l$, then $\lim_{x \rightarrow a} |f(x)| = |l|$. Does the converse hold? (2015) -19
- Q6. Prove that the image of a closed interval, interval $[a, b]$ under a continuous function is a closed interval. (2016, 2018) -20
- Q7. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions and let

$$h(x) = \begin{cases} f(x) & \text{if } x \text{ is rational} \\ g(x) & \text{if } x \text{ is irrational} \end{cases}$$
 Show that h is continuous only at points C in $\mathbb{R}$ for which $f(C) = g(C)$. (2016) -20
- Q8. Let $\lim_{x \rightarrow a} f(x) = l$, where $l (\neq 0) \in \mathbb{R}$,
 then prove that $\lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{l}$. (2017) -20
- Q9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Define limit f as $x \rightarrow \infty$. Show -21

that

$$\lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = 0. \quad (2017)$$

- Q10. Determine the set of points of continuity of $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by -21

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is irrational} \end{cases} \quad (2017)$$

- Q11. Let $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$ (2018) -22

- Q12. If a function is continuous at $x = a$ & $f(a) \neq 0$, then $\exists$ a number $\delta > 0$ such that $f(x)$ has same sign as $f(a)$ for all values of x in $(a - \delta, a + \delta)$. (2016) -23

- Q13. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on a interval $[a, b]$ & k be any real number between $f(a)$ and $f(b)$, then $\exists$ a real, number c in (a, b) such that $f(c) = k$. (2018, 2019) -23

- Q14. Evaluate $\lim_{x \rightarrow 0} x \left[\frac{1}{x} \right]$, where $[y]$ denotes the largest integer less than or equal to y . (2019) -23

- Q15. Prove that the sequence $\left\langle \left[1 + \frac{1}{n} \right]^2 \right\rangle \forall n \in \mathbb{N}$ converges. (2013) -24

- Q16. Evaluate $\lim_{x \rightarrow \infty} \frac{1}{x}$. (2012) -25

- Q17. Evaluate $\lim_{n \rightarrow \infty} \frac{\cos n}{n}$ (2012) -25

- Q18. Does $\lim_{x \rightarrow 0} \sin \left[\frac{1}{x} \right]$ exist? If yes, find it. (2013) -26

- Q19. Let $f: \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ be a map, where $f(x) = |x|^+$, then show that $\lim_{x \rightarrow 0} f(x)$ doesnot exists. (2010) -27

- Q20. Show that the function -27

$$f(x) = (1 + 2x)^{1/x} \text{ for } x \neq 0 \text{ and}$$

$$f(x) = e^2 \text{ for } x = 0 \text{ is continous at } x = 0. \quad (2010)$$

Chapter-3

Differentiability

- Q1. If a function f is derivable on a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then show that there exists at least one point c between a & b such that $f'(c) = 0$. (2014) -28

- Q2. State & Prove Cauchy mean value theorem. (2014) -28

- Q3. Examine the validity of the hypothesis and conclusion of Rolle's theorem: -29

$$f(x) = 1 - (x - 1)^{2/3} \text{ on } [0, 2] \quad (2014)$$

- Q4. State and Prove that chain rule for differentiation. (2015, 2017) -30

- Q5. Finds the value of c for which cauchy's mean value theorem -31

$$\text{hold for functions } \sin x \text{ and } \cos x \text{ on } \left[\frac{-\pi}{2}, 0 \right]. \quad (2015)$$

- Q6. Examine the validity of the hypothesis and conclusion of Cauchy's mean value theorem: -31

$$f(x) = \exp x, g(x) = \exp (-x) \text{ on } [0, 1]. \quad (2015)$$

- Q7. Give an example (with details) of a continuous function $f: [a, b] \rightarrow \mathbb{R}$, which is differentiable at end points. (2016) -32

- Q8. State and prove the Rolle's theorem. (2016, 2018) -33

- Q9. State the Lagrange's mean value theorem. (2016, 2018, 2019) -33

- Q10. Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined -34

$$\text{by } f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable function. (2017)

- Q11. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function such that f is differentiable over (a, b) . If $f'(x) > 0 \forall x \in (a, b)$, then prove that f is strictly monotonic increasing. (2017) -35

- Q12. Show that polynomial $x^5 - 7x - 2$ has exactly one real root. (2017) -36

- Q13. Let a function f is differentiable at point $x = x_0$ and $c \in \mathbb{R}$, then function cf is also differentiable at x_0 and prove that $(cf)'(x_0) = cf'(x_0)$. (2018) -36

- Q14. Determine values of a, b, c so that function f defined by - -37

$$f(x) = \begin{cases} 4x & ; \text{if } x \leq 0 \\ ax^2 + bx + c & ; \text{if } 0 < x < 1 \\ 3 - 2x & ; \text{if } x \geq 1 \end{cases}$$

(2019)

is differentiable on R.

- Q15. A function f is continuous in $[0, 1]$ & differentiable in $(0, 1)$.
Prove that $f'(c) = f(1) - f(0)$, where $c \in (0, 1)$. (2019) - 38

- Q16. Prove or disprove that $f(x) = x|x|$ is not differentiable at $x = 0$. (2013) - 38

- Q17. Let f be a function from $\mathbb{R} \rightarrow \mathbb{R}$ such that $|f(x) - f(y)| \leq (x - y)^2$
 $\forall x, y \in \mathbb{R}$ show that it is differentiable and its derivative is 0 at all points. Deduce that it is constant. (2011) - 38

- Q18. Show that $\log(1+x)$ lies between $x - \frac{x^2}{2}$ and $x - \frac{x^2}{2(1+x)}$,
for all $x > -0$. (2010) - 39

- Q19. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$ show that $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ will give at least one root between 0 & 1. (2012) - 40

- Q20. Verify lagrange's Mean value theorem for function
 $f(x) = x(x-1)(x-2) \ln \left[0, \frac{1}{2} \right]$. (2012) - 40

Chapter- 4

Application of derivative

- Q1. If $y = \sin(m \sin^{-1} x)$, then show that $(1-x^2) \frac{d^2 y}{dx^2} - \frac{xdy}{dx} + m^2 y = 0$. (2014) - 42

- Q2. Define convexity, concavity and points of inflexion of a function. Show that the points of inflexion of the curve
 $y^2 = (x-a)^2 \cdot (x-b)$ lie on the line $3x + a = 4b$. (2014) - 42

- Q3. Show that $\sin^p \theta \cos^q \theta$ attains a maximum value when $\theta = \tan^{-1} \sqrt{\frac{p}{q}}$. (2014) - 43

- Q4. State and prove leibnitz theorem for differentiation. (2015) - 44
Q5. Find maximum and minimum values of the function $(1-x^2)e^x$. (2015) - 44

- Q6. Define radius of curvature. (2015) - 45
Q7. Find the radius of curvature at the point $(0, a)$ of the curve
 $(x^2 + y^2)^2 = a^2 (y^2 - x^2)$ (2015) - 45

- Q8. State the Taylor's theorem with Cauchy's form of remainder. (2016, 2018) - 47

- Q9. Determine critical points of
 $f(x) = 3x^5 - 5x^3 + 1$ on $\mathbb{R}$. (2016) - 48

- Q10. Evaluate $\lim_{x \rightarrow \infty} \left[1 - \frac{1}{x^2} \right]^x$. (2016) - 49

- Q11. Derive the formula for the curvature of a plane curve. (2016, 2019) - 49

- Q12. If $y = e^{\tan^{-1} x}$, then find the value of
 $(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n$. (2017, 2019) - 51

- Q13. State & prove Darboux's theorem on intermediate value properties of derivatives. (2017, 2019) - 52

- Q14. Find the values of a, b & c such that, - 53

$$\lim_{x \rightarrow 0} \frac{a \sin x - bx + cx^2 + x^3}{x^4 - 2x^3 + 2x^2 \log(1+x)}$$

maybe finite. (2017)

- Q15. Find radius of curvature of curve - 54

$$x^3 + y^3 = 3axy \text{ at } \left[\frac{3a}{2}, \frac{3a}{2} \right] \quad (2017)$$

- Q16. Find points of inflexion of curve $x = \log \left(\frac{y}{x} \right)$ (2017) - 55

- Q17. Find local extrema of function
 $f(x) = x^3 - 6x^2 + 9x + 1$ on $\mathbb{R}$. (2016) - 56

- Q18. Evaluate (2016) - 57

$$\lim_{x \rightarrow \infty} \left[1 + \frac{1}{x} \right]^x \quad (2018)$$

Chapter - I

Properties of Integers & Sequences

Q. 1. Show that every Cauchy sequence in $\mathbb{R}$ converges.

(2014)

Ans. Let $\langle u_n \rangle$ is a Cauchy sequence
i.e., $|u_{n+p} - u_n| < \epsilon$, when $n \geq m$ & $p \geq 1$
then $u_n - \epsilon < u_{n+p} < u_n + \epsilon$
Thus $\langle u_{n+p} \rangle$ is bounded.
Let N & M be lower & upper bounds of u_{n+p}
respectively, then
 $N \leq u_{n+p} \leq M$
So, $N > u_{n+p} - \epsilon$
 $M < u_{n+p} + \epsilon$
Consequently, $|M - N| < |u_{n+p} + \epsilon - u_{n+p} - \epsilon| = 2\epsilon$
Hence, $M - N = 0$ i.e., $M = N$
 $|x| \epsilon \Rightarrow |x| = 0$
So, $M - \epsilon < u_{n+p} < M + \epsilon$
i.e., $|u_{n+p} - M| < \epsilon$
i.e., $u_{n+p} \rightarrow M$
Hence, $\langle u_n \rangle$ is convergent.

Q. 2. If r is a real number, then show that there is a unique integer m for which $m \leq r < m + 1$.

Ans. If r is any real number then $\exists m_1, m_2 \in \mathbb{Z}$
 $\Rightarrow m_1 < r < m_2$
Let $m = \max \{m_1, m_2 + 1, \dots, m_2\}$, $m \leq r$
then, $m + 1 > r$ i.e. $r < m + 1$
So, $m \leq r < m + 1$
Hence proved.

Q. 3. Set of all positive real number does not have an upper bound. Prove.

(2014)

Ans. Since, set of positive real numbers
 $\mathbb{R}^+ \{x : x > 0\}$
if possible let $\mathbb{R}^+$ be bounded above and u be its upper bound.
Since $1 \in \mathbb{R}^+$, then $1 \leq u$, which means
 $u > 0$ so $u + 1 > 0$

- Q19 Find the curvature of curve $y = x^3$ at $(1, 1)$ (2019)
Q20 Evaluate

(i) $\lim_{x \rightarrow 1} \left(\frac{1}{x} - 1 \right)$

(ii) $\lim_{x \rightarrow 1} \frac{x^h - h^x}{x^x - h^h}$ (2019)

Q21 Expand $\tan^{-1} x$ by Machaurin's theorem. (2010)

Q22 Find the minimum value of $(1 - x^2)e^x$. (2012)

Q23 If $y = c \cosh \left[\frac{x}{c} \right]$, prove that $\rho = \frac{y^2}{c}$. (2012)

Q24 Prove or disprove that function $f(x) = (x-2)^3(x+1)^2$ has maxima or minima at $x = 2$ (2011)

Q25 Find the shrotest distance of $(1, 0)$ from the curve $y^2 = 4x$. (2013)

Consequently, $u + 1 \in R'$.

but $u + 1 > u$, so u cannot be upper bound of R' . So R' does not have an upper bound.

Q. 4. Let $f: X \rightarrow Y$ be a map. Let $A, B \subseteq X$. Then show that,

$$(i) f(A_1 \cup A_2) = f(A_1) \cup f(A_2).$$

$$(ii) f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

(2014)

Ans.

(i)

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

$$y \in f(A_1 \cup A_2) \Leftrightarrow y = f(x) \text{ for some } x \in A_1 \cup A_2$$

$$\Leftrightarrow y = f(x) \text{ for some } x \in A_1 \text{ or } x \in A_2$$

$$\Rightarrow y \in f(A_1) \text{ or } y \in f(A_2)$$

$$\Rightarrow y \in f(A_1) \cup f(A_2)$$

$$\text{Hence, } f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

(ii)

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

$$y \in f(A_1 \cap A_2) \Rightarrow y = f(x) \text{ for some } x \in A_1 \cap A_2$$

$$\Rightarrow y = f(A_1) \text{ \& } x \in f(A_2)$$

$$\Rightarrow y \in f(A_1) \cap f(A_2).$$

$$\text{Hence, } f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

Q. 5.

Define lower bound & greatest lower bound of a set of real numbers. Prove that the greatest number of a set, if it exists is the supremum (l.u.b.) of the set. (2014)

Ans. Lower bound of a subset of R :-

A subset S or R set of real number is said to be bounded below if there exists a real number l such that $x \geq l \forall x \in S$.

The real number l is said to be lower bound of S .

If there exists no such l , then the set is said to be unbounded below.

Ex. - (i) The set N of natural number is bounded below.

(ii) The set $\{1, 2, 3, 4, 5\}$ is bounded below by 1.

(iii) The set Z^+ of positive integer is bounded below.

Greatest lower bound (infimum) -

A real number l is said to be a greatest lower bound (g.l.b.) of set S if every member of the set S is smaller or equal to l i.e., if

(i) l is lower bound of S &

(ii) for any other lower bound $l' < l$

In other words, no real number greater than l can be lower bound of

$\Rightarrow$ Let G be the greatest numbers of the set S , clearly,

$$x \leq G \forall x \in S$$

so, that G is an upper bound of S .

Again no numbers less than G can be an upper bound of S for if y be any number less than G , there exists at least one member of S which is greater than y .

Thus, G is the least of all the upper bounds of S i.e., G is supremum of S .

Q. 6. Prove, if R and S are two equivalence relation on a set A , then $R \cap S$ is also an equivalence relation. (2014, 2017)

Ans. If R and S are two equivalence relation on a set A , then $R \cap S$ is also an equivalence relation.

Proof. By definition

$$R \subseteq A \times A, S \subseteq A \times A \Rightarrow R \cap S \subseteq A \times A$$

$$\Rightarrow R \cap S \text{ is relation in } A$$

Now, we shall show that it is an equivalence relation on A .

Reflexivity- Let a be an arbitrary element of A , then

$$a \in A \Rightarrow (a, a) \in R \text{ and } (a, a) \in S$$

($\because$ R and S are reflexive)

$$\Rightarrow (a, a) \in R \cap S$$

Thus $(a, a) \in R \cap S, \forall a \in A$. So $R \cap S$ is reflexive on A .

Symmetry: Let $a, b \in A$ such that $(a, b) \in R \cap S$, then

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ and } (a, b) \in S$$

$$\Rightarrow (b, a) \in R \text{ and } (b, a) \in S \quad (\because R \text{ and } S \text{ are symmetric})$$

$$\Rightarrow (b, a) \in R \cap S$$

$$\text{Thus } (a, b) \in R \cap S \Rightarrow (b, a) \in R \cap S, \forall (a, b) \in R \cap S$$

So, $R \cap S$ is symmetric on A .

Transitivity: Let $a, b, c \in A$, such that

$$(a, b) \in R \cap S \text{ and } (b, c) \in R \cap S, \text{ then}$$

$$(a, b) \in R \cap S \text{ and } (b, c) \in R \cap S$$

$$\Rightarrow [(a, b) \in R \text{ and } (a, b) \in S] \text{ and } [(b, c) \in R \text{ and } (b, c) \in S]$$

$$\Rightarrow [(a, b) \in R \text{ and } (b, c) \in R] \text{ and } [(a, b) \in S \text{ and } (b, c) \in S]$$

$$\Rightarrow (a, c) \in R \text{ and } (a, c) \in S$$

($\because$ R and S are transitive)

$$\Rightarrow (a, c) \in R \cap S$$

$$\text{Thus } (a, b) \in R \cap S \text{ and } (b, c) \in R \cap S \Rightarrow (a, c) \in R \cap S$$

So, $R \cap S$ is transitive on A .

Hence $R \cap S$ is an equivalence relation on A .

Q. 7. Let P be a compound statement which is a tautology. Show that $(P \vee Q)$ is also a tautology for any statement Q . (2014)

Ans. $p \wedge q$ has a true value (T) if both have true values and $p \vee q$ has the truth value false (F) if both are false. Also conditional statement has the truth value false (F) only if antecedent (p) is true (T) and consequent (q) is false (F). Thus we have

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

In the last column all the truth values of T. Hence it is a tautology.

Q. 8. Define an equivalence relation on a set. (2015)

Ans. A relation in a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

For an equivalence relation R in A , we have

(i) $a R a \quad \forall a \in A$ (Reflexive relation)

(ii) $a R b \Rightarrow b R a : \forall a, b \in A$

(Symmetric relation)

(iii) $a R b$ and $b R c \Rightarrow a R c, \quad \forall a, b, c \in A$

(transitive relation)

Q. 9. Let $f: X \rightarrow Y$ be map let A & B be subsets of Y . Prove : $F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$. (2015)

Ans. Since, $A \subseteq A \cup B$ & $B \subseteq A \cup B$.

$$\Rightarrow F^{-1}(A) \subseteq F^{-1}(A \cup B)$$

$$\text{and } F^{-1}(B) \subseteq F^{-1}(A \cup B)$$

$$\Rightarrow F^{-1}(A) \cup F^{-1}(B) \subseteq F^{-1}(A \cup B)$$

$$\text{Let } x \in F^{-1}(A \cup B)$$

$$\Rightarrow f(x) \in A \cup B$$

$$\Rightarrow f(x) \in A \text{ or } f(x) \in B$$

$$\Rightarrow x \in F^{-1}(A) \text{ or } x \in F^{-1}(B)$$

$$\Rightarrow x \in F^{-1}(A) \cup F^{-1}(B)$$

$$\Rightarrow F^{-1}(A \cup B) \subseteq F^{-1}(A) \cup F^{-1}(B)$$

...(1)

...(2)

from eq (1) & (2)

$$F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$$

Q. 10. Show that every open interval (a, b) in $\mathbb{R}$ contains a rational number.

Or Between any two distinct real number there exists infinitely many rational number. (2015, 2018)

Ans. Let x and y be two distinct real numbers such that $x < y$ then $y - x > 0$. So by Archimedian property, there exists $n \in \mathbb{N}$, such that

$$\frac{1}{n} < y - x \Rightarrow n(y - x) > 1 \Rightarrow ny > nx + 1$$

Also for $nx \in \mathbb{R}$, there exists a unique integer m such that $m > nx \geq m - 1$

$$\text{Thus } ny > nx + 1 \geq m > nx$$

$$\Rightarrow y > \frac{m}{n} > x \quad n \in \mathbb{N}, m \in \mathbb{Z}$$

$$\Rightarrow \frac{m}{n} \text{ is a rational number, which lies between } x \text{ and } y.$$

Repeating the above argument for x and $\frac{m}{n}$ and $\frac{m}{n}$ and y we get

another

$$\text{rational numbers say } \frac{m_1}{n_1} \text{ and } \frac{m_2}{n_2}$$

$$\text{such that } x < \frac{m_1}{n_1} < \frac{m}{n} \text{ and } \frac{m}{n} < \frac{m_2}{n_2} < y$$

$$\Rightarrow x < \frac{m_1}{n_1} < \frac{m}{n} < \frac{m_2}{n_2} < y$$

Continuing this process, we get infinitely many rational numbers between two distinct real numbers x and y .

11. State & Prove the fundamental theorem of arithmetic.

(2015, 2017, 2019)

Ans. Every positive integer greater than 1 can be expressed as a unit times product of positive primes, except for the order of their occurrence.

Proof. Let a ($a < 1$) be a positive integer. If a is prime, then it can be expressed as unit times the product of positive prime. But if a is composite possible, let there be two ways of prime factorization of a .

$$a = p_1 \cdot p_2 \cdot p_3 \cdots p_r \quad \dots(1)$$

$$\text{and } a = q_1 \cdot q_2 \cdot q_3 \cdots q_s \quad \dots(2)$$

$$\text{Hence } p_1 \cdot p_2 \cdot p_3 \cdots p_r = q_1 \cdot q_2 \cdot q_3 \cdots q_s \quad \dots(3)$$

Since p_1 divides a i.e., $p_1 | a$ or p_1 is a factor of a .

From (2), we can say that $p_1 | q_1 \cdot q_2 \cdots q_s$

i.e., p_1 is a factor of $(q_1 \cdot q_2 \cdots q_s)$

Thus p_1 divides some q_i . Let $p_1 | q_i$.

$\therefore p_1$ and q_i are positive primes.

$$\therefore p_1 = q_i$$

Thus from (1), we have

$$p_1 \cdot p_2 \cdots p_r = q_1 \cdot q_2 \cdots q_{i-1} \cdot q_{i+1} \cdots q_s$$

Repeating the same argument, we can show that

$$p_2 = q_j \text{ for some } j, \text{ therefore, we have}$$

$$p_1 \cdot p_2 \cdots p_r = q_1 \cdot q_2 \cdots q_{i-1} \cdot q_{i+1} \cdots q_s$$

This process may be continued till then one side of (3) is being reduced to 1.

Since p 's and q 's are integers, the other sides must also become 1.

Thus, there exists a prime factorization of a and the two factorizations of a given by (1) and (2) are identical, except the order of their occurrence. The factors are unique and non repeated.

Hence, every positive integer greater than one can be expressed as unit times the product of primes, except for the order of their occurrence.

Q. 12. Define partial order relation. The relation R on the set of natural number defined by $a R b$ if a divides b is partial ordered relation on N . (2016, 20)

Ans. A relation R on the set A is said to be a partial order relation satisfies the following three conditions.

(i) $a R a, \forall a \in R$ (reflexivity)

(ii) $a R b$ and $b R a \Rightarrow a = b$ (anti symmetry),
 $\forall a, b \in A$.

(iii) $a R b$ and $b R c \Rightarrow a R c$ (Transitivity),

$$\forall a, b, c \in A.$$

$\Rightarrow$ (i) Since, a is a divisor of a for all $a \in N$ i.e., $a R a$
Thus, R is reflexive.

(ii) $a R b$ and $b R a \Rightarrow a | b$ & $b | a$ & $b | c \Rightarrow a = b$.

Thus anti-symmetric.

(iii) Let, $a, b, c \in N$, then

$$a R b \text{ and } b R c \Rightarrow a | b \text{ & } b | c \Rightarrow a | c \Rightarrow a R c$$

Thus R is transitive.

Hence, R is partial ordered relation on N

Q. 13. Prove: If $f : A \rightarrow B$ and $g : B \rightarrow C$ are each surjective, then $g \circ f$ is also surjective. (2016)

Ans. Let Z be an arbitrary element of C , then

$$Z \in C \Rightarrow \exists a y \in B \text{ such that } g(y) = Z$$

($\because g$ is surjective)

$$\& y \in B \Rightarrow \exists a x \in B \text{ such that } f(x) = y$$

($\because f$ is surjective)

Thus for every $Z \in C, \exists x \in A$ such that

$$z = g(y) = g(f(x)) = g \circ f(x).$$

Hence $g \circ f$ is a surjective map.

Q. 14. State the field axioms of the real number system. Prove that for

$$a, b \in R, ab = 0 \Rightarrow [a = 0 \text{ or } b = 0] \quad (2016)$$

Ans. R is a field with respect to the binary operations $+$ and $\cdot$; that is, if $a, b, c \in R$, then

$$(A_1) a + (b + c) = (a + b) + c \text{ (Addition is associative)}$$

$$(A_2) a + b = b + a \text{ (addition is commutative)}$$

$$(A_3) \exists 0 \in R \text{ such that } a + 0 = a \quad \forall a \in R$$

0 is called the additive identity.

$$(A_4) \forall a \in R \text{ there exists } -a \in R, \text{ such that}$$

$$a + (-a) = 0, -a \text{ is called additive inverse or negative of } a.$$

By the above axioms A_1, A_2, A_3, A_4 , we say is short that $(R, +)$ is an Abelian group.

$$(M_1) a \cdot (b \cdot c) = (a \cdot b) \cdot c \text{ (multiplication is associative)}$$

$$(M_2) a \cdot b = b \cdot a \text{ (multiplication is commutative)}$$

(M₃) $\exists 1 (\neq 0) \in R$, such that $a \cdot 1 = a \quad \forall a \in R$ is called the multiplication identity.

(M₄) $\forall a (\neq 0) \exists a^{-1} \in R$ such that $a \cdot a^{-1} = 1$ also written as $1/a$ is called the multiplicative inverse of a .

By axioms M_1, M_2, M_3, M_4 we can say in short that $(R - \{0\})$ is an

abelian group.

$$(D) a \cdot (b + c) = a \cdot b + a \cdot c$$

That is multiplication is distributive over addition.

Let $a \cdot b = 0$ and $a \neq 0$, then a^{-1} exists

$$\text{Now, } a \cdot b = 0$$

$$\Rightarrow a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$$

$$\Rightarrow (a^{-1} \cdot a) \cdot b = 0$$

$$\Rightarrow 1 \cdot b = 0$$

$$\Rightarrow b = 0.$$

Q. 15. State the division algorithm in integrals and prove it. (2016)

Ans. Division Algorithm - If a and b ($\neq 0$) are two integers, then there exists unique integers q and r such that $a = bq + r$, where $0 \leq r < |b|$, q is quotient and r is remainder.

Proof. Let X be the set of multiples of $|b|$, which are either less or equal

to a .

Let the greatest element of the set X be $p|b|$, then $p|b| \leq a$.

$$\text{Thus } a = p|b| + r, r \geq 0 \quad \dots(1)$$

Also, we have $(p+1)|b| > a \Rightarrow p|b| + |b| > a$

$$\Rightarrow p|b| + |b| > p|b| + r$$

$$\Rightarrow |b| > r \text{ or } r < |b|$$

Thus, from (1) and (2), we have

$$a = p|b| + r, 0 \leq r < |b|$$

$$\text{Putting } q = p \text{ if } b > 0 \text{ i.e. } |b| = b$$

$$q = -p \text{ if } b < 0 \text{ i.e. } |b| = -b$$

Thus $p|b| = qb$ or bq in either case

$$\text{Hence } a = bq + r, 0 \leq r < |b|$$

Now, we shall show that q and r are unique

Let, if possible, $\exists$ another pair of integers q_1 and r_1 such that

$$a = bq_1 + r_1, 0 \leq r_1 < |b|$$

$$\text{So, } a = bq + r \Rightarrow bq + r = bq_1 + r_1$$

$$\Rightarrow b(q - q_1) = r_1 - r, r < |b|, r_1 < |b|$$

$$\text{So that } r_1 - r < |b|$$

Thus shows that $b|r_1 - r$, which is impossible because $r_1 - r < |b|$

$$\text{Hence } r_1 - r = 0 \Rightarrow r_1 = r$$

$$\text{Consequently } q_1 = q \text{ as } b \neq 0$$

Thus q and r are unique.

Hence $a = bq + r$ is unique.

Q. 16. State well ordering property of N . Prove, if a is non zero rational numbers & $\sqrt{b}$ is a irrational, then $a\sqrt{b}$ is irrational.

(2016)

Ans. Well ordering property of set N -

Every non-empty subset of the set of natural number N has a first element i.e., for every $x \in N$, $\exists$ a number l such that $l \leq x$. This is known as well ordering property of N .

Now, let a be non-zero rational number & $\sqrt{b}$ be irrational, if possible let $a\sqrt{b}$ be rational then $a\sqrt{b} = \frac{p}{q}$, where p & q are non-zero integers having no common factor other than 1.

$$\text{Now, } a\sqrt{b} = \frac{p}{q} \Rightarrow \sqrt{b} = \frac{p}{aq}$$

But p & aq both are rational & $aq \neq 0$

$$\therefore \frac{p}{aq} \text{ is rational} \Rightarrow \sqrt{b} \text{ is rational}$$

Which contradicts the fact that $\sqrt{b}$ is irrational.

Hence $a\sqrt{b}$ is irrational.

Q. 17. Let $\langle u_n \rangle$ & $\langle v_n \rangle$ are two convergent sequences such that $u_n \leq v_n \forall n$ and $\lim_{n \rightarrow \infty} u_n = a$ & $\lim_{n \rightarrow \infty} v_n = b$, then prove $a \leq b$. Or Let $\langle a_n \rangle$ & $\langle b_n \rangle$ be convergent sequences on R with limit a & b respectively. If $a_n < b_n$ for all n , prove that $a \leq b$. (2016)

Ans. Proof. Let, if possible, $a > b$. Suppose $a - b = 3\varepsilon$. So that nbd of a and b are distinct. Since $\lim_{n \rightarrow \infty} u_n = a$ and $\lim_{n \rightarrow \infty} v_n = b$, therefore for $\varepsilon > 0$, $\exists$ two positive integers k_1 and k_2 respectively, such that

$$a - \varepsilon < u_n < a + \varepsilon \forall n \geq k_1$$

$$\text{and } b - \varepsilon < v_n < b + \varepsilon \forall n \geq k_2$$

$$\text{Let } K = \max(K_1, K_2)$$

$$\therefore a - \varepsilon < u_n < a + \varepsilon \forall n \geq K$$

$$\text{and } b - \varepsilon < v_n < b + \varepsilon \forall n \geq K$$

$$\text{Thus } u_n < v_n \forall n \geq K, \text{ which is contrary to } u_n \leq v_n \forall n$$

Hence our supposition is wrong, so $a \leq b$.

Q. 18. Define a Cauchy sequence. Prove that every Cauchy sequence in R is bounded. (2017, 2019)

Ans. Cauchy Sequence. A sequence $\langle u_n \rangle$ is said to be a Cauchy's se-

quence or a fundamental sequence, if for each $\varepsilon > 0$, $\exists$ a positive integer in such that

$$\begin{aligned} & |u_n - u_m| < \varepsilon \quad \forall n \geq m \\ \text{or} & |u_p - u_q| < \varepsilon \quad \forall p, q \geq m \\ \text{or} & |u_{n,p} - u_n| < \varepsilon \quad \forall n \geq m \text{ and } p > 0 \end{aligned}$$

Every Cauchy sequence is bounded

Proof. Let $\langle u_n \rangle$ be a Cauchy sequence

Taking $\varepsilon = 1$, a positive integer m such that

$$|u_n - u_m| < 1 \quad \forall n \geq m$$

$$\text{i.e., } u_m - 1 < u_n < u_m + 1 \quad \forall n \geq m$$

$$\text{Let } m = \min(u_1, u_2, \dots, u_{m-1})$$

$$\text{and } M = \max(u_1, u_2, \dots, u_{m-1})$$

$$\text{Then } m \leq u_n \leq M \quad \forall n \in \mathbb{N}$$

Hence the sequence $\langle u_n \rangle$ is bounded.

Q. 19. If $f: X \rightarrow Y$ be any arbitrary mapping & $B \subseteq Y$, then
 $f(f^{-1}(B)) = B$, if f is bijective. (2017, 2019)

Ans. Let f be a bijection, then
 For each $y \in B$, $x \in f^{-1}(B)$, where $y = f(x)$.
 $\Rightarrow f(x) \in f(f^{-1}(B))$
 $\Rightarrow y \in f(f^{-1}(B))$.
 Thus $y \in B \Rightarrow y \in f(f^{-1}(B))$.
 i.e., $B \subseteq f(f^{-1}(B))$.
 Now, let $y \in f(f^{-1}(B)) \Rightarrow y = f(x) \& x \in f^{-1}(B)$
 $\Rightarrow y = f(x) \& f(x) \in B$
 $\Rightarrow y \in B$.
 Thus $f(f^{-1}(B)) \subseteq B$.
 from (1) & (2), we conclude that,
 $f(f^{-1}(B)) = B$.

Q. 20. Prove that every bounded monotonic sequence in $\mathbb{R}$ is convergent sequence. (2017, 2018)

Ans. The condition is necessary for we know that every convergent sequence is bounded.

The condition is sufficient. Let a bounded sequence (S_n) be monotonic increasing. Let S denote its range, which is evidently bounded. By the completeness property, S has the least upper bound (the supremum), say M . We shall show that (S_n) converges to M .

Let ε be any pre-assigned positive number.

Now since $M - \varepsilon$ is a number less than the supremum M , there exists at least one member say S_m such $S_m > M - \varepsilon$.

As (S_n) is a monotonic increasing sequence,

$$\therefore S_n \geq S_m > M - \varepsilon, \quad \forall n \geq m$$

Again since M is the supremum,

$$S_n \leq M < M + \varepsilon, \quad \forall n$$

$$\text{Thus } M - \varepsilon < S_n < M + \varepsilon, \quad \forall n \geq m,$$

$$\Rightarrow |S_n - M| < \varepsilon \quad \forall n \geq m$$

$$\Rightarrow (S_n) \text{ converges and } \lim S_n = M.$$

We may similarly consider the case of a bounded monotonic decreasing sequence.

Q. 21. Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \quad n \in \mathbb{N}.$$

is monotonic increasing or convergent. (2017)

$$\text{Ans. } S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0, \quad \forall n$$

$\therefore$ the sequence (S_n) is monotonic increasing.

$$\text{Again } S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

Corollary - If function f is derivable on a closed interval (a, b) and $f'(a) \neq f'(b)$ and k a number lying between $f'(a)$ and $f'(b)$ then $\exists$ at least one point $c \in]a, b[$ such that $f'(c) = k$.

$$\text{Let } g(x) = f(x) - kx, \quad x \in [a, b]$$

Clearly $g(x)$ is derivable on $[a, b]$ and

$$g'(a) = f'(a) - k.$$

$$g'(b) = f'(b) - k$$

Since k lies between $f'(a)$ and $f'(b)$.

$\therefore g'(a)$ and $g'(b)$ are of opposite signs.

$\therefore g(x)$ satisfies the conditions of Darboux's theorem.

Thus, there exists at least one point $c \in]a, b[$ such that

$$g'(c) = 0 \text{ or } f'(c) = k.$$

Q. 22. Prove that every non-empty subset of the set of positive integer has at least an element. (2018)

Ans. Let x & y be any two real number, with $x > 0$, then there exists a positive integer n such that $nx > y$.
when $y \leq 0$, then above is evident, for $y > 0$. Let it possible above statement is not true then,

$$nx \leq y, \forall n \in \mathbb{N}.$$

$$\text{we define } S = \{nx : n \in \mathbb{N}\}.$$

S is non-empty & bounded above.

$$\text{since } nx \leq y.$$

Therefore by completeness axiom, S has l.v.b

Let l.v.b of S be α , then

$$nx \leq \alpha, \forall n \in \mathbb{N}$$

$$\Rightarrow (n+1)x \leq \alpha, \forall n \in \mathbb{N}$$

$$\Rightarrow nx \leq \alpha - x, \forall n \in \mathbb{N}$$

Thus $\alpha - x$ is also an upper bound of S , but $\alpha - x < \alpha$

which contradicts that $\sup(S) = \alpha$

Hence, assumption $nx \leq y, \forall n \in \mathbb{N}$ is false

$\therefore nx < y$, this implies $nx > y$.

Q. 23. Construct a truth table for $(P \Rightarrow Q) \cup \Rightarrow \neg(Q)$ where P & Q are mathematical statements. (2019)

Ans.

P	Q	$\neg Q$	$(P \Rightarrow Q)$	$\neg(P \Rightarrow Q)$	$P \wedge (\neg Q)$	$\neg(P \Rightarrow Q) \Rightarrow P \wedge (\neg Q)$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	F

Q. 24. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where, A, B, C are sets. (2019)

Ans. Let x be an arbitrary element of $A \cap (B \cup C)$ then,
 $x \in A \cap (B \cup C) \Leftrightarrow x \in A \text{ and } x \in B \cup C$

$$\Rightarrow x \in A \text{ \& } (x \in B \text{ or } x \in C)$$

$$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C).$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C).$$

$$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Q. 25. For $x, y \in \mathbb{R}$. Prove that

$$|x + y| \leq |x| + |y|.$$

(2019)

Ans. We know that

$$|x + y|^2 = (x + y)^2$$

$$= x^2 + 2xy + y^2$$

$$\leq x^2 + 2|xy| + y^2 \quad (\because |xy| \geq xy)$$

$$= |x|^2 + 2|xy| + |y|^2 \quad (\because |x|^2 = x^2)$$

$$= |x|^2 + 2|x||y| + |y|^2$$

$$(\because |xy| = |x||y|).$$

$$= (|x| + |y|)^2$$

Since, $|x + y|$ & $|x| + |y|$ are non negative, so on taking positive square root of both sides we get

$$|x + y| \leq |x| + |y|.$$

Q. 26. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n \rightarrow \infty} u_n = l$, where $|l| < 1$, then prove $\lim_{n \rightarrow \infty} u_n = 0$. (2019)

Ans. Proof: Since $|l| < 1$, so we can choose a positive small number ε such that $|l| + \varepsilon < 1$

Now, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$ therefor for $\varepsilon > 0$, $\exists$ a positive number m such

that

$$\left| \frac{u_{n+1}}{u_n} \right| < \varepsilon, \forall n \geq m$$

$$\text{or } \left| \frac{u_{n+1}}{u_n} - l \right| - |l| \leq \left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \forall n \geq m$$

$$\text{or } \left| \frac{u_n}{u_m} \right| < |l| + \varepsilon = k \text{ (say), } k < 1, \forall n \geq m$$

Now, putting $n = m, m+1, \dots, n-1$ in above inequality and multiplying them, we get

$$\left| \frac{u_{n+1}}{u_n} \right| < k^{n-m} \quad \forall n \geq m$$

or $|u_n| < |u_m| k^{n-m} \quad \forall n \geq m$

or $|u_n| < |u_m| \frac{k^n}{k^m}$

As $k < 1$, So $k^n \rightarrow 0$, when $n \rightarrow \infty$

$$\therefore \lim_{n \rightarrow \infty} u_n = 0.$$

Q. 27. Let R be a relation in the set of integers Z given by -
 $R = \{(x, y) : x = y \text{ or } x - y = 1\}$. Is R on equivalence relation?
Support. (2012)

Ans. (i) Let $x \in Z$
 $x R x \Rightarrow x = x \text{ or } x - x = 0 \in Z$
 $\therefore x R x$ is reflexive.
 (ii) Let $x, y \in Z$, then
 $x R y \Rightarrow y R x$
 as $(2, 1) \in R$ but $(1, 2) \notin R$ because
 $2 - 1 = 1$
 but $1 - 2 \neq 1$
 $\therefore R$ is not symmetric
 R is not transitive.
 Hence R is not an equivalence relation.

Q. 28. Give an example of symmetric, transitive relation on the set $\{0, 1, 2\}$, which is not reflexive. (2013)

Ans. Let $A = \{0, 1, 2\}$ then
 $R = A \times A = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.
 Let $R_1 = \{(0, 0), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1)\}$.
 Here, $(0, 1) \in R_1$; $(1, 0) \in R_1$
 $(0, 2) \in R_1$; $(2, 0) \in R_1$
 $\& (1, 2) \in R_1$; $(2, 1) \in R_1$
 Thus $x R_1 y \Rightarrow y R_1 x$
 $\therefore R_1$ is symmetric.
 Also,
 $(0, 1) \in R_1$; $(1, 2) \in R_1 \Rightarrow (0, 2) \in R_1$

Thus $x R_1 y$ and $y R_1 z \Rightarrow x R_1 z$

$\therefore R_1$ is transitive.

But it is not reflexive as $(0, 0)$, $(1, 1)$, and $(2, 2)$ does not related to R_1
 Hence R_1 is symmetric, transitive but not reflexive.

Q. 29. Let $X = \{a, b, c\}$, then show that $\Delta = \{(x, x), (y, y), (z, z)\}$ is an equivalence relation. (2013)

Ans. $X = \{a, b, c\}$
 $\Delta = \{(x, x), (y, y), (z, z)\} = \{(x, y) : x = y\}$.
 $(x) \times (x) = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

We have to show that relation Δ is an equivalence relation, For this

$(x, x) \in \Delta$ ($\because x = x$) [Reflexive].

$(x, y) \in \Delta \Leftrightarrow x = y \Leftrightarrow y = x \Leftrightarrow (y, x) \in \Delta$ [Symmetric]

and $(x, y) \in \Delta$ and $(y, z) \in \Delta$

$\Leftrightarrow x = y$ and $y = z$

$\Leftrightarrow x = z$

$\Leftrightarrow (x, z) \in \Delta$ [Transitive]

Hence, Δ is an equivalence relation

Q. 30. A relation R in the set of real number R is given by $x R y$ if $x - y + \sqrt{3}$ is an irrational number. Is R an equivalence relation? (2012)

Ans. (i) Let $x \in R$, then
 $x R x \Leftrightarrow x - x + \sqrt{3} = \sqrt{3}$, which is an irrational
 $\therefore x R x$ is reflexive.
 (ii) Let $x, y \in R$ then
 $x R y \Rightarrow x - y + \sqrt{3}$ is irrational
 $\Rightarrow y - x + \sqrt{3}$, may not be irrational
 Example, taking $x = \sqrt{3}$, $y = 3$.
 $x R y = \sqrt{3} - 3 + \sqrt{3} = 2\sqrt{3} - 3$ (irrational)
 $y R x = 3 - \sqrt{3} + \sqrt{3} = 3$ (rational)
 $\therefore x R y \not\Rightarrow y R x$
 $\therefore R$ is not symmetric
 Let $x, y, z \in R$, then
 $x R y$ and $y R z$
 $\Rightarrow x - y + \sqrt{3}$ and $y - z + \sqrt{3}$ are irrational

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Elementry Analysis (paper-2)

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$$\Rightarrow (x-y+\sqrt{3})+(y-z+\sqrt{3})$$

$$\Rightarrow (x-z+2\sqrt{3})$$

$$\Rightarrow xRz$$

$\therefore R$ is not transitive

Q. 31. Find the negative of $(p \vee q) \wedge (p \Rightarrow q)$. (2012)

Ans.

P	q	$p \vee q$	$p \Rightarrow q$	$(p \vee q) \wedge (p \Rightarrow q)$	$[(p \vee q) \wedge (p \Rightarrow q)]$
T	T	T	T	T	F
T	F	T	F	F	T
F	T	T	T	T	F
F	F	F	T	F	T

Q. 32. Let $f: R \rightarrow R$ be a map defined by $f(x) = \sin x$, $x \in R$, find $f^{-1}(0)$. (2013)

Ans. Let $f^{-1}(0) = x$
 $\Rightarrow f(x) = 0$
 $\Rightarrow \sin x = 0$
 $\Rightarrow \sin x = \sin 0$
 $x = 0$, which is in R
 So $f^{-1}(0) = 0$.

**Q. 33. Let $f: X \rightarrow Y$ be an arbitrary mapping and $A \in X$, then
 (i) $A \subseteq f^{-1}(f(A))$
 (ii) $A = f^{-1}(f(A))$, if f is injective. (2012)**

Ans. (i) Let $x \in A \Rightarrow f(x) \in f(A)$
 $\Rightarrow x \in f^{-1}(f(A))$
 Thus $A \subseteq f^{-1}(f(A))$ (1)
 (ii) $x \in f^{-1}(f(A)) \Rightarrow f(x) \in f(A)$
 $\Rightarrow f(a) \in f(A)$ for some $a \in A$ ($\because f$ is injective)
 $x = a \Rightarrow f(x) = f(a)$
 $\Rightarrow x \in A$
 Thus, $f^{-1}(f(A)) \subseteq A$... (2)
 From (1) & (2), we conclude
 $f^{-1}(f(A)) = A$.

Chapter - 2

Limits & Continuity

Q. 1. If a function f is continuous on a closed interval $[a, b]$ then show that it attains its bound at least once in $[a, b]$. (2014)

Ans. Proof. Since the function f is continuous on the closed and bounded in the interval (a, b) , therefore it is bounded.

This implies supremum M and infimum m of f exist in (a, b)

Now, to show, there exist two points x_1 and x_2 in the closed interval (a, b) such that $f(x_1) = m$ and $f(x_2) = M$

By definition of supremum, we have

$$f(x) \leq M \quad \forall x \in (a, b)$$

Let, if possible $f(x) \neq M$ for any $x \in (a, b)$, then

$$f(x) < M \quad \forall x \in (a, b), \text{ therefore}$$

$$M - f(x) > 0 \quad \forall x \in (a, b)$$

$\therefore f$ is continuous on (a, b) and M is constant, therefore $M - f(x)$ is continuous on (a, b) .

$$\text{Also } M - f(x) \neq 0 \text{ for any } x \in (a, b)$$

$$\therefore \frac{1}{M - f(x)} \text{ is continuous on } (a, b)$$

$$\Rightarrow \frac{1}{M - f(x)} \text{ is bounded on } (a, b)$$

$$\Rightarrow \exists \text{ a number } K \text{ such that } \frac{1}{M - f(x)} \leq K \quad \forall x \in (a, b)$$

$$\Rightarrow M - f(x) \leq \frac{1}{K} \quad \forall x \in (a, b)$$

$$\Rightarrow f(x) \leq M - \frac{1}{K} \quad \forall x \in (a, b)$$

$$\Rightarrow M - \frac{1}{K} \text{ is an upper bound if } f \text{ on } (a, b) \text{ such that}$$

$$M - \frac{1}{K} < M = \text{supremum of } f$$

Which is contradiction.

Thus, $\exists$ a point $x_2 \in (a, b)$ such that $M = f(x_2)$

Similarly, we can prove that if $m = \inf(f)$, $\exists$ a point x_1 such that $m = f(x_1)$.

(x).

Q. 2. Show that $\lim_{x \rightarrow 0} \frac{1}{x} \sin \frac{1}{x}$, $x \neq 0$ does not exist. (2014)

Ans. $f(x) = \frac{1}{x} \sin \frac{1}{x}$, $x \neq 0$.

then, $\lim_{x \rightarrow 0} f(x) = \lim_{h \rightarrow 0} f(0+h)$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \sin \frac{1}{h}$$

$$= x \times \text{finite value}$$

$$= x$$

This means $\lim_{x \rightarrow 0} \frac{1}{x} \sin \frac{1}{x}$ doesn't exist.

Q. 3. Show that the function f defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$.

(2014, 2019)

Ans. Test for continuity at $x = 0$.

$$f(0) = 0$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0^+} f(0+h) = \lim_{h \rightarrow 0^+} \begin{cases} -h & , \text{ if } h \in \mathbb{Q} \\ -(h) & , \text{ if } h \in \mathbb{Q}' \end{cases}$$

$$\text{R.H.L.} = 0$$

Similarly,

$$\text{L.H.L.} = \lim_{h \rightarrow 0^-} f(0-h) = \lim_{h \rightarrow 0^-} \begin{cases} -h & , \text{ if } -h \in \mathbb{Q} \\ -(h) & , \text{ if } -h \in \mathbb{Q}' \end{cases}$$

$$\text{L.H.L.} = 0$$

$$\text{Thus R.H.L.} = \text{L.H.L.} = f(0) = 0$$

So, if f is continuous at $x = 0$.

Q. 4. If $\langle u_n \rangle$ and $\langle v_n \rangle$ are two convergent sequences such that

$$\lim_{n \rightarrow \infty} u_n = l_1 \text{ \& \> } \lim_{n \rightarrow \infty} v_n = l_2, \text{ then } (u_n \cdot v_n) = l_1 \cdot l_2 \text{ (2014, 2016)}$$

Ans. Let $\varepsilon > 0$ be given, then

$$|u_n v_n - l_1 l_2| = |u_n v_n - l_1 v_n + l_1 v_n - l_1 l_2|$$

$$= |v_n (u_n - l_1) + l_1 (v_n - l_2)|$$

$$\leq |v_n| |u_n - l_1| + |l_1| |v_n - l_2|$$

Since $\langle v_n \rangle$ is convergent, therefore it is bounded, so there exists a positive $k > 0$ such that

$$\langle v_n \rangle \leq k \quad \forall n \in \mathbb{N}$$

Since $\langle u_n \rangle$ and $\langle v_n \rangle$ both are convergent, so $\exists$ positive integers m_1 and m_2 such that

$$|u_n - l_1| < \varepsilon/2k \quad \forall n \geq m_1$$

$$\text{and } |v_n - l_2| < \varepsilon/2k \quad \forall n \geq m_2$$

$$\text{Let } m = \max\{m_1, m_2\}$$

Thus, from above, we have

$$|u_n v_n - l_1 l_2| < k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{\varepsilon}{2k} \quad \forall n \geq m$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq m$$

$$\text{Hence } \lim_{n \rightarrow \infty} (u_n v_n) = l_1 l_2 = \left(\lim_{n \rightarrow \infty} u_n \right) \cdot \left(\lim_{n \rightarrow \infty} v_n \right)$$

i.e. Product of two convergent sequence is a convergent sequence.

Q. 5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a map show that $\lim_{x \rightarrow a} f(x) = l$, then $\lim_{x \rightarrow a} |f(x)| = |l|$. Does the converse hold? (2015)

Ans. Since, $\lim_{x \rightarrow a} f(x) = l$

Then, by definition for given $\varepsilon > 0$, $\exists$ a positive number $\delta > 0$.

$$\text{Such that } |f(x) - l| < \varepsilon \text{ for } 0 < |x - a| < \delta \quad \dots(1)$$

Also, we have

$$|f(x) - l| \geq ||f(x)| - |l|| \quad \forall x \in \mathbb{R} \quad \dots(2)$$

From (1) & (2), we have

$$||f(x)| - |l|| \leq |f(x) - l| < \varepsilon$$

$$\text{for } 0 < |x - a| < \delta$$

$$\Rightarrow \lim_{x \rightarrow a} |f(x)| \text{ exists}$$

$$\text{Hence } \lim_{x \rightarrow a} |f(x)| = |l|.$$

Converse of the above theorem is need not be true.

Q. 6. Prove that the image of a closed interval $[a, b]$ under a continuous function is a closed interval. (2016, 2018)

Ans. Let f be a continuous function on a closed interval $I = [a, b]$.
Let $M = \text{l.u.b of } f \text{ and } I$ & $m = \text{g.l.b of } f \text{ on } I$.

$$\text{Then } m \leq f(x) \leq M \quad \forall x \in I \quad \dots(1)$$

$$\Rightarrow f(I) \subset [m, M]$$

Since f is continuous on I , f assumes Every value between m & M .

$\therefore$ If $m \leq c \leq M$, then for some $x \in I$, $f(x) = c$ so that $[m, M] \subset f(I)$... (2)

From eq (1) & (2), we get

$$f(I) = [m, M].$$

Q. 7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions and let

$$h(x) = \begin{cases} f(x) & \text{if } x \text{ is rational} \\ g(x) & \text{if } x \text{ is irrational} \end{cases}$$

Show that h is continuous only at point c in $\mathbb{R}$ for which $f(c) = g(c)$. (2016)

Ans. R.H.L. $\lim_{x \rightarrow c^+} h(x) = \lim_{x \rightarrow c^+} h(c+h)$

$$\text{R.H.L.} = f(c).$$

Similarly-

$$\text{L.H.L.} = \lim_{x \rightarrow c^-} h(x) = \lim_{x \rightarrow c^-} h(c-h)$$

$$\text{L.H.L.} = g(c).$$

for $h(x)$ to be continuous at $x = c$

if & only if L.H.L. = R.H.L.

$$\therefore f(c) = g(c)$$

Q. 8. Let $f(x) = \frac{1}{x}$, where $x \neq 0 \in \mathbb{R}$,

then prove that $\lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{f(a)}$ (2017)

Ans. $|f(x) - \frac{1}{a}| < \epsilon$ when ever $0 < |x - a| < \delta$
So, we have to prove that

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

$$\text{Now, } \left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| = \left| \frac{f(a) - f(x)}{f(x)f(a)} \right| = \frac{|f(x) - f(a)|}{|f(x)||f(a)|}$$

$$< \frac{2|f(x) - f(a)|}{|f(a)|^2} : \frac{1}{2}|f(a)| < |f(x)|$$

$$< \frac{2}{|f(a)|^2} \epsilon$$

$$< \epsilon, \text{ when ever } 0 < |x - a| < \delta$$

$$\text{taking } \epsilon = |f(a)|^2 \frac{\epsilon}{2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{f(a)}$$

Q. 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Define f as $x \rightarrow \infty$. Show that

$$\lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = 0 \quad (2017)$$

Ans. Given, $x \rightarrow \infty$, $\Rightarrow x$ is very large.

$$\Rightarrow \frac{1}{x^2 + 1} \text{ is very small \& hence is close to zero.}$$

$$\text{Thus as } x \rightarrow \infty \Rightarrow \frac{1}{x^2 + 1} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = 0$$

Q. 10. Determine the set of points of continuity of $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is irrational} \end{cases} \quad (2017)$$

Ans. R.H.L. = $\lim_{x \rightarrow h^+} f(o+h) = (o+h)^2 - 1$
 R.H.L. = $h^2 - 1$
 L.H.L. = $\lim_{x \rightarrow h^-} f(o-h) = 0$
 L.H.L. = 0
 For function $f(x)$ to be continuous,
 R.H.L. = L.H.L.
 $h^2 - 1 = 0$
 $h = +1$.
 $\therefore$ for $x = +1$ & -1 , $f(x)$ is a continuous functions

Q. 11. Let $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$ (2018)

Ans. Case I. If a is a non-zero rational number.

$$\text{Then } f(a-0) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} (a-h) \text{ or } \lim_{h \rightarrow 0} (a-h)$$

according as $(a-h)$ is rational or irrational
 $= a$ or $-a$ i.e. a is not unique.

$\therefore f(a-0)$ does not exist.

Hence $\lim_{h \rightarrow 0} f(x)$ does not exist.

Case II. If $a = 0$.

$$\text{then } f(0-0) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) \text{ or } \lim_{h \rightarrow 0} (-h)$$

according as $(a-h)$ is rational or irrational = 0

$$\therefore \text{R.H.L.} = \text{L.H.L.} = 0$$

Hence, $\lim_{x \rightarrow 0} f(x)$ exist and is equal to zero.

Case III. If a is an irrational number.

$$\text{L.H.L.} = f(a-0) = \lim_{h \rightarrow 0} f(a-h)$$

$$= \lim_{h \rightarrow 0} (a-h) \text{ or } \lim_{h \rightarrow 0} -(a-h)$$

according to $(a-h)$ is rational or irrational = a or $-a$ i.e. a is not unique

$\therefore f(a-0)$ does not exist.

Hence, $\lim_{x \rightarrow a} f(x)$ exists only when $a = 0$.

Q. 12. If a function is continuous at $x = a$ & $f(a) \neq 0$, then $\exists$ a number $\delta > 0$ such that $f(x)$ has same sign as $f(a)$ for all values of x in $(a - \delta, a + \delta)$. (2016)

Ans. Proof: $f(x)$ is continuous $x = a$, then for a given $\epsilon > 0$, we can find a positive number δ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta$$

$$\Rightarrow f(a) - \epsilon < f(x) < f(a) + \epsilon, \text{ whenever } a - \delta < x < a + \delta$$

$$\text{Now } f(a) \neq 0 \Rightarrow |f(a)| > 0. \text{ If we choose } 0 < \epsilon < |f(a)|.$$

then we have $f(a) - \epsilon$ and $f(a) + \epsilon$ having the same sign as $f(a)$

$$\Rightarrow f(x) \text{ has the same sign as } f(a) \text{ for all } x \in (a - \delta, a + \delta).$$

Q. 13. Let $f: R \rightarrow R$ be a continuous function on a interval $[a, b]$ & k be any real number between $f(a)$ and $f(b)$, then $\exists$ a real number c in (a, b) such that $f(c) = k$. (2018, 2019)

Ans. Let us suppose that

$$f(a) < K < f(b)$$

Now define a function $g(x)$ such that

$$g(x) = f(x) - K, x \in [a, b]$$

Since f is continuous in $[a, b]$ and K is constant, therefore g is continuous on $[a, b]$

Now, since K lies between $f(a)$ and $f(b)$, therefore either

$$f(a) < K < f(b) \text{ or } f(b) < K < f(a)$$

$$\text{Again, } g(a) = f(a) - K < 0$$

$$g(b) = f(b) - K > 0$$

$$\Rightarrow g(a) \cdot g(b) < 0$$

Thus, $\exists$ a point $c \in (a, b)$ such that

$$g(c) = 0 \Rightarrow f(c) - K = 0 \Rightarrow f(c) = K.$$

Hence, $\exists$ a point c such that $a < c < b$ and $f(c) = K$

Q. 14. Evaluate $\lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor$, where $[y]$ denotes the largest integer less than or equal to y . (2019)

Ans. Let $\frac{1}{x} = y$

$$x \rightarrow 0^+ \Rightarrow y \rightarrow \infty$$

$$\Rightarrow \lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor$$

$$\Rightarrow \lim_{y \rightarrow +\infty} \frac{[y]}{y}$$

$$\Rightarrow \lim_{y \rightarrow +\infty} \frac{y - \{y\}}{y}$$

$$\Rightarrow 1 - \lim_{y \rightarrow +\infty} \frac{\{y\}}{y}$$

$$\Rightarrow 1 - 0^+$$

$$\Rightarrow 1.$$

Q. 15. Prove that the sequence $\left\langle \left[1 + \frac{1}{n}\right]^2 \right\rangle \forall n \in \mathbb{N}$ converges.

(2013)

Ans. Let $\langle u_n \rangle = \left\langle \left[1 + \frac{1}{n}\right]^2 \right\rangle$

$$\therefore u_1 = 1 + \frac{1}{1} = 2$$

Now, $u_n = \left[1 + \frac{1}{n}\right]^2 = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \dots$

$$+ \frac{n(n-1)(n-2) \dots 1}{n!} \cdot \frac{1}{n^n}$$

(Using binomial expansions).

$$= 1 + 1 + \frac{1}{2!} \left[1 - \frac{1}{n}\right] + \frac{1}{3!} \left[1 - \frac{1}{n}\right] \left[1 - \frac{2}{n}\right] \cdot \frac{1}{n^2} + \dots + \frac{1}{n^2}$$

Similarly, $u_{n+1} = 1 + \frac{1}{2!} \left[1 - \frac{1}{n+1}\right] + \frac{1}{3!} \left[1 - \frac{1}{n+1}\right] \left[1 - \frac{2}{n+1}\right] \cdot \frac{1}{n^2} + \dots +$

$$\frac{1}{(n+1)!} \left[1 - \frac{1}{n+1}\right] \left[1 - \frac{2}{n+1}\right] \left[1 - \frac{2}{n+1}\right] \dots \left[1 - \frac{n}{n+1}\right]$$

... (1)

Q. 17. Evaluate $\lim_{n \rightarrow \infty} \frac{\cos n}{n}$

(2012)

Ans. we have

$$\lim_{n \rightarrow \infty} \frac{\cos n}{n} = \lim_{n \rightarrow \infty} m \cos \left(\frac{1}{m} \right)$$

$$\left[\because n = \frac{1}{m} \right]$$

... (2)

From (2) & (3), we have

$$u_{n+1} > u_n \quad \forall n \in \mathbb{N}$$

$\therefore$ The sequence $\langle u_n \rangle$ is monotonically increasing

Also from (1), we have

$$2 < u_n < 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

which is a geometric progression

$$= 1 - \frac{1 - \frac{1}{2n}}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3$$

so $\langle u_n \rangle$ is bounded

Hence sequence $\langle u_n \rangle$ is monotonically increasing and is bounded above by 3, so it is convergent.

Q. 16. Evaluate $\lim_{x \rightarrow \infty} \frac{1}{x}$.

(2012)

Ans. Given $x \rightarrow \infty \Rightarrow x$ is very large

$\Rightarrow \frac{1}{x}$ is very small and is hence close to zero.

$$\text{Thus, } x \rightarrow \infty \Rightarrow \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\text{Let } f(m) = m \cos\left(\frac{1}{m}\right)$$

$$\text{Now, RHL} = f(0+0) = \lim_{h \rightarrow 0} f(0+h).$$

$$= \lim_{h \rightarrow 0} h \cos\left(\frac{1}{h}\right)$$

$= 0 \times \text{a finite quantity between } -1 \text{ and } 1 = 0$

$$\text{LHL} = f(0-0) = \lim_{h \rightarrow 0} f(0-h)$$

$$= \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} (-h) \cos\left[\frac{-1}{h}\right]$$

$= 0 \times \text{a finite quantity lying between } -1 \text{ and } 1$
 $= 0$

$$\text{Thus, } f(0+0) = f(0-0) = 0$$

$$\text{Hence, } \lim_{h \rightarrow 0} = 0$$

Q. 18. Does $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ exist? If yes, find it. (20)

Ans. we have

$$f(x) = \sin\left[\frac{1}{x}\right]$$

$$\text{Now, RHL} = f(0+0) = \lim_{h \rightarrow 0} f(0+h)$$

$$= \lim_{h \rightarrow 0} \sin \frac{1}{(0+h)} = \lim_{h \rightarrow 0} \sin \frac{1}{h}$$

$= \text{not a finite quantity.}$

$$\text{Similarly, LHL} = f(0-0) = \lim_{h \rightarrow 0} f(0-h)$$

$$= \lim_{h \rightarrow 0} \sin\left[\frac{-1}{h}\right]$$

$= \text{not a finite quantity.}$

Thus RHL and LHL does not exist

Hence $\lim_{x \rightarrow 0} f(x)$ does not exist.

Q. 19. Let $f: R - \{0\} \rightarrow R$ be a map, where $f(x) = |x|^{-1}$, then show that $\lim_{x \rightarrow 0} f(x)$ does not exist. (2010)

Ans. We have $f(x) = |x|^{-1} = \frac{1}{|x|}$

$$\text{Now, RHL} = f(0+0)$$

$$= \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{1}{|0+h|}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} = \infty$$

$$\text{LHL} = f(0-0) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{1}{|0-h|}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} = \infty$$

Thus

$$\text{RHL} = \text{LHL} = \infty \text{ (not a finite quantity)}$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

20. Show that the function

$$f(x) = (1+2x)^{1/x} \text{ for } x \neq 0 \text{ and}$$

$$f(x) = e^2 \text{ for } x = 0 \text{ is continuous at } x = 0.$$

(2010)

Ans. Given $f(0) = e^2$

$$\text{Also, } f(0+h) = [1+2(0+h)]^{1/(0+h)} \\ = [1+2h]^{1/h}$$

$$\therefore f(0+0) = \lim_{h \rightarrow 0} (1+2h)^{1/h}$$

$$= \lim_{h \rightarrow 0} [(1+2h)^{1/2h}]^2 = e^2$$

Similarly,

$$f(0-0) = \lim_{h \rightarrow 0} (1-2h)^{-1/h} = \lim_{h \rightarrow 0} [(1-2h)^{-1/2h}]^2 = e^2$$

$$\text{Thus, } f(0+0) = f(0-0) = e^2$$

Hence, the given function is continuous at $x = 0$.

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Elementry Analysis (paper-2)

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Chapter - 3

Differentiability

Q. 1. If a function f is derivable on a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then show that there exists at least one point c between a & b such that $f'(c) = 0$ (2014)

Ans. Let $f'(a) > 0$ and $f'(b) < 0$ and, evidently, $\exists \delta_1, \delta_2 > 0$ such that

$$0 < h < \delta_1 \Rightarrow \left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| < f'(a)$$

$$\Rightarrow f(a) < f(a+h)$$

$$\text{and } 0 < h < \delta_2 \Rightarrow \left| \frac{f(b+h) - f(b)}{h} - f'(b) \right| < -f'(b)$$

$$\Rightarrow f(b) < f(b-h)$$

Since f is derivable on $[a, b]$ it is continuous on $[a, b]$ and it attains its supreme on $[a, b]$. The supreme is not attained at a or b , as $f(a) < f(a-h)$ & $f(b) < f(b-h)$ hence f attains supreme at some point, now we shall show that $f'(c) = 0$. If possible $f'(c) > 0$, since f is derivable at c .

For $\epsilon = f'(c) > 0$, $\exists \delta_3 > 0$ such that,

$$0 < h < \delta_3 \Rightarrow \left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < f'(c).$$

$$\Rightarrow f(c) < f(c+h)$$

Therefore $f(c) \neq \sup f$ on $[a, b]$ thus $f'(c) \neq 0$

Similarly, $f'(c) \neq 0$, so that $f'(c) = 0$.

Q. 2. State & Prove Cauchy mean value theorem. (2014)

Ans. Cauchy's Mean Value Theorem (or Second Mean Value Theorem)

Let f and g be two functions such that

(i) f and g are continuous on $[a, b]$

(ii) f and g are derivable on (a, b)

(iii) $g'(x) \neq 0$ for any point of (a, b)

then there exist a real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Let us define a function

$$F(x) = f(x) + A \cdot g(x)$$

Where A is constant to be determined such that

$$F(a) = F(b)$$

Now, the function F is the sum of two continuous and differentiable functions, therefore

(i) F is continuous on $[a, b]$ (ii) F is differentiable on (a, b) (iii) $F(a) = F(b)$, then by Rolle's Theorem, there exist a real number c between a and b such that

$$F'(c) = 0$$

$$F'(x) = f'(x) + A \cdot g'(x)$$

$$\Rightarrow F'(c) = f'(c) + A \cdot g'(c)$$

$$\Rightarrow f'(c) + A \cdot g'(c) = 0 \quad (\because F'(c) = 0)$$

$$-A = \frac{f'(c)}{g'(c)}$$

...(3)

$$\text{Now } F(a) = F(b) \Rightarrow f(a) + A \cdot g(a) = f(b) + A \cdot g(b)$$

$$\Rightarrow -A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

....(4)

$$\text{From (3) and (4), we have, } \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Q. 3. Examine the validity of the hypothesis and conclusion of Rolle's theorem:

$$f(x) = 1 - (x-1)^{2/3} \text{ on } [0, 2]$$

(2014)

Ans. Since the function, $(x-1)^{2/3}$ is continuous in $[0, 2]$, so

$$f'(x) = -\frac{2}{3}(x-1)^{-1/3}$$

so, $f'(x)$ exists on $[0, 2]$

$$f(0) = 0$$

$$f(2) = 1 - (2-1)^{2/3}$$

$$f(2) = 1 - 1 = 0$$

$$\therefore f(0) = f(2).$$

Thus f satisfies all the three conditions of Rolle's theorem, hence $\exists c \in (0, 2)$ such that

$$f'(c) = 0$$

$$\frac{2}{3}(c-1) = 0$$

$$c = 1$$

Since $c = 1 \in (0, 2)$, where $f'(1) = 0$.

Q. 4. State and Prove that chain rule for differentiation. (2015, 2017)

Ans. (Chain Rule): Let f and g be two functions such that range of f is contained in the domain of g . If f is differentiable at $x = x_0$, and g is differentiable at $f(x_0)$ and $(g \circ f)$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

Proof. the range of f is contained in the domain of g

$(g \circ f)$ has the same range as that of f .

Now, let $y = f(x)$ and $y_0 = f(x_0)$

If f is differentiable at $x = x_0$, we have $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

$$\text{or } f(x) - f(x_0) = (x - x_0)[f'(x_0) + A(x)]$$

Where $A(x) \rightarrow 0$ as $x \rightarrow x_0$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\text{or } g(y) - g(y_0) = (y - y_0)[g'(y_0) + B(y)] \quad \dots(2)$$

Where $B(y) \rightarrow 0$ as $y \rightarrow y_0$

Now, $(g \circ f)(x) - (g \circ f)(x_0) = g(f(x)) - g(f(x_0))$

$$= (y - y_0)[g'(y_0) + B(y)] \quad [\text{using (2)}]$$

$$= [f(x) - f(x_0)][g'(y_0) + B(y)]$$

$$= (x - x_0)[f'(x_0) + A(x)][g'(y_0) + B(y)] \quad [\text{using (1)}]$$

$$\text{Thus, if } x \neq x_0, \text{ then } \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = [g'(y_0) + B(y)][f'(x_0) + A(x)]$$

(x)

Also f being differentiable at $x = x_0$ is continuous at $x = x_0$ and

hence, $x \rightarrow x_0, f(x) \rightarrow f(x_0)$ i.e. $y \rightarrow y_0$

This gives $B(y) \rightarrow 0$ as $x \rightarrow x_0$ and $A(x) \rightarrow 0$ as $x \rightarrow x_0$

Now taking the limit as $x \rightarrow x_0$, we have from (3)

$$\lim_{x \rightarrow x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(y_0) f'(x_0)$$

Hence $g \circ f$ is differentiable at $x = x_0$ and $(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0)$

Q. 5. Find the value of c for which Cauchy's mean value theorem

hold for functions $\sin x$ and $\cos x$ on $\left[-\frac{\pi}{2}, 0\right]$. (2015)

Ans. Let $f(x) = \sin x$ and $g(x) = \cos x$. It can be easily seen that the functions $\sin x$ and $\cos x$ both are continuous on $\left[-\frac{\pi}{2}, 0\right]$ and differentiable on (a, b) .

Also $g'(x) = -\sin x \neq 0$ for any point on $\left[-\frac{\pi}{2}, 0\right]$.

Then, by C.M.V.T., $\exists$ at least one $c \in \left[-\frac{\pi}{2}, 0\right]$.

such that

$$\frac{f(0) - f\left(-\frac{\pi}{2}\right)}{g(0) - g\left(-\frac{\pi}{2}\right)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sin 0 + \sin \frac{\pi}{2}}{\cos 0 - \cos \frac{\pi}{2}} = \frac{\cos c}{-\sin c}$$

$$\Rightarrow \frac{0+1}{1-0} = \frac{\cos c}{-\sin c}$$

$$\Rightarrow -\tan c = 1 = \tan \frac{\pi}{4}$$

$$\Rightarrow c = -\frac{\pi}{4} \text{ Since } c = -\frac{\pi}{4} \text{ lies in } \left[-\frac{\pi}{2}, 0\right]. \text{ Hence C.M.V.T. is satisfied for}$$

$$c = -\frac{\pi}{4}$$

Q. 6. Examine the validity of the hypothesis and conclusion of Cauchy's mean value theorem:

$f(x) = \exp x$, $g(x) = \exp(-x)$ on $[0, 1]$. (2015)

Ans. It can be easily seen that the functions both are continuous on $[0, 1]$ & differentiable on $(0, 1)$.

Let $f(x) = \sin x$ and $g(x) = \cos x$. It can be easily seen that the functions

$\sin x$ and $\cos x$ both are continuous on $\left[-\frac{\pi}{2}, 0\right]$ and differentiable on (a, b)

Then, by C.M.V.T., $\exists$ at least one $c \in (0, 1)$

Such that

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{e - 1}{e^{-1} - 1} = \frac{e^x}{-e^{-x}}$$

$$\Rightarrow \frac{(e - 1)}{(1 - e)} e = \frac{e^x}{-e^{-x}}$$

$$\Rightarrow e = \frac{e^c}{e^{-c}}$$

$$\Rightarrow e = e^{2c}$$

$$\Rightarrow 1 = 2c$$

$$\Rightarrow c = \frac{1}{2}$$

$\Rightarrow$ Since, lies in $(0, 1)$. Hence

C.M.V.T. is satisfied for $c = \frac{1}{2}$.

Q. 7. Give an example (with details) of a continuous function $f: [a, b] \rightarrow \mathbb{R}$, which is differentiable at end points. (2016)

Ans. For a radius $r > 0$, consider the function

$$f(x) = \sqrt{r^2 - x^2}, x \in [-r, r]$$

Its graph is the upper semicircle centered at the origin.

This function is continuous on the closed interval $[-r, r]$ & differentiable at the endpoints $-r$ and r . Since $f(-r) = f(r)$, Rolle's theorem applies and indeed; there is a point where derivative of f is zero. Note that the theorem applies even when the function cannot be differentiated at the endpoints because it only requires the function to be differentiable in the open interval.

Q. 8. State and prove the Rolle's theorem. (2016, 2018)

Ans. Rolle's Theorem: If a function f is

(i) continuous in the closed interval (a, b)

(ii) derivable in the open interval (a, b)

(iii) $f(a) = f(b)$

then there exist at one value c of x in (a, b) such that $f'(c) = 0$

Proof: Since, function $f(x)$ is continuous on (a, b) . So it is bounded as every continuous function is bounded. Also, $f(x)$ attains its bounds, as function which is continuous on closed bounded interval $[a, b]$, attains a bound on $[a, b]$.

Let M and m are least upper bound and greatest upper bound of $f(x)$ respectively. Now there are two possibilities.

(i) $M = m$ (ii) $M \neq m$

(i) If $M = m$, then obviously $f(x)$ is a constant function.

so its derivative $f'(x) = 0 \quad \forall x \in (a, b)$.

(ii) If $M \neq m$, then at least one of members M and m must be different from the equal values $f(a)$ and $f(b)$.

Let $M \neq f(a)$

Since every continuous function on a closed interval attains its l.u.b. therefore there exist a real number $c \in [a, b]$ such that $f(c) = M$.

Also, $f(a) \neq m \neq f(b)$

Now $f(c)$ is l.u.b. of f on $[a, b]$

$\therefore f(x) \leq f(c)$ for all $x \in [a, b]$... (1)

(by definition of l.u.b.)

In particular $f(c - h) \leq f(c)$, $h > 0$

$$\frac{f(c - h) - f(c)}{-h} \geq 0 \quad \dots (2)$$

$\therefore f(x)$ exist at each point of (a, b) , Thus $f'(c)$ exist

Hence, $L f'(c) \geq 0$ (from (2)) ... (3)

Similarly $f(c + h) \leq f(c)$, $h > 0$ (from (1)) ... (4)

Then by the same arguments $R f'(c) \leq 0$... (5)

From (3), (4) and (5), we have, $f'(c) = 0$

Similarly we can consider the case $M = f(a) \neq m$.

Thus we find that $f'(c) = 0$, where $c \in (a, b)$.

Q. 9. State the Lagrange's mean value theorem. (2016, 2018, 2019)

Ans. **Statement:**

If a function f is

- (i) continuous in the closed interval $[a, b]$
- (ii) derivable in the open interval (a, b) .

then there exist a real number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

Proof: Let us define a function $g(x)$ such that

$$g(x) = f(x) + Ax \quad \forall x \in [a, b] \quad \dots(1)$$

where A is a constant such that

$$g(a) = g(b)$$

$$\text{i.e. } f(a) + Aa = f(b) + Ab$$

$$\text{i.e. } -A(b - a) = f(b) - f(a)$$

$$\text{i.e. } -A = \frac{f(b) - f(a)}{b - a} \quad \dots(2)$$

Now (i) Since f is continuous on $[a, b]$ and Ax being polynomial is continuous in $[a, b]$, therefore g is continuous in $[a, b]$

(ii) g is differentiable $\because f(x)$ and Ax both are differentiable in (a, b)

(iii) $g(a) = g(b)$

Hence g satisfies all the conditions of Rolle's Theorem in $[a, b]$, therefore there exists at least one real number c in (a, b) such that $g'(c) = 0$

We have,

$$g(x) = f(x) + Ax$$

$$\Rightarrow g'(x) = f'(x) + A$$

$$\Rightarrow g'(c) = f'(c) + A$$

$$\Rightarrow f'(c) = -A \quad (\because g'(c) = 0)$$

From (2) and (3), we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Q. 10. Show that the function $f: R \rightarrow R$ defined

$$\text{by } f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable function.

(2017)

Ans. We have

$$f(0+0) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} (0+h)^2 \sin \frac{1}{0+h} = \lim_{h \rightarrow 0} h^2 \sin \frac{1}{h} = 0$$

and

$$f(0-0) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} (0-h)^2 \sin \frac{1}{0-h} = \lim_{h \rightarrow 0} h^2 \sin \frac{1}{h} = 0$$

also $f(0) = 0$

$$\therefore f(0+0) = f(0) = f(0-0)$$

Hence the given function is continuous at $x = 0$

$$\text{Now } Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{(-h)^2 \sin \left(-\frac{1}{h} \right) - 0}{-h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

$$\text{and } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

$$\therefore Lf'(0) = Rf'(0)$$

Hence $f(x)$ is differentiable at $x = 0$.

Q. 11. Let $f: [a, b] \rightarrow R$ be a continuous function such that f is differentiable over (a, b) . If $f'(x) > 0 \quad \forall x \in (a, b)$, then prove that f is strictly monotonic increasing. (2017)

Ans. Let $f'(x) > 0$ for all $x \in (a, b)$

Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$, then, clearly f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) therefore by mean value theorem, $\exists$ a real number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\because f'(x) > 0$, for all $x \in (a, b)$ in particular $f'(c) > 0$

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Elementry Analysis (paper-2)

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$$\text{Now, } f'(c) > 0 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$$\Rightarrow f(x_2) - f(x_1) > 0 \quad (\because x_2 - x_1 > 0 \text{ when } x_1 < x_2)$$

$$\Rightarrow f(x_2) > f(x_1) \text{ for } x_2 > x_1$$

$$\Rightarrow f(x_1) < f(x_2)$$

$$\text{Thus } x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Hence f is strictly increasing function on $[a, b]$

Q. 12. Show that polynomial $x^5 - 7x - 2$ has exactly one real root. (2017)

Ans. Given function $f(x) = x^5 - 7x - 2$ is continuous as well as derivable on every interval being a polynomial function

Now,

$$f'(x) = 5x^4 - 7$$

$$\therefore f'(c) = 5c^4 - 7$$

$$\text{Now, } f'(c) = 0$$

$$5c^4 = +7$$

$$c^4 = \frac{7}{5}$$

$$C^2 = \pm \sqrt{\frac{7}{5}} = \pm 1.183$$

$$c = \pm 1.087, \pm 1.087i$$

Hence, given function has real roots.

Q. 13. Let a function f is differentiable at point $x = x_0$ and $c \in \mathbb{R}$, then function cf is also differentiable at x_0 and prove that $(cf)'(x_0) = cf'(x_0)$. (2018)

Ans. By definition of derivative of a functions f at $x = x_0$, we have

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\text{Now consider } \lim_{x \rightarrow x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{cf(x) - cf(x_0)}{x - x_0}$$

$$= \lim_{x \rightarrow x_0} \left[c \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \right] = c = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x_0)$$

Hence function cf is differentiable at $x = x_0$ and $(cf)'(x_0) = cf'(x_0)$.

Q. 14. Determine values of a, b, c so that function f defined by -

$$f(x) = \begin{cases} 4x & ; \text{if } x \leq 0 \\ ax^2 + bx + c & ; \text{if } 0 < x < 1 \\ 3 - 2x & ; \text{if } x \geq 1 \end{cases}$$

very differentiable
function is also
continuous function.
(2019)

is differentiable on $\mathbb{R}$.

Ans. For differentiability at $x = 0$.

$$Lf'(0) = \frac{d}{dx} f(x) = \frac{d}{dx} (4x) = 4$$

$$Rf'(0) = \frac{d}{dx} f(x) = \frac{d}{dx} (ax^2 + bx + c)$$

$$= 2ax + b = b \rightarrow x=0 \text{ putting}$$

$$\therefore Lf'(0) = Rf'(0)$$

$$\Rightarrow 4 = b$$

$$[b = 4]$$

For differentiability at $x = 1$.

$$Lf'(1) = \frac{d}{dx} f(x) = \frac{d}{dx} (ax^2 + bx + c)$$

$$= 2ax + b = 2a + 4. \leftarrow \text{put } x=1$$

$$Rf'(1) = \frac{d}{dx} f(x) = \frac{d}{dx} (3 - 2x)$$

$$= -2$$

$$\Rightarrow Lf'(1) = Rf'(1)$$

$$\Rightarrow 2a + 4 = -2$$

$$\Rightarrow 2a = -6$$

$$\Rightarrow a = -3$$

If $f(x)$ is differentiable at $x = 1$, then it is also continuous at $x = 1$.

$\therefore \text{LHL} = \text{RHL}$

$$\Rightarrow \lim_{x \rightarrow 1} (ax^2 + bx + c) = \lim_{x \rightarrow 1} (3 - 2x)$$

$$\Rightarrow -3(1)^2 + 4(1) + c = 3 - 2(1)$$

$$\Rightarrow 1 + c = 1$$

$$\Rightarrow c = 0$$

Q. 15. A function f is continuous in $[0, 1]$ & differentiable in $(0, 1)$.
Prove that $f'(c) = f(1) - f(0)$, where $c \in (0, 1)$. (2018)

Ans. As the function $f(x)$ is continuous in $[0, 1]$ & differentiable in $(0, 1)$,
 $f(1) \neq f(0)$.

Hence the condition of L.M.V. theorem are satisfied.
Now, by applying L.M.V. theorem, to $f(c)$ in $0 < c < 1$, we get

$$\frac{f(1) - f(0)}{1 - 0} = f'(c), \text{ where } 0 < c < 1$$

Hence, $f'(c) = f(1) - f(0)$.

Q. 16. Prove or disprove that $f(x) = x|x|$ is not differentiable at $x = 0$. (2013)

Ans. We have,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-h|h| - 0}{-h} = \lim_{h \rightarrow 0} h = 0$$

$$\text{and } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h|h| - 0}{h}$$

$$= \lim_{h \rightarrow 0} h = 0$$

$$\therefore Lf'(0) = Rf'(0).$$

Hence, given function differentiable at $x = 0$

Q. 17. Let f be a function from $R \rightarrow R$ such that

$$|f(x) - f(y)| \leq (x-y)^2 \quad \forall x, y \in R$$

show that it is differentiable and its derivative is 0 at all points. Deduce that it is constant. (2011)

Ans. We have

$$|f(x) - f(y)| \leq (x-y)^2 \Rightarrow \left| \frac{f(x) - f(y)}{x-y} \right| \leq x-y$$

Taking limits of both sides as $y \rightarrow x$, we have

$$\lim_{y \rightarrow x} \left| \frac{f(x) - f(y)}{x-y} \right| \leq \lim_{y \rightarrow x} (x-y) = 0$$

$$\Rightarrow |f'(x)| = 0$$

$$\text{as } \lim_{y \rightarrow x} \frac{f(x) - f(y)}{x-y} = f'(x)$$

$$\Rightarrow f(x) = \text{constant}$$

Q. 18. Show that $\log(1+x)$ lies between $x - \frac{x^2}{2}$ and $x - \frac{x^2}{2(1+x)}$,
for all $x > -0$. (2010)

Ans. Let us consider,

$$f(x) = \log(1+x) - \left[x - \frac{x^2}{2} \right]$$

$$\Rightarrow f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{1 - (1-x^2)}{1+x} = \frac{x^2}{1+x} > 0, \quad \forall x > 0$$

which implies function $f(x)$ is an increasing function for all $x > 0$. Also,

we have

$$f(0) = \log(1+0) - (0) = \log 1 = 0$$

Thus,

$$f(x) > 0, \quad \forall x > 0.$$

Hence,

$$\log(1+x) > x - \frac{x^2}{2(1+x)}, \quad \forall x > 0$$

Further by considering the function

$$\log(1+x) > x - \frac{x^2}{2}$$

$$\phi(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

It can be shown that

$$\log(1+x) < x - \frac{x^2}{2(1+x)}, \forall x > 0 \quad \dots(2)$$

From (1) & (2), we have

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, \forall x > 0$$

Rolle's theorem

Q. 19. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$ show that $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ will give at least one root between 0 & 1. (2012)

Ans. Let $f(x) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$ Here, $f(x)$ is polynomial, so $f(x)$ is continuous and differentiable for all x . So, $f(x)$ is continuous in $[0, 1]$ and derivable in $(0, 1)$.

Also, we have

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$$

and $f(0) = 0$.

Thus $f(1) = f(0) = 0$

Hence all the three condition of Rolle's theorem are satisfied.

Now,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$$

$$\therefore f'(c) = 0$$

$$\Rightarrow a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + \dots + a_n = 0$$

Hence, there lies one point $c \in (0, 1)$

Q. 20. Verify lagrange's Mean value theorem for function

$$f(x) = x(x-1)(x-2) \ln \left[0, \frac{1}{2} \right] \quad (2012)$$

Ans. we have

$$f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

$$\therefore f'(x) = 3x^2 - 6x + 2$$

$$\Rightarrow f'(c) = 3c^2 - 6c + 2$$

$$\text{Let } a = 0 \text{ and } b = \frac{1}{2}$$

$$f(a) = f(0) = 0$$

$$f(b) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) = \frac{3}{8}$$

From L.M.V.T., we have

$$f(b) = f(a) + (b-a)f'(c)$$

$$\Rightarrow f\left(\frac{1}{2}\right) - f(0) = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 0$$

$$\Rightarrow c = \frac{24 \pm \sqrt{21}}{24} = 1 \pm \sqrt{\frac{7}{12}}$$

Taking -ve sign, $c = 1 - \sqrt{\frac{7}{12}}$, which lies in, hence L.M.V.T. is verified.

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